Lesson: A Review of Solving Radical Equations and Equations with Rational Exponents

What is the general strategy with solving radical equations?

What do you have to watch out for when solving radical equations?

<u>Examples</u> Solve each equation.

a. $x - 3\sqrt{x+2} = 2$

b. $\sqrt{6x+7} - 9 = x - 7$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Solve: $\sqrt{3x+4}-5=\sqrt{3x-11}$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Solve each equation.

a. $\sqrt[3]{2x+1} = \sqrt[3]{3x+5}$

b. $\sqrt[4]{x+3} = 2$

Example Solve each equation.

a. $x^{\frac{5}{4}} = 32$

b. $(2x+1)^{\frac{2}{3}} = 36$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Solve for x: $y = k\sqrt{x}$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: A Review of Solving Equations Quadratic in Form (Using Substitution)

<u>Examples</u> Solve each equation.

a. $x^4 - 10x^2 + 9 = 0$

b. $(x+1)^{\frac{2}{5}} - 3(x+1)^{\frac{1}{5}} + 2 = 0$

c. $x^{-\frac{2}{3}} - x^{-\frac{1}{3}} - 6 = 0$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: A Review of Basic Inequalities

Please check out the examples video on how to graph on a number line, use interval notation, or use set notation in case you forgot!

The Inequality Signs

$<$ Less than

$\leq$ Less than or equal to

$>$ Greater than

$\geq$ Greater than or equal to

The Properties of Inequalities

If a, b, and c are real numbers then:

1. If $a < b$, then $a + c < b + c$

2. If $a < b$ and $c > 0$ then $ac < bc$

3. If $a < b$ and $c < 0$, then $ac > bc$

These properties also hold if you switch to $>, \leq, \geq$

<u>Examples</u> Solve the following. Put your final answer in interval notation.

$-2x + 8 < 10$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Solve the following. Put your final answer in interval notation.

a. $-\frac{1}{3}(x+3)+\frac{1}{3}x+\frac{2}{5}x \leq \frac{1}{10}$

b. $-5 \leq 1+2x < 13$

c. $2 \leq -6x-3 < 9$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Quadratic Inequalities

A quality inequality is an inequality in the form: $ax^2 + bx + c < 0$ where a, b, and c are real numbers. You can interchange $<$ with $\leq, >, \geq$.

There are 3 steps to solving a quadratic inequality. Write out each step as you solve the next example.

<u>Example</u> Write out the steps that you use to solve the following: $x^2 - x - 6 > 0$. Write your answer in interval notation.

Explain Step 1:

Explain Step 2:

Explain Step 3:

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Solve: $3x^2 - 4 \leq x$. Write your answer in interval notation.

Example Solve $x^2 \geq 25$. Write your answer in interval notation.

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Review: Piecewise Functions for Calc Students

Piecewise functions look like this:

Explain how to graph the following examples. It would be best to actually write out directions for yourself in case you ever need to reference this again.

a. $f(x) = \begin{cases} 2, & x \geq 3 \\ -3, & x < 3 \end{cases}$

b. $f(x) = \begin{cases} 4, & x \geq 2 \\ x+2, & -1 \leq x < 2 \\ -2x+1, & x < -1 \end{cases}$

c. $f(x) = \begin{cases} 3, & x > 4 \\ 2, & -1 < x \leq 1 \\ x^2, & x < -1 \end{cases}$

d. $f(x) = \begin{cases} 3x+1, & x \geq 3 \\ x+1, & x < -2 \\ 3, & x = 0 \end{cases}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Review: Basic Graphs and Translations

Sketch the basic graphs from the video.

Copyright 2024, Dr. Jennifer Hill, DivideandConquerMath.com

Explain the basic translations.

<u>Examples</u> State what translations have occurred and sketch the graph.

a. $f(x) = (x + 2)^2 - 3$

b. $g(x) = -|x - 2| + 1$

c. $f(x) = \sqrt{x + 3}$

Copyright 2024, Dr. Jennifer Hill, DivideandConquerMath.com

Review: Zeros vs Factors vs Intercepts

Use the following functions. Find the zeros, factors, and x-intercepts of each function.

a. $f(x) = x^2 + 6x + 8$

b. $g(x) = x^2 - 9$

c. $f(x) = x^2 - 7x + 12$

Copyright 2024, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Rational Inequalities

Rational inequalities look like this:

$\frac{5}{x+3} > 3$ or $\frac{2x+5}{4x-1} \leq 2$

We will out the steps to solve a rational inequality by using the following example.

<u>Example</u> Solve $\frac{4}{x+2} \geq 6$ Put your final answer in interval notation.

Explain Step 1 and show how to do it:

Explain Step 2 and show how to do it:

Explain Step 3 and show how to do it:

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

What do you have to watch out for with rational inequalities?

<u>Example</u> Solve $\frac{3}{x+6} \leq 2$ Put your final answer in interval notation.

<u>Example</u> Solve $\frac{x+3}{x-5} \geq 1$ Put your final answer in interval notation.

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: A Review of Absolute Value Equations and Inequalities

We can summarize how to solve each case with the following chart:

Case (k is a number)	How to Setup
$\lvert x \rvert = k$	
$\lvert x \rvert < k$ (or $\leq k$)	
$\lvert x \rvert > k$ (or $\geq k$)	

<u>Examples</u> Solve the following. Put your answer on a number line and in interval notation when you have an inequality.

a. $|x + 3| = 5$

b. $|2x - 4| \geq 6$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Examples Solve the following. Put your answer on a number line and in interval notation when you have an inequality.

a. $|4x - 2| = |6x + 8|$

b. $|5x - 2| + 1 < 9$

c. $|2 - x| - 1 > 9$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Special Cases!

<u>Example</u> Explain what is special about each case below and how to solve the problem.

a. $|4x - 1| = -7$

b. $|4x - 1| = 0$

c. $|4x - 1| < -7$

d. $|4x - 1| > -7$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Explain what is special about each case below and how to solve the problem.

a. $|4x - 1| \geq 0$

b. $|4x - 1| \leq 0$

c. $|4x - 1| < 0$

d. $|4x - 1| > 0$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Piecewise Functions

What is a piecewise function? Draw an example of what they look like.

<u>Example</u> Given the function: $f(x) = \begin{cases} 3x - 1, & x \geq 2 \\ \frac{1}{2}x - 4, & x < 2 \end{cases}$, evaluate the following:

a. $f(3) =$

b. $f(-1) =$

c. $f(2) =$

<u>Example</u> Given the function: $f(x) = \begin{cases} 4, & x > 3 \\ 4x - 2, & -2 < x \leq 3 \\ x^2 - 5, & x \leq -2 \end{cases}$ evaluate the following:

a. $f(4) =$

b. $f(-5) =$

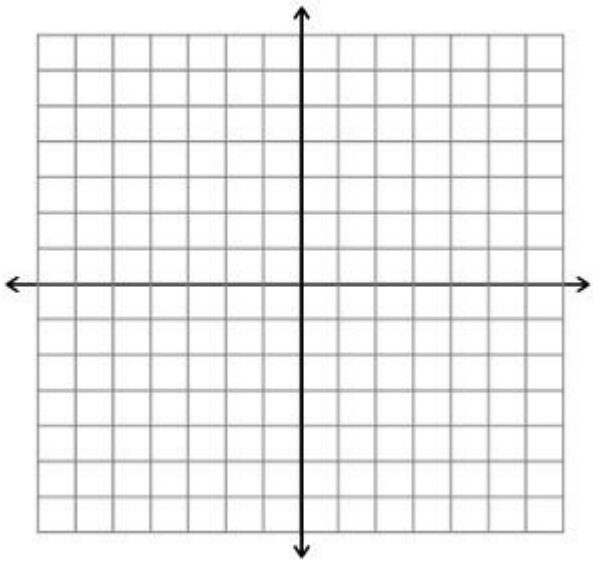
c. $f(-2) =$

d. $f(3) =$

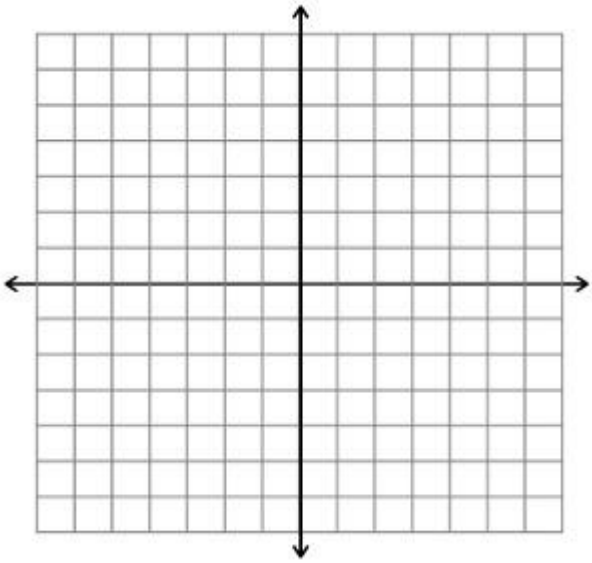
Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Show how to graph the following piecewise functions.

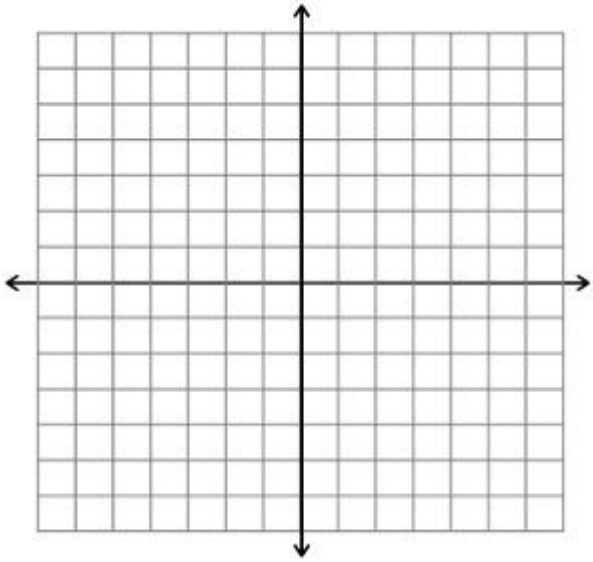
a. $f(x) = \begin{cases} 4, & x > 5 \\ -1, & x \leq 5 \end{cases}$



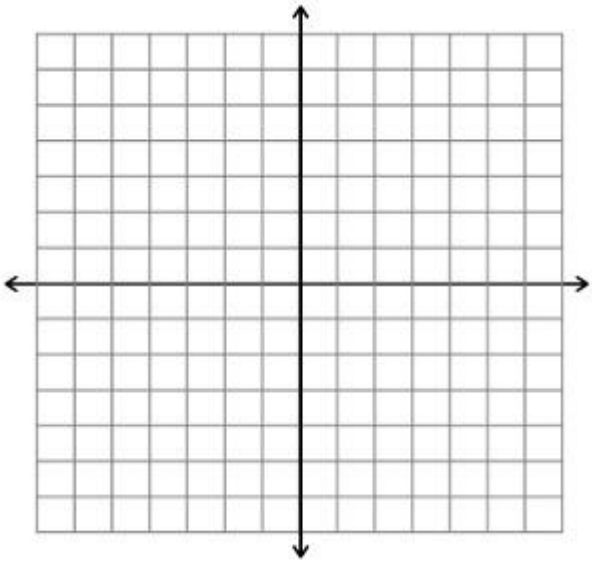
b. $f(x) = \begin{cases} 3x + 1, & x > -3 \\ -2x, & x \leq -3 \end{cases}$



c. $f(x) = \begin{cases} 3, & x > 2 \\ 2x, & -1 < x \leq 2 \\ -x + 3, & x \leq -1 \end{cases}$



d. $f(x) = \begin{cases} 3x + 1, & x \geq 1 \\ 6, & -1 \leq x < 1 \\ x^2, & x < -1 \end{cases}$



Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: The Greatest Integer Function

Show how to graph the greatest integer function $f(x) = [\![x]\!]$

<u>Example</u> Graph $f(x) = \left[\!\left[\frac{1}{2}x - 2\right]\!\right]$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: A Quick Review of Some Things About Graphing

Remind yourself how to graph the following points. Draw your own graph.

Point 1: $(-3, 4)$ Point 2: $(0, -2)$

Draw another rectangular coordinate system below and label the quadrants and origin.

Often when graphing, we use equations in two variables.

<u>Example</u> Find the ordered pair associated with the equation $3x + y^2 = 7$ when $x = 1$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Find at least 3 ordered pairs for the following equations.

a. $y = \sqrt{x+2}$

b. $2x + 3y = 6$

Finding 3 points can be a strategy for graphing.

Example Sketch a graph for the previous 2 examples.

What is the x-intercept? And how do you find it?

What is the y-intercept? And how do you find it?

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Examples Find the x- and y-intercepts for the following equations.

a. $y = 3x - 1$

b. $x = \sqrt{y - 1}$

c. $x^2 - 3x + 2 = y$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: The Distance Formula

Graph the points $(2, 4)$ and $(-4, -6)$. Explain how you can find the distance between these points by summarizing the explanation given in the video (make a decent effort to explain this – you will have an easier time remembering the formula if you understand WHY it works).

State the distance formula:

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Find the distance between the following points: $(4, -1)$ and $(7, 3)$

There are 2 ways we can use the distance formula.

Example Show how to use the distance formula to determine whether the following points are the vertices of a right triangle.

a. $(0, -2), (-4, 3)$ and $(5, 1)$

b. $(-2, -8), (0, -4)$ and $(-4, -7)$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

What does it mean for 3 points to be collinear?

<u>Example</u> Show how to use the distance formula to determine if the 3 points are collinear.

a. $(-1, 4), (-2, -1), (1,14)$

b. $(-2, -4), (-6, 11), (0, -12)$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: The Midpoint Formula

What is the midpoint formula?

So what is the midpoint formula effectively doing?

<u>Example</u> Find the midpoint between the points $(-5, -6)$ and $(-3, 2)$

<u>Example</u> The midpoint of a line segment is $(5, 8)$ and the endpoint of that line segment is $(11, 10)$. What is the OTHER endpoint on this line segment?

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example The midpoint of a line segment is $(3, 1)$ and the endpoint of that line segment is $(6, 8)$. What is the OTHER endpoint on this line segment?

Example The midpoint of a line segment is $(2a, 5b)$ and the endpoint of that line segment is $(-1a, 9b)$. What is the OTHER endpoint on this line segment?

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Review of Functions

How do you represent a function?

What is a function often interchangeable with? How can you rewrite $f(x) = 3x + 8$?

So what IS a function? What does it actually represent?

What's another handy feature of functions?

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

What are the domain and range?

Does domain and range only work for functions?

<u>Examples</u> Are the following functions, and what is the domain and range?

a. $\{(2,3), (5, -1), (-3, 7)\}$

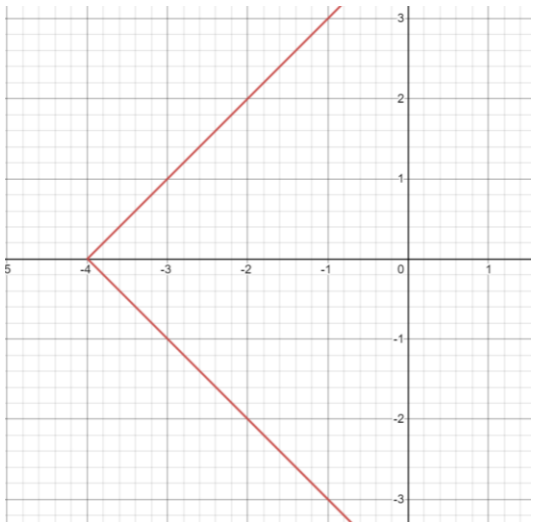
b. $\{(-1, 5), (-8, 3), (-1, 6)\}$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

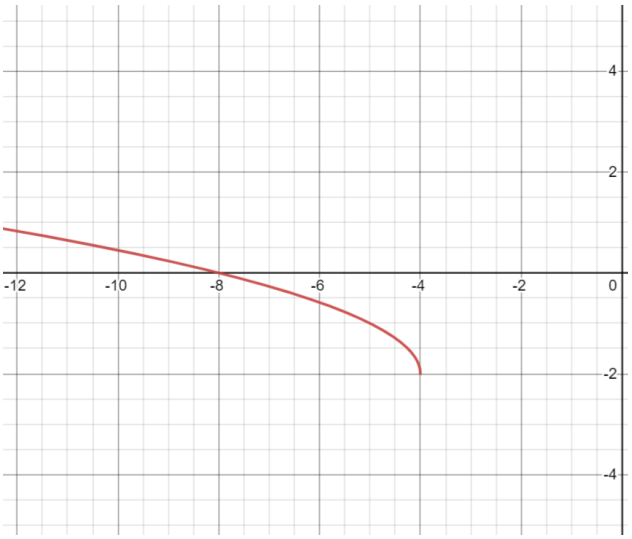
What is the Vertical Line Test, and how does it help you determine if you have a function?

<u>Example</u> State the domain and range of the following, and use the vertical line test to determine if the graph represents a function.

a.



b.



Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Examples Determine if the following represent functions.

a. $y = 3x^2$

b. $x = y^2$

c. $y = \frac{5}{x^2}$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Write the following in function notation: $3x + 2y = 6$

Example Suppose $f(x) = 3x - 5$ and $g(x) = x^2 + 4x - 1$ and $h(x) = \{(2,5),(6,-3),(-1,-4)\}$. Find the following:

a. $f(7) =$

b. $g(3) =$

c. $f(k) =$

d. $g(x + h) =$

e. $h(6) =$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Increasing, Decreasing, and Constant Functions

Below are the "formal math" definitions of whether a function is increasing, decreasing, or constant. Let I be an open interval and let x_1 and x_2 be two points in that interval.

- A function is **increasing** over I if when $x_1 < x_2, f(x)_1 < f(x_2)$
- A function is **decreasing** over I if when $x_1 < x_2, f(x)_1 > f(x_2)$
- A function is **constant** over I if when $x_1 < x_2, f(x)_1 = f(x_2)$

While these are the formal definitions, what we really want to know are the intuitive meanings of these definition.

Sketch a graph that is increasing:

Decreasing:

And constant:

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Sketch the graph in the video, and then identify where the graph is increasing, decreasing, or constant.

a.

b.

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Graphs of Basic Functions

In this section, we will review some basic graphs that are vital to know for this class and calculus. Knowing the "basic shapes" of certain graphs will make your life much easier and give you greater intuition when graphing.

Show hot to graph the following and highlight the **key points** and **key patterns.** This will come in handy for the next lesson. Take your time to sketch your own NEAT CLEAN CLEAR graphs.

$f(x) = x$

$f(x) = x^2$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

$f(x) = x^3$

$f(x) = \sqrt{x}$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

$f(x) = \sqrt[3]{x}$

$f(x) = |x|$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Translations and Reflections of Functions

Using the graph $f(x) = |x|$ as the "base function," sketch how vertical translations work (summarize what was discussed in the video).

How do you describe vertical translations in function notation?

Using the graph $f(x) = x^2$ as the "base function," sketch how horizontal translations work (summarize what was discussed in the video).

How do you describe horizontal translations in function notation?

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Use the graph $f(x) = \sqrt{x}$ to show how reflections over the x- and y-axis work (summarize what was discussed in the video).

How do you describe reflections over the x-axis and y-axis using function notation.

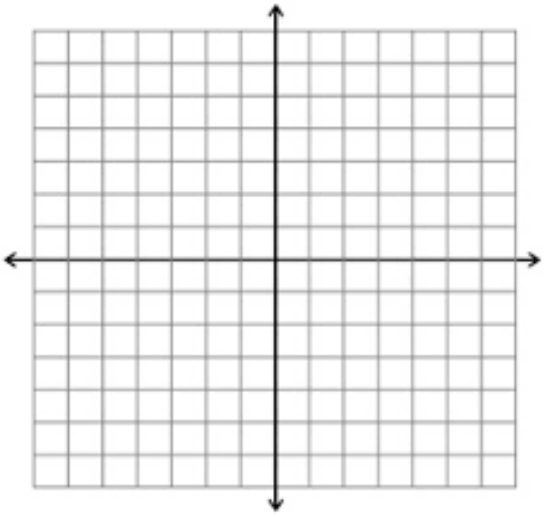
Use the graph $f(x) = |x|$ to show how stretching and shrinking works (summarize what was discussed in the video).

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

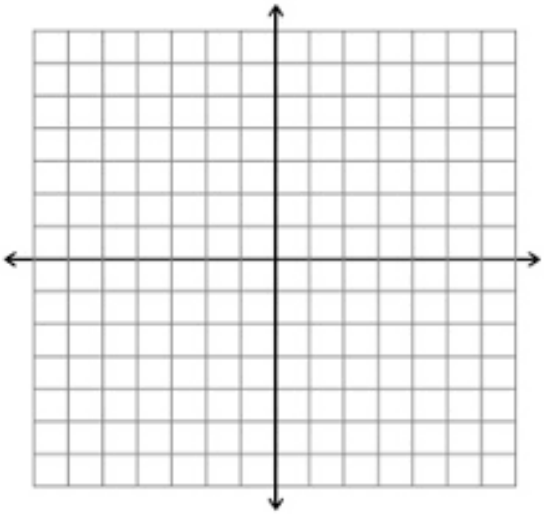
Explain how stretching and shrinking works with function notation.

<u>Example</u> Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $f(x) = -x^2 + 3$



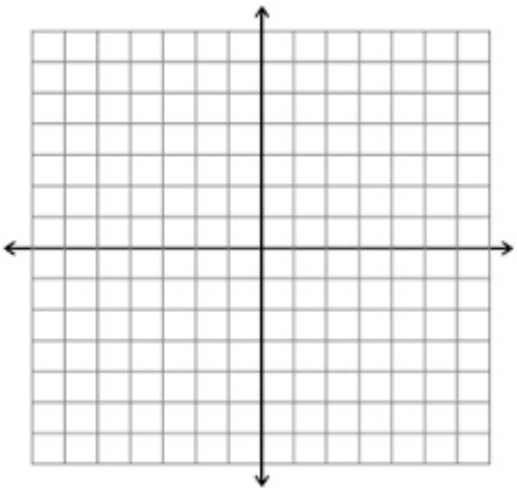
b. $g(x) = (x + 2)^3 - 3$



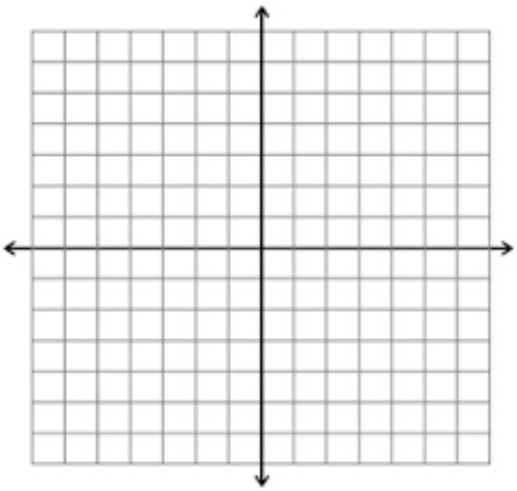
Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Example Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $g(x) = -2(x+1)^2 - 5$



b. $h(x) = 2\sqrt{-x} + 1$



Examples Write the equation of the given functions with the following characteristics.

a. The function $f(x) = x^2$ moved 3 units left and 5 units down.

b. The function $g(x) = \sqrt[3]{x}$ reflected over the y-axis, translated 2.5 units right and 1.75 units up.

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

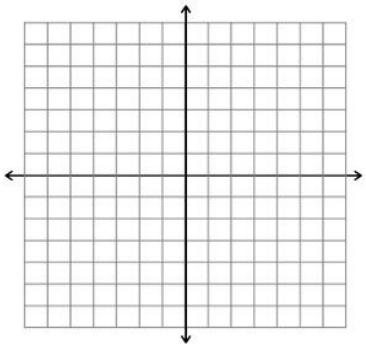
Transformations of Functions

These are looking at graphs of things like:

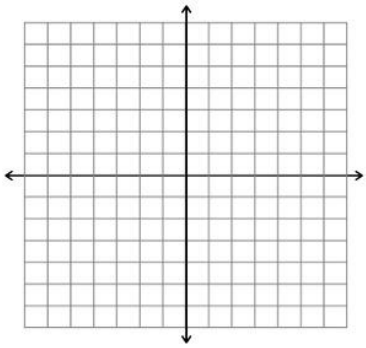
$$f(x) = x^2 - 5 \quad \text{or} \quad f(x) = |x+3| - 2 \quad \text{or} \quad f(x) = \sqrt{x-3}$$

First we need to discuss the shape of a few basic functions. Write these down and note the important points.

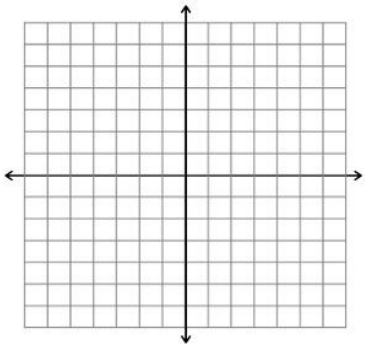
a. $f(x) = x^2$



b. $f(x) = |x|$



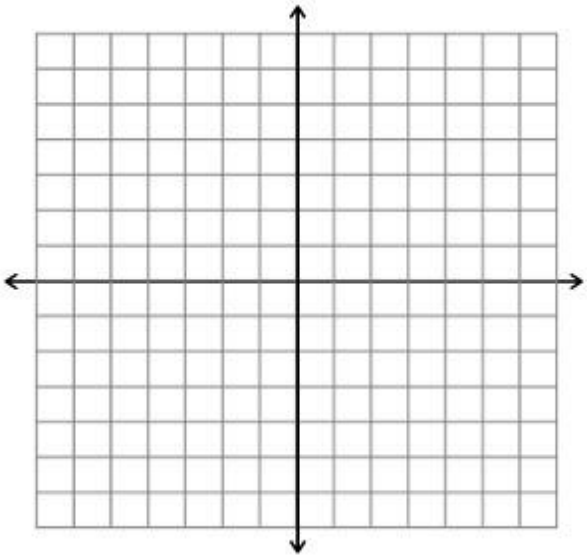
c. $f(x) = \sqrt{x}$



Using a few different colors, sketch out what the graphs of

a. $f(x) = x^2$ versus b. $g(x) = x^2 + 3$ versus c. $h(x) = x^2 - 1$

look like all on the same graph below:



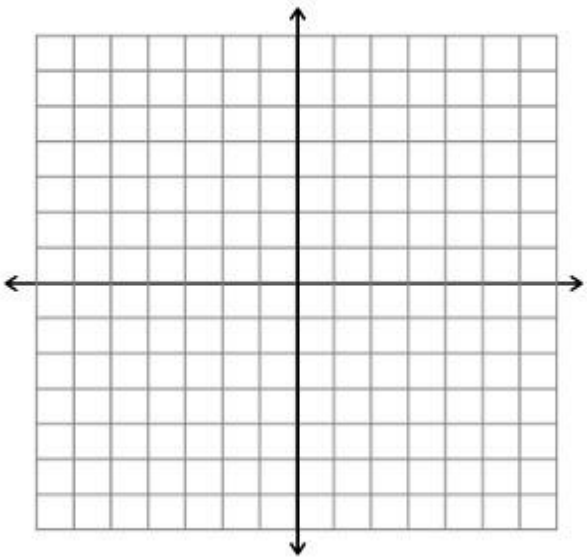
Copyright 2020, Jennifer Hill, DivideandConquerMath.com

This is an example of a vertical shift. Explain what this means in mathematical terms:

Using a few different colors, sketch out what the graphs of

a. $f(x) = |x|$ versus b. $g(x) = |x - 1|$ versus c. $h(x) = |x + 2|$

look like all on the same graph below:



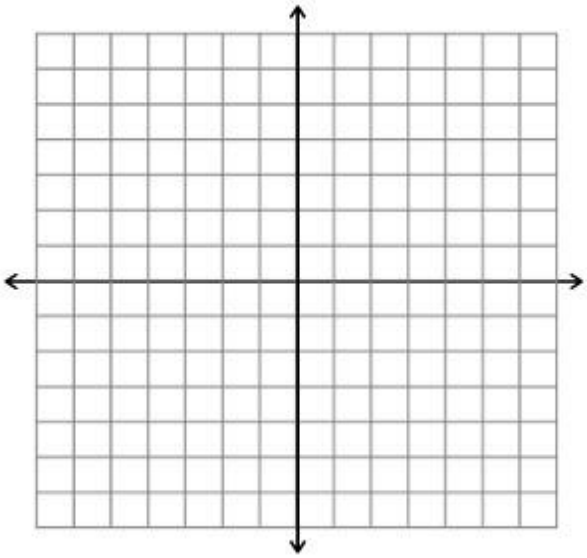
This is an example of a horizontal shift. Explain what this means in mathematical terms:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Using a few different colors, sketch out what the graphs of

a. $f(x) = |x|$ versus b. $g(x) = -|x|$

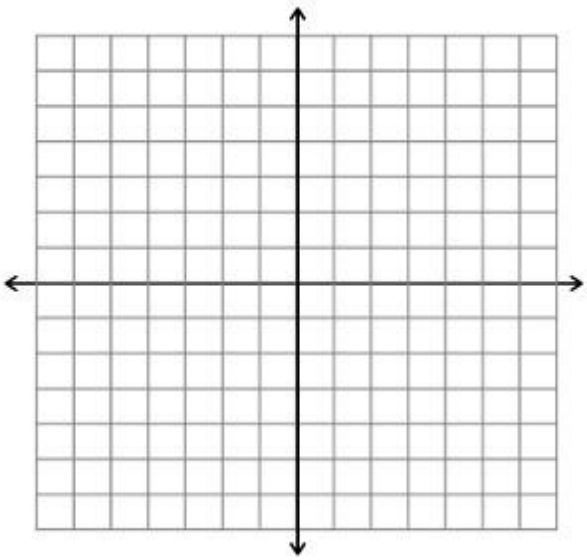
look like all on the same graph below:



This is an example of a reflection over the x-axis. Explain what this means in mathematical terms:

<u>Example</u> Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

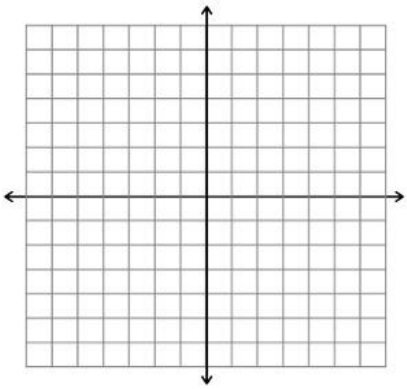
$$f(x) = |x| + 3$$



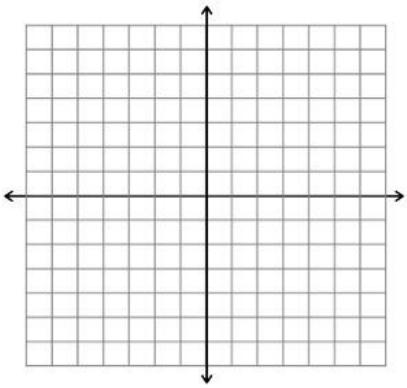
Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

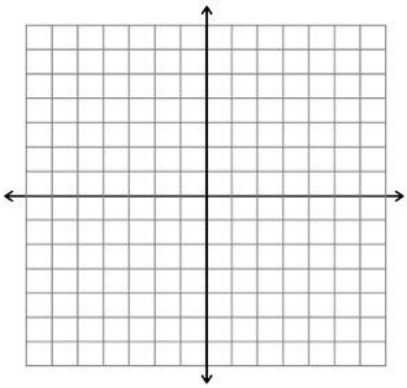
a. $f(x) = (x-1)^2$



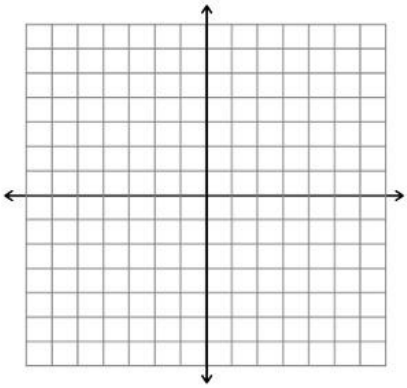
b. $g(x) = \sqrt{x} - 2$



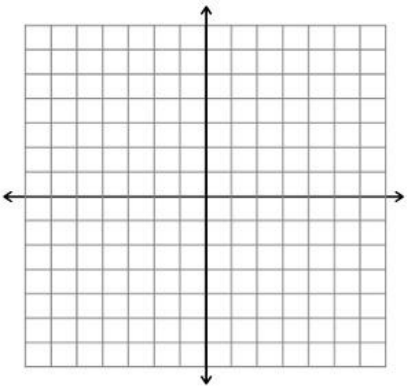
c. $h(x) = |x+1| - 2$



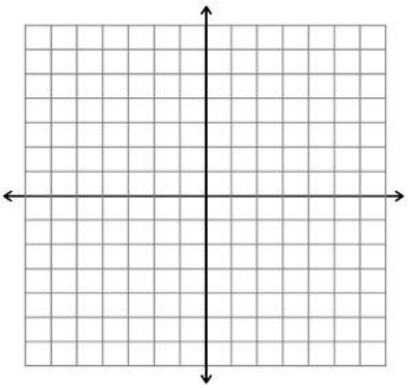
d. $k(x) = (x-2)^2 + 4$



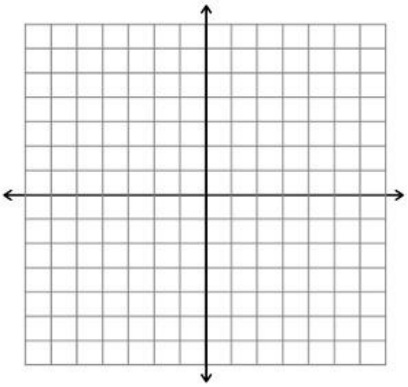
e. $f(x) = -(x+3)^2 - 2$



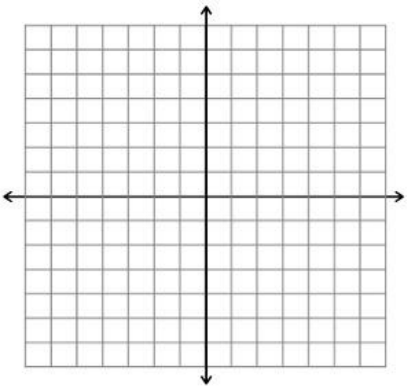
f. $g(x) = -|x+3| - 1$



g. $h(x) = -\sqrt{x-2}$



h. $k(x) = -\sqrt{x+1} - 3$



Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Write the equations that represent the listed transformations.

a. A function $f(x) = \sqrt{x}$ is shifted 7 units down.

b. A function $g(x) = x^3$ is shifted 3 units left and 2 units up.

c. A function $h(x) = |x|$ is shifted 2.5 units right and reflected over the x-axis.

d. A function $f(x) = e^x$ is shifted 4 units up and reflected over the x-axis.

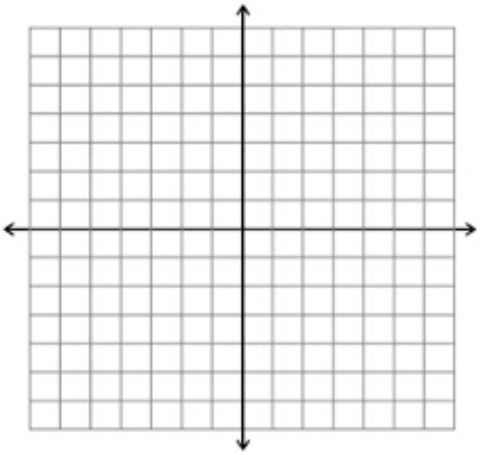
e. A function $g(x) = \sqrt[3]{x}$ is shifted 3.2 units right, 5.7 units up, and reflected over the x-axis

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

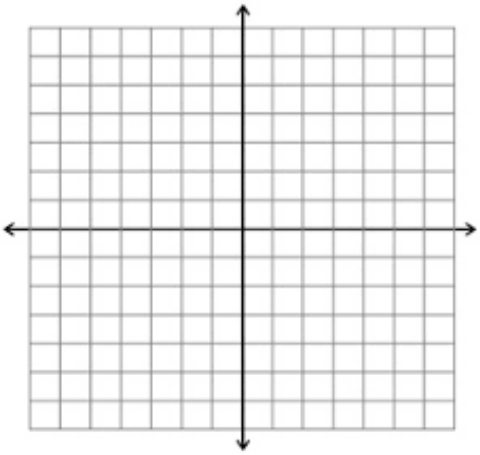
Examples: Translations and Reflections of Functions

Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

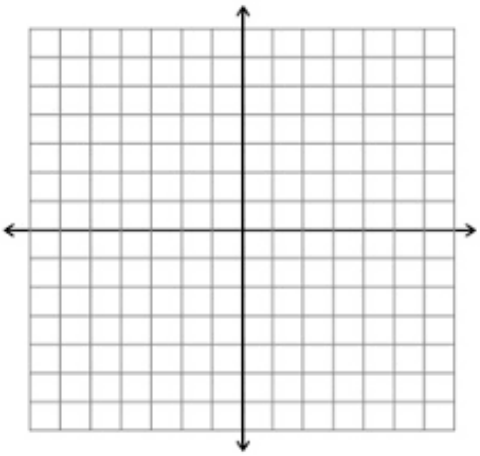
a. $f(x) = -(x+2)^2 - 3$



b. $g(x) = \frac{1}{2}(x-1)^3 + 2$



c. $h(x) = -\frac{1}{3}\sqrt{x+1} + 1$



Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Symmetry and Even or Odd Functions

Explain when a graph is symmetric with respect to the y-axis:

Explain when a graph is symmetric with respect to the x-axis:

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Explain when a graph is symmetric with respect to the origin:

What makes a function odd?

What makes a function even?

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

So what is the test for even or odd functions?

<u>Examples</u> Determine what type of symmetry the following functions have:

a. $y = x^2 + 4$

b. $x^2 + y^2 = 25$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Examples Determine what type of symmetry the following functions have:

a. $y = x^3 - 5$

b. $y = 7x^2 + 2x - 4$

c. $y = 2x^{\frac{1}{3}} - 3x$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Review of Function Operations

Given that f and g are functions, how do we define the function operations?

$(f+g)(x) =$

$(f-g)(x) =$

$(fg)(x) =$

$\left(\frac{f}{g}\right)(x) =$

<u>Example</u> Let $f(x) = 3x+1, g(x) = x^2-4, h(x) = 3x^2+7x+2$, and $k(x) = x^2+6x+9$. Find each of the following:

a. $(f+g)(x) =$

b. $(f+g)(3) =$

c. $(g-k)(1) =$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = 3x^2 + 7x + 2$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $\left(\frac{h}{f}\right)(x) =$

b. $(gh)(x) =$

c. $\left(\frac{k}{h}\right)(1) =$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Review of Composition of Functions

For two functions f and g, what is the composition $(f \circ g)(x)$?

<u>Example</u> Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(f \circ g)(x) =$

b. $(g \circ f)(x) =$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(k \circ f)(1) =$

b. $(g \circ h)(4) =$

c. $(h \circ k)(x) =$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: The Difference Quotient

Explain what the difference quotient is and draw some sketches to explain how it works. In other words, watch the video explanation and then explain it in your own words on this page.

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

State the formula for the difference quotient here:

<u>Example</u> Calculate the different quotient for the following.

a. $f(x) = 4x + 5$

b. $f(x) = x^2 - 1$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: The Difference Quotient

Find the difference quotient.

a. $f(x) = \frac{1}{x+3}$

b. $f(x) = x^2 - 4x + 1$

c. $f(x) = -x^2 + 3$

Copyright 2022, Jennifer Hill, DivideandConquerMath.com

Lesson: Review of Triangles and Similar Triangles

The sum of angle measures of any triangle is:

Draw the different types of triangles:

<u>Examples</u> Draw the triangles in the example, then classify the triangles as acute, right, or obtuse. Also classify as equilateral, isosceles, or scalene.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

What makes 2 triangles similar?

What makes 2 triangles congruent?

What is the key relationship between similar and congruent triangles?

<u>What are the 2 conditions for similar triangles?</u>

In other words, triangle ABC is similar to triangle DEF if:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Draw the similar triangles in the video, and then find any missing angle measures.

Example Draw the similar triangles in the video, and then find any missing side lengths.

Example Draw the similar triangles in the video, and then find any missing measures.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: The Pythagorean Theorem, The Distance Formula, and Intro to Trig Functions

State the Pythagorean theorem:

State the distance formula:

How does the Pythagorean theorem relate to the distance formula?

Now we will introduce **trigonometric functions!** Explain what makes trig functions unique:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

How do trig functions relate to the distance formula? Draw the picture associated with this.

Explain the 6 trig functions and how they relate to what we discussed above.

Sometimes you will see expressions such as $\sin^2 90°$. What does this mean?

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example The terminal side of an angle θ in standard position passed through the point (-12, -5). Find the values of the 6 trig functions of angle θ.

Example Find the 6 trig function values of an angle θ in standard position if the terminal side of θ is defined by

$3x + 5y = 0, x \geq 0$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Suppose the point (x, y) is in the indicated quadrant. Determine whether given ratio is positive or negative.

$$III, \frac{y}{r}$$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Right-Triangle Based Definitions of Trigonometric Functions and Cofunctions

Instead of using standard position to describe trig functions, we can alternatively use right triangles.

Draw a right triangle to orient yourself.

What does A represent?

Show how to define the 6 trig functions with this right triangle.

1.

2.

3.

4.

5.

6.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

What mnemonic device can help you remember the trig functions?

Example Draw the triangle in the video, and then show how to find the values of sine, cosine, and tangent.

Example Using the same example above, show how to find the values of cosecant, secant, and cotangent.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Draw a new right triangle, and show an alternative angle measure we can find from this picture:

These relationships are called the **cofunction identities,** summarize these below:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples Write each function in terms of its cofunction

a. $\sin 10°$

b. $\cot 65°$

c. $\csc 34°$

Example Find one solution for each equation. Assume all angles involved are acute angles.

a. $\cos\theta = \sin(2\theta - 30°)$

b. $\sec(\theta + 15°) = \csc(2\theta + 25°)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Radian Measure

Radian measure is another way to describe angle measure, as opposed to degrees.

What does radian measure allow us to do?

What is the definition of a radian?

Are radians a unit of measure like feet or meters? Circle: Yes No

What is the formula for the distance around a circle?

What does this formula show?

Write a way to relate radians to degrees:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

So what is π radians?

Show a conversion rate between radians and degrees:

<u>Examples</u> Convert each degree to radians:

a. $35°$

b. $-175°$

c. $27.8°$

Convert each radian measure to degrees:

a. $\frac{7}{4}\pi$

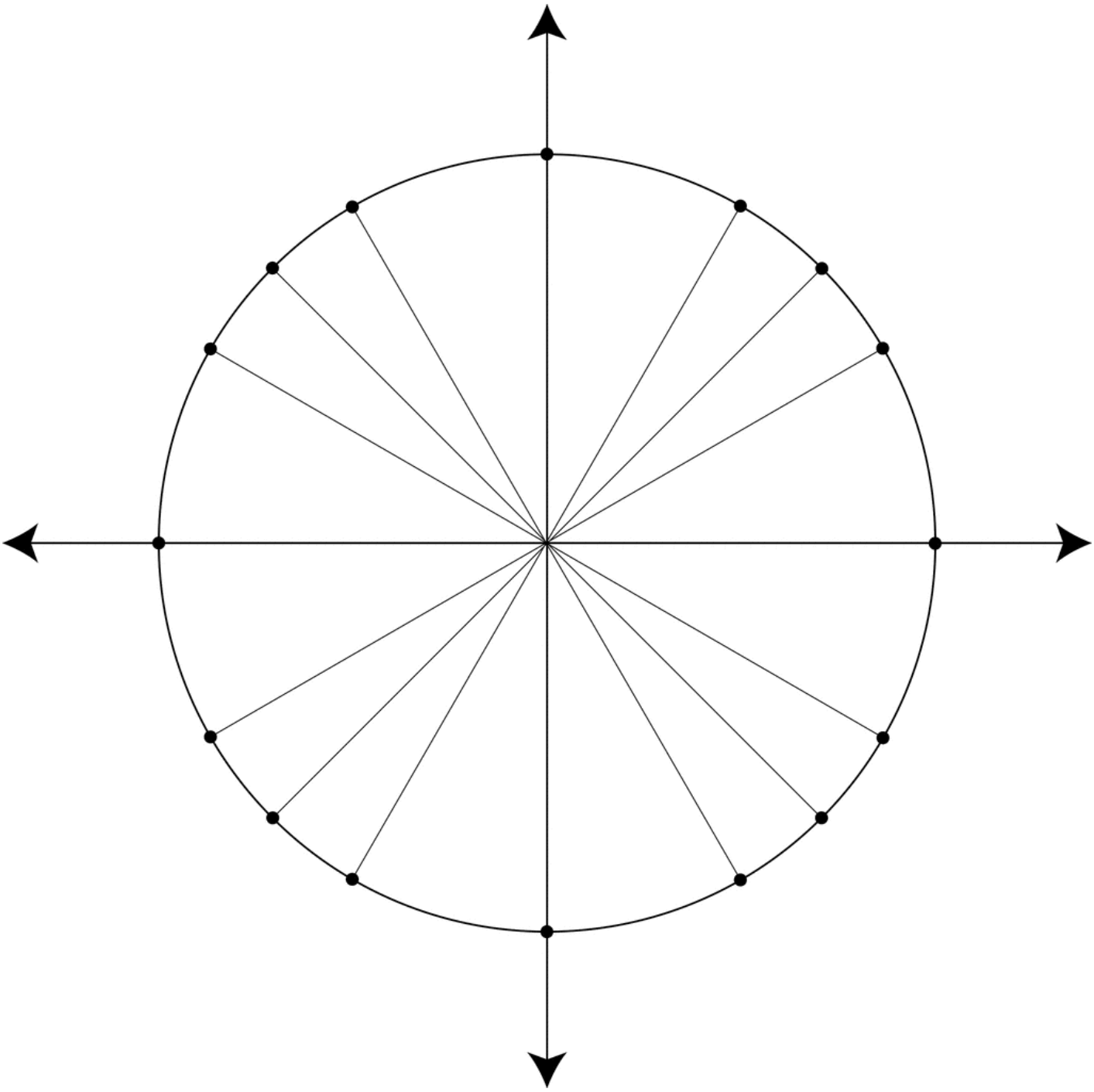
b. $-\frac{2\pi}{3}$

c. 7

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

If no unit of angle measure is specified, what should you assume?

Below we will construct an illustration of all the common degree measures we have discussed, and then write their radian equivalent. Take your time to copy this perfectly!



Copyright 2021, Jennifer Hill, DivideandConquerMath.com

You will need to know all of those common degrees AND radian measures for the exam!

You may want to reference cheat sheets to help you with the next few problems.

<u>Examples</u> Find the value of each function.

a. $\sin\frac{5\pi}{6}$

b. $\tan\frac{3\pi}{4}$

c. $\csc\frac{5\pi}{3}$

d. $\cos\left(-\frac{2\pi}{3}\right)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: The Unit Circle and Circular Functions

Explain how we create the unit circle:

What is the equation for the unit circle?

Recall the formula for arc length is $s = r\theta$. When the radius of the circle is 1, what does this formula become?

Why does this matter / what is cool about this?

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Since r=1 on the unit circle, what can we say about our previously studied trig functions? Just summarize what we discussed in the video as best as you can.

Write out the **circular functions**

1.

2.

3.

4.

5.

6.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

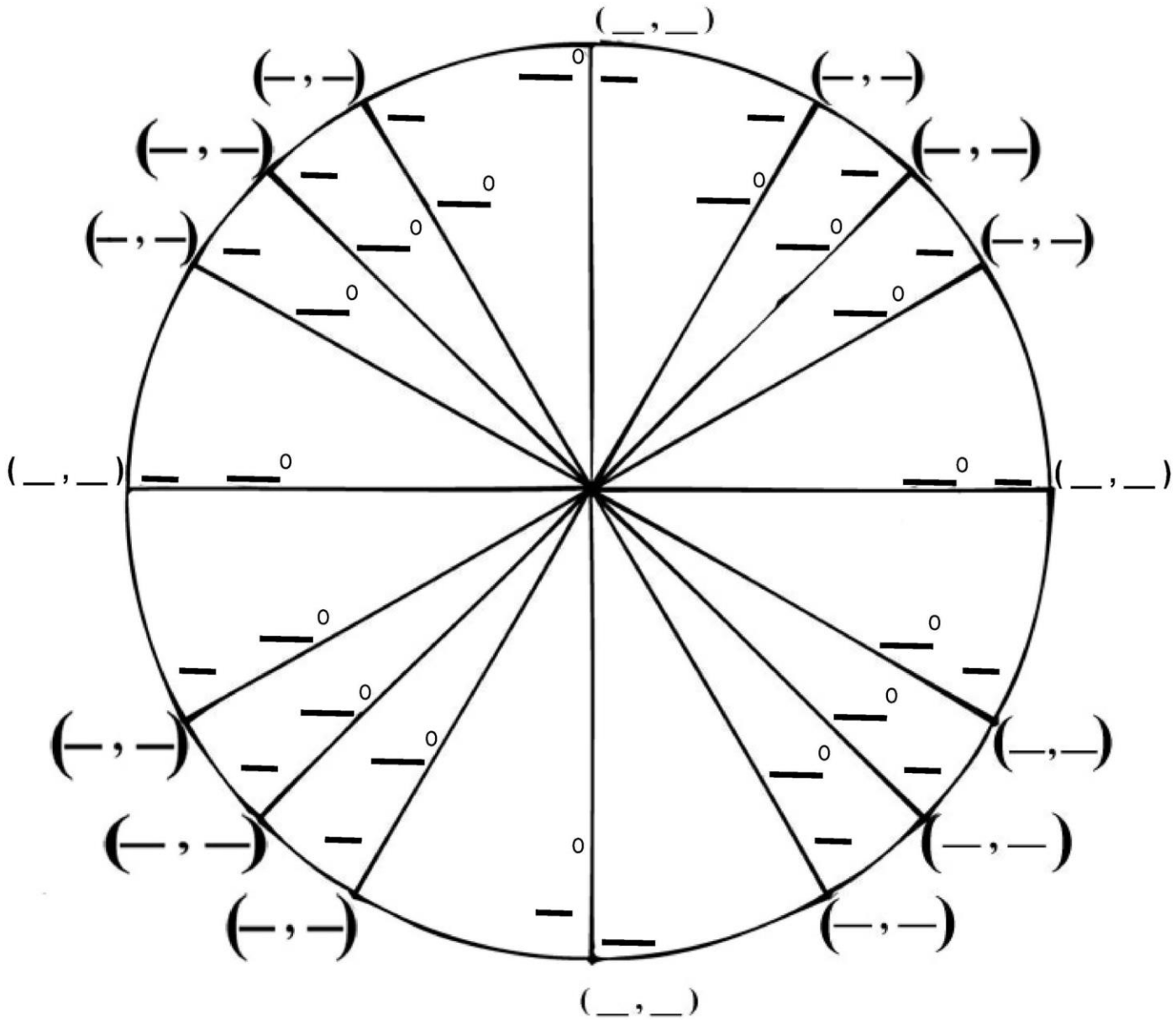
Explain the symmetry of the unit circle:

If a point (a, b) in on the unit circle, what other points are guaranteed to be on the unit circle?

How can we use symmetry with drawing the unit circle? How is it handy?

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Label the unit circle. Note: you will have to have this memorized! These are the standard angle measures that you'd be expected to do without a calculator!



Now, going back to the circular functions we discussed, how do you view the x-coordinates?

How do you view the y-coordinates?

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples Use the unit circle to find the values of the following:

a. $\sin\frac{\pi}{3}$

b. $\cos\frac{5\pi}{4}$

c. $\sin\frac{11\pi}{6}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Arc Length on a Circle and Some Applications

Using the unit circle, we can also determine the domains of our circular functions.

What is the domain of sine and cosine, and why?

What is the domain of tangent and secant, and why?

What is the domain of cotangent and cosecant, and why?

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Values of Circular Functions

We have previously discussed the unit circle (you will want to have it handy). One the most important things about the unit circle is that it will quickly tell you the value of the sine and cosine functions.

On the unit circle, what does the y-coordinate represent?

On the unit circle, what does the x-coordinate represent?

When we think about the values of these functions, what do we assume?

What does this mean for your calculator?

<u>Examples</u> Use the unit circle to find the values of:

a. $\sin\frac{\pi}{2}$ b. $\cos\frac{\pi}{2}$ c. $\tan\frac{\pi}{2}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples Use the unit circle to find the values of:

a. $\sin\frac{5\pi}{4}$ b. $\cos\frac{5\pi}{4}$ c. $\tan\frac{5\pi}{4}$

Example Compare methods. Use reference angles and degrees to find the exact value of $\sin\frac{2\pi}{3}$, then confirm your answer with the unit circle.

For approximate values, we still must use a calculator. Remember, when you are not specifically told you are using degrees, then you assume you are using radians.

What must you ensure with your calculator?

Examples Use a calculator to approximate the function values.

a. $\cos(1.72)$

b. $\sin(-3.54)$

c. $\cot(1.438)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Values of Inverse Circular Functions

What do the functions $\sin^{-1} x$, $\cos^{-1} x$, $\cot^{-1} x$, etc. represent?

Your calculator makes a few key assumptions if it uses these functions (the reasons why are explained later):

1. For all x in $[-1, 1]$, using $\sin^{-1} x$ with a calculator will return a value between:

2. For all x in $[-1, 1]$, using $\cos^{-1} x$ with a calculator will return a value between:

3. For all real numbers x, using $\tan^{-1} x$ with a calculator will return a value between:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Sometimes you will be asked to find answers in different quadrants than what your calculator will return!

<u>Example</u> Solve for s, given that s is in the interval $\left[0, \frac{\pi}{2}\right]$: $\sin s = 0.3311$

<u>Example</u> Solve for s, given that s is in the interval $\left[0, \frac{\pi}{2}\right]$: $\tan s = 0.2155$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Solve for s, given that s is in the interval $\left[\pi, \frac{3\pi}{2}\right]$: $\tan s = \sqrt{3}$

Example Solve for s, given that s is in the interval $[0, 2\pi)$: $\cos^2 s = \frac{1}{2}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: The Unit Circle and Circular Functions as Line Segments

The unit circle is really just an extension of how we have previously thought about trig functions as line segments / right triangles.

Draw the illustration in the video, then show how these sides relate to our trig functions.

From your illustration, how do we define the following?

$\cos\theta =$ $\tan\theta =$ $\csc\theta =$

$\sin\theta =$ $\cot\theta =$ $\sec\theta =$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Using your illustration from the previous page, suppose the measurement of TVU is $45°$. Find the lengths of the following line segments:

OQ

OU

OV

VR

US

PQ

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Fundamental Identities of Trig

Review the test for symmetry (and the results):

We know that the sine function is odd, show what this means:

We know that the cosine function is even, show what this means:

These are called **identities.**

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Using these identities, what other identity can we determine about the tangent function:

These are called the **even-odd identities.**

We have already seen a handful of identities, and we call these the fundamental identities. Summarize them below.

Reciprocal Identities

Quotient Identities

Pythagorean Identities

Even-Odd Identities

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example If $\cos\theta = \frac{3}{4}$ and θ is in quadrant I, find $\sin\theta$.

Example If $\tan\theta = -\frac{1}{4}$ and θ is in quadrant IV, find the values of all the other 5 trig functions.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

We use the identities to work on problems called transformations.

Example Write $\tan x$ in terms of $\sec x$.

Example Write the following in terms of sine and cosine, and then simplify until no quotients appear and all functions are of θ only.

a. $\csc\theta \cos\theta \tan\theta$

b. $\cos\ \theta(\cos\theta - \sec\theta)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples: Using Fundamental Trig Identities to Find Values of Other Trig Functions

Find the five remaining trig functions given the following information.

a. $\csc\theta = -\frac{5}{2}$, θ in quadrant III.

b. $\sec\theta = \frac{7}{2}$, $\tan\theta < 0$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples: Writing Trig Functions in Terms of Other Trig Functions (Fundamental Identities)

a. Write $\cot x$ in terms of $\sin x$

b. Write $\csc x$ in terms of $\cos x$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples: Using Fundamental Identities to Write Expressions in Terms of Sine and Cosine

Write each expression in terms of sine and cosine, then simplify so that no quotients appear and all functions are in θ only.

a. $\cos\theta \csc\theta$

b. $\frac{(1+\cot\theta)}{\cot\theta}$

c. $\frac{\sec^2\theta-1}{csc^2\,\theta-1}$

d. $\frac{\csc\theta}{\cot(-\theta)}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Proofs of the Sum and Difference Identities for Cosine

Write out the proof for $\cos(A - B) = \cos A \cos B + \sin A \sin B$. Do your best to understand the proof and then explain it in a way that you will understand.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Write out the identity for $\cos\,(A - B)$ below (yes I know I stated it above, but YOU should write it so you have a better chance of remembering it):

Write out the proof for $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Write out the identity below (yes, write it out so you don't forget!):

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Using Sum and Difference Identities for Cosine

<u>Examples</u> Show how to use the cosine of a sum or difference to find the exact value of the expressions:

a. $\cos 105°$

b. $\cos \frac{\pi}{12}$

c. $\cos 40° \cos 50° - \sin 40° \sin 50°$

Show how the cofunction identities are an extension of the cosine of a difference:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Write out all the **cofunction identities**

<u>Examples</u> Find one value of θ that satisfies the following:

a. $\sec\theta = \csc 72°$

b. $\tan\theta = \cot(-44°)$

c. $\cos\theta = \sin\frac{5\pi}{6}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Write $\cos(180° - \theta)$ as a trigonometric function of θ alone.

Example Suppose $\sin s = \frac{3}{5}$ and $\sin t = -\frac{12}{13}$, where s is in quadrant I and t is in quadrant III. Find $\cos\,(s + t)$ and $\cos(s - t)$.

Example Verify that $\sec(\pi + x) = -\sec x$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Sum and Difference Identities for Cosine

Write out the proof for $\sin(A + B)$

Write out the proof for $\sin(A - B)$

Write out the identities for the sum and difference of sine below:

Write out the proof for $\tan\,(A + B)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Write all identities for the sum and difference of tangent below:

<u>Examples</u> Find the exact value of each expression.

a. $\sin(-15°)$

b. $\tan\left(-\frac{5\pi}{12}\right)$

c. $\sin\left(\frac{5\pi}{9}\right)\cos\left(\frac{\pi}{18}\right) + \cos\left(\frac{5\pi}{9}\right)\sin\left(\frac{\pi}{18}\right)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Write each function as an expression involving functions of θ alone.

a. $\cos(30° + \theta)$

b. $\sin\,(45° + \theta)$

c. $\tan\left(\frac{\pi}{4} + x\right)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples Suppose A and B are angles in standard position such that $\sin A = -\frac{3}{5}\ \frac{3\pi}{2} < A < 2\pi$ and

$\cos B = -\frac{7}{25}$, $\pi < B < \frac{3\pi}{2}$. Find the following.

a. $\sin(A + B)$

b. $\tan(A - B)$

c. the quadrant of A-B

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Verify the identities.

a. $\sin(x+y)+\sin(x-y)=2\sin x\cos y$

b. $\frac{\sin(x-y)}{\sin(x+y)}=\frac{\tan x-\tan y}{\tan x+\tan y}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples: Sine of a Sum and Tangent of a Sum

Let $\cos s = \frac{5}{12}$, $\sin t = \frac{4}{5}$ where s and t are in quadrant I. Find:

$\sin(s + t)$

$\tan\,(s + t)$

The quadrant of $(s + t)$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Double-Angle Identities

Show how we derive the first identity for $\cos 2A$:

Show how we derive the second identity for $\cos 2A$:

Show how we derive the third identity for $\cos 2A$:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Show how we derive the identity for $\sin 2A$:

Show how we derive the identity for $\tan 2A$:

Summarize the Double-Angle Identities Below:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Give $\sin\theta = \frac{5}{8}$ and $\cos\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

Example Find the values of the six trig functions give $\cos 2\theta = -\frac{11}{13}$ and $180° < \theta < 270°$.

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Verify the following identities:

a. $(\sin x + \cos x)^2 = \sin 2x + 1$

b. $\frac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples Simplify the expressions.

a. $\sin^2 15° - \cos^2 15°$

b. $\sin^2 \frac{3\pi}{5} - \cos^2 \frac{3\pi}{5}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Write $\cos 3x$ in terms of $\cos x$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Product-to-Sum Identities

Show what happens when we add the identities $\cos(A+B)$ to $\cos(A-B)$

Now show what happens when we subtract them:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples Write the following as the sum or difference of 2 trig functions.

a. $2\sin 48° \cos 104°$

b. $8\sin 9x \sin 5x$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples Write the expression as a product of trigonometric functions.

a. $\cos 6x + \cos 9x$

b. $\sin 106° - \sin 96°$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Half-Angle Identities

Show how we derive the identity for $\sin\frac{A}{2}$:

Show how we derive the identity for $\cos\frac{A}{2}$:

Show the first way we can derive an identity for $\tan\frac{A}{2}$:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Show the second way we can derive an identity for $\tan\frac{A}{2}$:

Show the third way we can derive an identity for $\tan\frac{A}{2}$:

Summarize the half-angle identities below:

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Show how to use the half-angle identity to evaluate: $\sin 12.5°$

Example Find the exact value of $\cos 165°$

Example Find the exact value of $\tan 75°$ using the identity $\tan A = \frac{\sin A}{1+\cos A}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Given $\cos s = \frac{2}{7}$ with $0 < s < \frac{\pi}{2}$, find $\sin\frac{s}{2}$, $\cos\frac{s}{2}$, and $\tan\frac{s}{2}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Simplify each expression.

a. $\sqrt{\frac{1-\cos 40°}{2}}$

b. $\frac{1-\cos 64.72°}{\sin(64.72°)}$

c. $\pm\sqrt{\frac{1+\cos 10x}{2}}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Verify the identities.

a. $\cot^2 \frac{x}{2} = \frac{(1+\cos x)^2}{\sin^2 x}$

b. $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Summarizing All of the Identities

Using all the trig identities that we've discussed, verify the following:

a. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

b. $1 - \tan^2 \frac{x}{2} = \frac{2\cos x}{1+\cos x}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

c. $\tan\left(\frac{a}{2}+\frac{\pi}{4}\right)=\sec a+\tan a$

d. $\sin 7x+\sin 5x+\sin 3x+\sin x=4\sin 4x\cos 2x\cos x$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

e. $\tan 4x = \frac{2\tan 2x}{2-\sec^2 2x}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Average Rates of Change

Calculate the average rate of change of the functions over the given intervals.

a. $f(x) = x^2 + 3x; [0, 3]$

b. $f(x) = 2 + \sin x; \left[\frac{\pi}{2},\ \pi\right]$

c. $f(x) = \sqrt{4x + 1}; [0, 2]$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Determining the Slope at a Point and the Tangent Line

Find the slope of the curve at the given point and find the equation of the tangent line at that point. Do this only using the formula for the slope of the secant line.

a. $f(x) = x^2 - 1, (2, 3)$

b. $f(x) = x^2 - 4x; (2, -4)$

c. $y = \frac{1}{x}; (2, \frac{1}{2})$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: An Intro to Limits

Graph $f(x) = \frac{(x^2-4)}{x-2}$ below. What does it look like this graph should equal when x=2?

Explain this previous question now in mathematical terms

<u>Example</u> Find $\lim_{x \to 1} f(x)$ when $f(x) = \begin{cases} x^2 + 1 & x \neq 1 \\ -2 & x = 1 \end{cases}$. Draw a picture of this situation.

What is the value of $f(1)$?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example What is $\lim_{x \to 2} f(x)$ when $f(x) = x$? Draw a graph to accompany your answer.

What do we call the function $f(x) = x$?

What can we say about the limit of these types of functions?

Example What is $\lim_{x \to -3} f(x)$ when $f(x) = 4$? Draw a graph to accompany your answer.

What do we call the function $f(x) = c$, where c is a constant?

What can we say about the limit of these types of functions?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Write out all the limit laws on this space:

Write out the other 2 handy theorems:

Theorem 1: Limits of Polynomials

Theorem 2: Limits of Rational Functions

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Using the limit laws, calculate the following limits:

a. $\lim_{x \to 3}(2x + 1)$

b. $\lim_{x \to 1} \frac{5x+2}{x-4}$

c. $\lim_{x \to 2} \sqrt{5x + 4}$

d. $\lim_{x \to -3} 5\sqrt{2 - 3x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Intro to Calculating Limits

Calculate the following limits:

a. $\lim_{x \to 3}(x^2 + 2x + 1)$

b. $\lim_{x \to 0}(8 - 2x)^{\frac{2}{3}}$

c. $\lim_{x \to \frac{2}{5}}(5x + 2)(3x - 1)$

d. $\lim_{x \to \frac{1}{3}} \sqrt[3]{3x + 7}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Limits that Do Not Exist

List the types of limits that do not exist, as well as an example and graph

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: One-Sided Limits

Use the following function to describe one-sided limits: $f(x) = \begin{cases} 3, & x > 2 \\ 1, & -1 \leq x \leq 2 \\ -2, & x < -1 \end{cases}$

Evaluate the following:

a. $\lim_{x \to 2^+} f(x)$

b. $\lim_{x \to 2^-} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to -1^+} f(x)$

e. $\lim_{x \to -1^-} f(x)$

f. $\lim_{x \to -1} f(x)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Draw the graph presented in the video then evaluate the following:

a. $\lim_{x \to 3^+} f(x)$

b. $\lim_{x \to 3^-} f(x)$

c. $f(3)$

d. $\lim_{x \to -2^+} f(x)$

e. $\lim_{x \to 4^-} f(x)$

f. $\lim_{x \to 4^+} f(x)$

g. $\lim_{x \to 4} f(x)$

h. $f(4)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Write down the theorem that explains when $\lim_{x \to c} f(x)$ exists:

<u>Examples</u> Evaluate the following:

a. $\lim_{x \to 2^+} \sqrt{\frac{x-2}{x+1}}$

b. $\lim_{x \to 2^-} \left(\frac{1}{x+1}\right)\left(\frac{x-3}{x}\right)\left(\frac{4}{4-x}\right)$

c. $\lim_{x \to -3^-} \frac{(x+1)|x+3|}{x+3}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Basic One-Sided Limits

a. $\lim_{x \to 2^+} \sqrt{x^2 - 4}$

b. $\lim_{x \to 0^+} \frac{\sqrt{2x^2+3x+1}-1}{x}$

c. $\lim_{x \to 3^-} \begin{cases} 4, & x > 3 \\ -2, & x \leq 3 \end{cases}$

d. $\lim_{x \to 5^-} \frac{x-5}{|x-5|}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: One-Sided Limits with Absolute Value

a. $\lim_{x \to 2^+} \frac{2-x}{|x-2|}$

b. $\lim_{x \to 5^-} \frac{(x-4)|x-5|}{x-5}$

c. $\lim_{x \to 2^+} \frac{\sqrt{3x}(x-2)}{|x-2|}$

d. $\lim_{x \to 0^-} \frac{\sin x}{|\sin x|}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The $\boldsymbol{lim_{x\to 0}\frac{sin\ x}{x} = 1}$

This lesson is about the tricky but very important limit:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Or equivalently:

$$\lim_{x\to 0}\frac{x}{\sin x} = 1$$

The proof is interesting and worth noting, you can write it down if you wish but you should minimally attempt to understand the reasoning behind it. It's difficult, but difficult things will expand your mind!

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\ x}{x} = 1}$

a. $\lim\limits_{x\to 0}\frac{\sin 3x}{5x}$

b. $\lim\limits_{x\to 0} 12x^2(\cot 2x)\ (\csc 3x)$

c. $\lim\limits_{x\to 0}\frac{x^3-x+\sin 5x}{3x}$

d. $\lim\limits_{x\to 0}\frac{\sin x}{\sin 3x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step-by-Step Examples: Limits using $\boldsymbol{lim_{x\to 0}\frac{sinx}{x}=1}$

a. $\lim\limits_{x\to 0}\frac{\sin 7x}{x}$

b. $\lim\limits_{x\to 0} x\cot 3x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

More Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\,x}{x} = 1}$

a. $\lim\limits_{x\to 0}\frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)$

b. $\lim\limits_{x\to 0}\frac{\sin x}{\sqrt{x}}$

c. $\lim\limits_{x\to 0}\frac{1-\cos 2x}{x}$

d. $\lim\limits_{x\to 0}\frac{\sin 3x}{\mathrm{x}\cos 5x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Troubleshooting with $\lim_{x \to 0} \frac{\sin x}{x}$ problems – examples of what not to do

a. $\lim_{x \to 0} \frac{\tan 5x}{x}$

b. $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

Lesson: The Precise Definition of a Limit

What is the issue with the idea of limits?

Explain the example that went with this (it's better to watch this then write this out after):

State the Precise Definition of a Limit

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Precise Definition of a Limit with a Specific Epsilon

Given the function $f(x)$, a point c, and a specific ϵ, find L and the specific δ that fulfills the definition of a limit.

a. $f(x) = 3x + 1, c = 2, \epsilon = 0.01$

b. $g(x) = x^2 + 1, c = 1, \epsilon = 0.1$

c. $f(x) = \sqrt{1 - 4x}, c = -2, \epsilon = 0.02$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Precise Definition of a Limit with a Specific Epsilon

Given the function $f(x)$, a point c, and a specific ϵ, find L and the specific δ that fulfills the definition of a limit.

$$f(x) = \frac{1}{x}, c = 1, \epsilon = 0.01$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Precise Definition of a Limit with a No Specific Epsilon

Find the specific δ for each limit that would correspond to the precise definition of a limit.

a. $\lim_{x \to 1} \sqrt{5 - x} = 2$

b. $\lim_{x \to 1} f(x) = 3$ when $f(x) = \begin{cases} 2x + 1, & x \neq 1 \\ 6, & x = 1 \end{cases}$

c. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Troubleshooting Limits With Algebraic Tricks (When You Get 0/0)

Trick #1 (name this trick here):

Note what happens if we try to evaluate $\lim_{x \to 2} \frac{x^2-4}{x-2}$

Draw what this situation looks like graphically.

How can we solve this problem?

<u>Example</u> Evaluate $\lim_{x \to 3} \frac{(x^3-27)}{x-3}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Trick #2 (name it):

<u>Examples</u> Evaluate:

a. $\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x}$

b. $\lim_{x \to 1} \frac{2-\sqrt{x^2+3}}{x-1}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Trick #3 (name it):

<u>Examples</u> Evaluate

a. $\lim_{x\to 3} \frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{x-3}$

b. $\lim_{x\to 1} \frac{\left(1-x^{-1}\right)}{1-x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Algebraic Tricks for Limits

Evaluate the following:

a. $\lim_{x \to -4} \frac{\left(\frac{1}{\sqrt{12+x}} - \frac{1}{3}\right)}{x+4}$

b. $\lim_{x \to 0} \frac{\left(\frac{1}{3+x} - \frac{1}{3}\right)}{x}$

c. $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Sandwich Theorem and Basic Trig Limits

Explain the idea of the Sandwich Theorem:

State the Sandwich Theorem

<u>Example</u> Suppose there is a function $f(x)$ where $3 - x^2 \leq f(x) \leq 3\cos x$. Find $\lim_{x \to 0} f(x)$.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Show how to use the Sandwich Theorem to evaluate the following limits:

a. $\lim_{x \to 0} \sin x$

b. $\lim_{x \to 0} \cos x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Basic Trig Limits

Evaluate the following:

a. $\lim_{x \to 0} \frac{3+x+\sin x}{4\cos x}$

b. $\lim_{x \to \frac{\pi}{3}} \sec x$

c. $\lim_{x \to \pi} \sin^2 x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Continuity

Determine where the following functions are continuous:

a. $f(x) = \frac{1}{(x+3)^2} - 2$

b. $g(x) = \begin{cases} 1 + x, & x \geq 3 \\ 4, & -2 \leq 0 < 3 \\ 2 - x, & x < -2 \end{cases}$

c. $h(x) = \begin{cases} |x + 1|, & x \geq -2 \\ -5, & x < -2 \end{cases}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Continuity

The idea of continuity is probably intuitive at the point. Draw the example in the video to demonstrate continuity.

For a point c, define when a function is:

1. Continuous

2. Right – continuous

3. Left – continuous

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Draw the graph in the video to demonstrate these ideas.

<u>Example</u> Draw the graph in the video, then describe the continuity at the following points:

$x = 1$ $x = 3$ $x = -4$ $x = -2$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

A function is continuous at a point c is it meets these 3 conditions:

1. f(c) exists

2. the limit $\lim_{x \to c} f(x)$ exists

3. $\lim_{x \to c} f(x) = f(c)$

Describe the different kinds of discontinuities below:

What does it mean for a function to be continuous?

If a function is not continuous, then it is _____

If $f(x)$ and $g(x)$ are continuous functions at the point x=c, then the following are continuous at c:

1. sum/differences:

2. constant multiples (for any number k):

3. products:

4. quotients:

5. powers:

6. roots:

7. compositions:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Describe where the following functions are continuous:

a. $f(x) = \frac{x+3}{x-2}$

b. $f(x) = \sqrt{2x}$

What can we say about the limits of a composition of continuous functions $f(x)$ and $g(x)$?

Example Find the limit $\lim_{x \to 0} (x+2)e^{\sin x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Using Continuity with Limits

Find the limits.

a. $\lim_{x \to 0} \cos\left(x + \cos\left(\frac{\pi}{2} + x\right)\right)$

b. $\lim_{x \to \frac{\pi}{2}} \sin(\cos(\tan x))$

c. $\lim_{x \to 0} \sin^{-1}\left(e^{\cos\left(\frac{\pi}{2} + x\right)}\right)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Intermediate Value Property and Continuous Functions

What is the Intermediate Value Property?

There are 2 consequences of this:

1. Connectedness:

2. Root Finding:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Determine the value of a that would make the function continuous at every x:

$$f(x) = \begin{cases} x^2 + 1, & x > 2 \\ ax, & x \leq 2 \end{cases}$$

Example Show that $x^3 - 14x + 2 = 0$ has 3 solutions in the interval $[-4, 4]$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Continuous Extensions

Determine the value of a that would make the following functions continuous:

a. $f(x) = \begin{cases} 2x, & x > 1 \\ ax^2, & x \leq 1 \end{cases}$

b. $g(x) = \begin{cases} \frac{2x+a}{a+2}, & x > 0 \\ x^2 + a, & x \leq 0 \end{cases}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step-by-Step Examples: Continuous Extensions

Use the space below to find a (or a and b) to make the following functions continuous:

a. $f(x) = \begin{cases} x + a, & x \geq 1 \\ 3, & x < 1 \end{cases}$

b. $f(x) = \begin{cases} 3, & x \geq 1 \\ ax + b, & -2 < x < 1 \\ -3, & x \leq -2 \end{cases}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Finite Limits as $x \to \infty$ and Limits That Go To Infinity, the Conceptual Idea

Recreate the graph below, then find the following limits:

a. $\lim_{x \to \infty} f(x)$

b. $\lim_{x \to -\infty} f(x)$

c. $\lim_{x \to 3^+} f(x)$

d. $\lim_{x \to 2^-} f(x)$

e. $f(2)$

f. $\lim_{x \to 2^+} f(x)$

g. $\lim_{x \to 3} f(x)$

h. $\lim_{x \to -4} f(x)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Finite Limits as $x \to \infty$ and Rational Functions

Do the limit laws apply to limits as $x \to \infty$? Yes No

How do you evaluate limits at infinity for rational functions?

In the following examples, we will see all the different ways this trick might be carried out:

<u>Examples</u> Evaluate the following:

a. $\lim_{x\to\infty} \frac{4x+3}{x^2+4x+3}$

b. $\lim_{x\to-\infty} \frac{5x^2+3x+1}{7x^2+5x-4}$

c. $\lim_{x\to\infty} \frac{6x^2+3x+1}{x+2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

This trick can be extended.

Examples

a. $\lim_{x\to\infty} \sqrt{\frac{x^2+5}{3x^2-6}}$

b. $\lim_{x\to-\infty} \frac{x^{\frac{2}{3}}-x^{\frac{1}{3}}+2}{x^2+3}$

c. $\lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+3}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Limits as $x \to \infty$

Find the horizontal asymptotes / evaluate the following:

a. $\lim_{x\to\infty} \frac{\sin x}{x^2}$

b. $\lim_{x\to\infty} \frac{7x^4-2x^3+8}{9x^4+2x^2+x-1}$

c. $\lim_{x\to-\infty} \frac{\sqrt{25x^2+x}}{x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

d. $\lim_{x\to\infty}\left(x-\sqrt{x^4+2x^2}\right)$

e. $\lim_{x\to-\infty}(5x^7+2x^3+8)$

f. $\lim_{x\to-\infty}\frac{x^3+2x+5}{x^2+3x+1}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Formal Definition of the Derivative

One of the things we learn in pre-calculus is about secant lines. You can write a review of this below:

When the distance between two points for a secant line goes to zero, what do we get?

What is the formal definition of the derivative?

What is the alternate formula for the derivative?

Which formula should you use?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

How do you find the derivative at a specific point?

What 4 things does the derivative at a point represent?

1.

2.

3.

4.

<u>Example</u> Calculate the derivative of $f(x) = x^2 + 2$

<u>Example</u> What is the value of the derivative of $f(x) = x^2 + 2$ at $(1, 3)$?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Find the equation of the tangent line for $f(x) = x^2 + 2$ at $(1, 3)$

List all the different ways to represent a derivative below:

Write out the theorem that discusses differentiability and continuity

What types of functions will not have a derivative at a point?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Calculating Derivatives Using the Formal Definition Only and Finding Tangent Lines

Use the formal definition of a derivative to calculate the follow derivatives of each function. Then determine the tangent line at the given point.

a. $f(x) = 4 - x^2; (-1, 3)$

b. $y = \frac{1-x}{2x}, (1, 0)$

c. $g(x) = x^{\frac{3}{2}}, (4, 8)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Using the Alternative Formula for Derivatives

Use the alternative formula for derivatives to calculate the derivative of the following function. Compare this calculation to the original formula.

a. $y = x^3 - x^2$

b. $y = \frac{1}{x+3}$

c. $f(x) = 2 + \sqrt{x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step-by-Step Examples: Comparing the Definition of the Derivative to the Alternative Formula

a. $f(x) = x + \frac{3}{x}$

b. $g(x) = x^3 - 5x^2 + 1$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Finding the Derivative of a Square Root with the Definition of the Derivative

Find the tangent line at (6,5) for $y = \frac{5}{\sqrt{7-x}}$ using only the definition of the derivative.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Other Notes about the Derivative

Define right-hand derivatives

Define left-hand derivatives

Explain how $y = |x|$ is differentiable on each side

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Interpreting Graphs of Derivatives

The derivative of $f(x) = x^2$ is $f'(x) = 2x$. Compare their graphs below:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Basic Differentiation Rules

Now we will introduce the basic differentiation rules, which will make life much easier. Let $f(x)$ and $g(x)$ be differentiable functions. Then:

1. Derivative of a Constant Function

Example Find the derivative of $f(x) = 6$

2. Power Rule

Example Find the derivative of $f(x) = x^{\frac{3}{2}}$

3. Constant Multiple Rule

Example Find the derivative of $f(x) = 4x^7$

4. Sum Rule

Example Find the derivative of $f(x) + 5x^2 + 2$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

5. The Natural Exponential Function

Example Find the derivative of $f(x) = 5e^x - 7\sqrt{x}$

6. Product Rule

Example Find the derivative of $f(x) = (3x^2 - 5)e^x$

7. Quotient Rule

Example Find the derivative of $f(x) = \frac{5x}{e^x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Tangent Lines

a. Find the line tangent to $y = 4x^2 + 3$ that has a slope of 16. State the point that this occurs at on the graph, and also state the equation of the tangent line.

b. Find the equation of any tangent lines perpendicular to the x-axis for $f(x) = x^3 - 4x^2 + 4$

c. Find the equation of the line that is tangent to the curve $f(x) = e^x$ and goes through the origin.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Finding a Point Tangent to One Graph, Perpendicular to Another Line

Find all the points on $f(x) = \frac{x}{x-2}$ where the tangent lines are perpendicular to $f(x) = 2x + 7$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Lesson: Higher Derivatives

How do you find the second derivative?

Notation for the second derivative:

How do you find and notate higher derivatives?

<u>Examples</u> Find the 4th derivatives of the following:

a. $y = x^7 - 5x^3 + 2x$

b. $f(x) = e^x - 7x^2 + x^3$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Derivative as a Rate of Change

We've talked about that the derivative at a point x_0 represents:

- the slope of the curve at x_0
- the derivative at x_0
- the slope of the tangent line at x_0
- the instantaneous rate of change at x_0

We will focus on the derivative as a rate of change in this lesson.

Often times, we omit the word "instantaneous" when we talk about this.

What represents position?

What is displacement?

What is average velocity?

How do we find velocity (instantaneous velocity)?

How is velocity different from speed?

How do you find speed?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

How do you find acceleration?

What comes after acceleration?

If I move at a constant speed, what would the function and graphs that model this situation from a position, velocity, and acceleration standpoint?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

If I ride down a hill, what would the function and graphs that model this situation from a position, velocity, and acceleration standpoint?

Now let's model the situation where I drop a tennis ball from a height of 6ft (1.8m). The equation that models this situation is $s(t) = -4.9t^2 + 1.8$

a. When do the ball hit the ground?

b. When the ball hits the ground, what was its velocity? What about its speed?

c. What is the acceleration of the ball at any time?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Position, Velocity, and Acceleration

Let $s = -t^3 + 9t^2 - 24t$ represent the position of a moving object, in feet, at time t, in seconds.

a. Find the acceleration of the object when the velocity is 0.

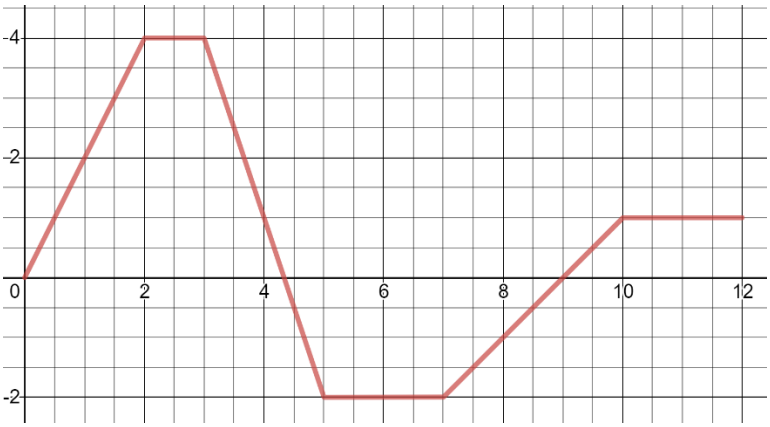
b. What is the speed of the object when the acceleration is 0?

c. What is the total distance traveled from t=0 to t=7?

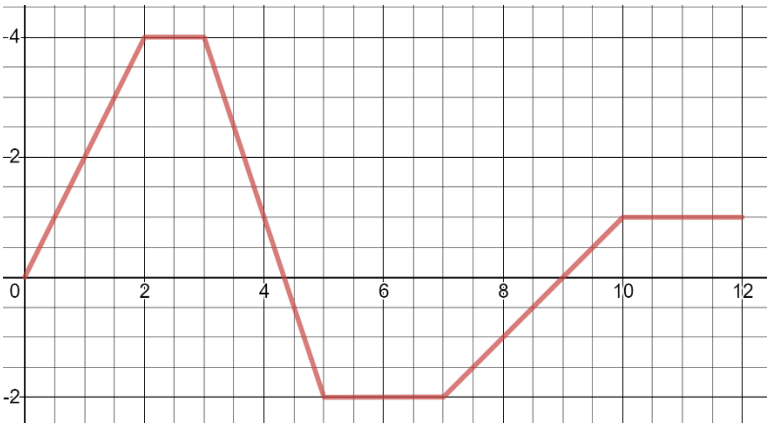
Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Position, Velocity, and Acceleration from a Graph

This is the graph of velocity for an object. The y-axis is m/s, the x-axis is seconds; the graph is from desmos.com. I have reprinted the graph for each question is case you want to mark it up.

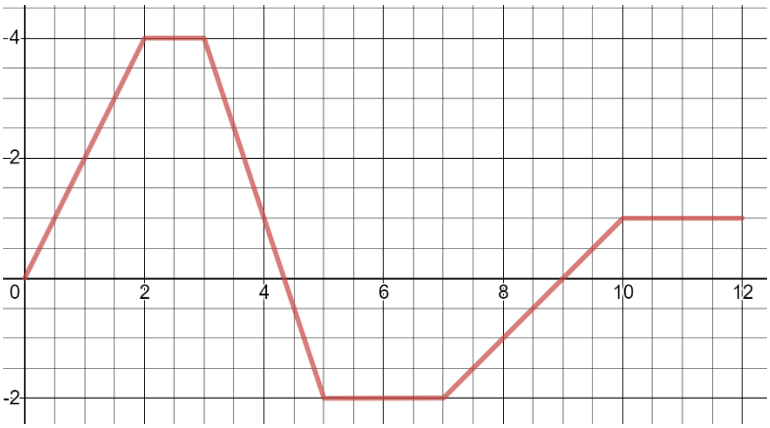


When does the object reverse its direction?

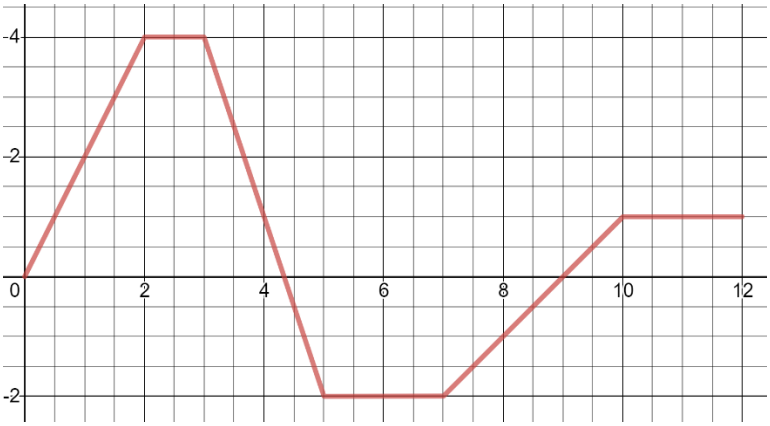


When is the object going a constant speed?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

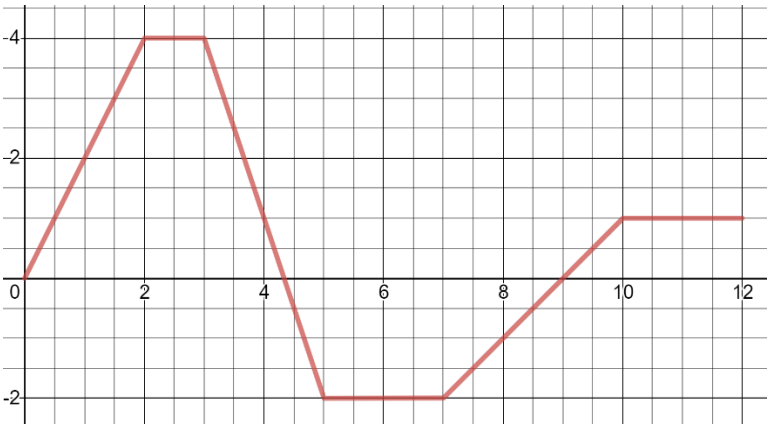


What would the graph of the object's speed look like?



What would the graph of the object's acceleration look like?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

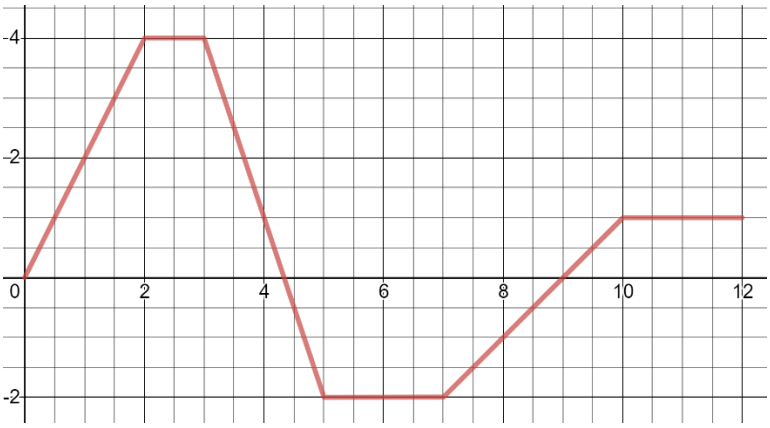


When is the object's acceleration …

Positive?

Negative?

Constant?



When does the object attain its greatest speed?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: The Most Basic Derivative Examples: Constant, Constant Multiple, Power Rule, and Sum Rule

Take the derivative of the following only using the constant, constant multiple, power, or sum rules:

a. $f(x) = 5x^7 - 3x^5 + \pi x^2$

b. $g(x) = 3x^{-5} - 2\sqrt{x} + \frac{1}{2}x^{\frac{9}{5}} + x^{\pi}$

c. $y = \frac{x^5 - 3x^2 - 5}{x^2}$

d. $y = (2x + 1)(5x - 4)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: The Product and Quotient Rules

Take the derivative of the following only the product and quotient rules. Do not simplify these expressions, the point is to ues the rules.

a. $f(x) = (3x^2 - 6x)(2x + 1)$

b. $g(x) = \frac{3x^7}{2x^5}$

c. $y = \left(5\sqrt{x} + 3x^{\frac{4}{5}}\right)(7e^x)$

d. $y = \frac{(\sqrt{x}-1)(x+1)}{(\sqrt{x}+1)(x-3)}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step-by-Step Examples: The Quotient Rule

Find the derivatives:

a. $y = \frac{1+x+6\sqrt{x}}{x}$

b. $y = \frac{\sqrt{x}+1}{\sqrt{x}-1}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Quotient Rule for Derivatives Onlyl

Take the derivatives:

a. $f(x) = \frac{\sin x}{\sqrt{x}}$

b. $y = \frac{9\cos x - 2\tan x}{e^x + \sqrt[3]{x} - 8\sin x}$

c. $f(x) = \frac{(5x^7 - 4x^3 + 3x)}{\sin x - x^{\frac{3}{4}}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Derivatives of Trigonometric Functions

Name of Function	Function Form	Derivative
Sine	$f(x) = \sin x$	
Cosine	$f(x) = \cos x$	
Tangent	$f(x) = \tan x$	
Cotangent	$f(x) = \cot x$	
Secant	$f(x) = \sec x$	
Cosecant	$f(x) = \csc x$	

<u>Examples</u> Find the following derivatives:

a. $f(x) = 3x^7 - \cos x$

b. $g(x) = e^x \sin x$

c. $y = \frac{5}{\cos x} - \frac{3}{\tan x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Derivatives of Trig Functions

Note: these examples assume you do not know the chain rule, you will strictly use the rules for trig derivatives here.

Find the derivatives of the following:

a. $f(x) = \sqrt{x}\csc x - \frac{1}{x^2}$

b. $g(x) = \sin^2 x$

c. $y = \frac{\sin x}{\cos^2 x}$

d. $y = x^2 \sin x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Chain Rule

The Chain Rule focuses on what types of functions?

State the Chain Rule:

<u>Examples</u> Take the derivatives of the following functions:

a. $y = (x^2 - 8)^{12}$

b. $f(x) = e^{\sin(5x)}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

The key to the chain rule is understanding how functions are being composed with one another. So before we start taking any derivatives, we will first focus on understanding compositions of functions.

The basic rules that we have for derivatives are (you should have this memorized, quiz yourself!):

Constant Rule $(k)'$ =	Sine Derivative $(\sin x)' =$
Power Rule $(x^n)' =$	Cosine Derivative $(\cos x)' =$
Constant Multiple Rule $\big(kf(x)\big)' =$	Tangent $(\tan x)' =$
Sum Rule $\big(f(x) + g(x)\big)' =$	Cotangent $(\cot x)' =$
Natural Exponential Rule $(e^x)' =$	Secant $(\sec x)' =$
Product Rule $\big(f(x) * g(x)\big)' =$	Cosecant $(\csc x)' =$
Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' =$	

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

The first thing we must do is identify when we actually need the chain rule.

Example Determine if the chain rule is needed for each function (is it a composition of the derivatives in the previous list?), and if so, then identify how many compositions of functions there are.

Function	Chain Rule Needed?	How Many/Which Functions are Being Composed?
a. $f(x) = \sin(3x)$	Yes No	
b. $y = e^{\sin 4x}$	Yes No	
c. $g(x) = (x^4 - 5)^7$	Yes No	
d. $y = \sin x \tan x$	Yes No	
e. $f(x) = \sin^2(\cos(5x))$	Yes No	
f. $g(x) = (x^2 + 7)^5(3x + 1)^{-2}$	Yes No	
g. $y = \frac{1}{3}\cos x\, e^x$	Yes No	

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Now take the derivative of each function:

a. $f(x) = \sin(3x)$

b. $y = e^{\sin 4x}$

c. $g(x) = (x^4 - 5)^7$

e. $f(x) = \sin^2(\cos(5x))$

f. $g(x) = (x^2 + 7)^5(3x + 1)^{-2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Chain Rule

Find the derivatives of the following functions:

a. $y = \left(\frac{1}{3}x^3 - 7x^2 + 1\right)^8$

b. $f(x) = (4x + 1)e^{3x+1}$

c. $y = \cos\left(\tan\left(e^{\sin 4x}\right)\right)$

d. $y = \sin\left(\frac{x}{\sqrt{x+2}}\right)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

e. $f(x) = x^3 e^{-2x} \sin 5x$

f. $y = 5x(e^{-x} - 4x + 2)^5$

g. $y = \left(\frac{\sqrt{x}-7}{e^{\sin x}}\right)^{-3}$

h. $f(x) = \frac{e^{\sin(4x)}}{x^2 \cos(3x-1)}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Help! The Chain Rule is messing with my head!

The key to the chain rule is understanding how functions are being composed with one another. So before we start taking any derivatives, we will first focus on understanding compositions of functions.

The basic rules that we have for derivatives are (you should have this memorized, quiz yourself!):

Chain Rule $(f(g(x)))' =$	Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' =$
Constant Rule $(k)'$ =	Sine Derivative $(\sin x)' =$
Power Rule $(x^n)' =$	Cosine Derivative $(\cos x)' =$
Constant Multiple Rule $(kf(x))' =$	Tangent $(\tan x)' =$
Sum Rule $(f(x) + g(x))' =$	Cotangent $(\cot x)' =$
Natural Exponential Rule $(e^x)' =$	Secant $(\sec x)' =$
Product Rule $(f(x) * g(x))' =$	Cosecant $(\csc x)' =$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Take the derivatives of the following:

a. $f(x) = \sin(7x)$

b. $g(x) = e^{x^2}$

c. $h(x) = (3x + 7)^{21}$

Example Take the derivatives of the following:

a. $f(x) = x\sin(7x)$

b. $g(x) = e^{x^2}\tan x + 77$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Example Take the derivatives of the following:

c. $h(x) = \frac{(3x+7)^{21}}{9x}$

d. $y = e^{x^2} \sin(7x)$

e. $f(x) = \frac{e^{x^2}}{(3x+7)^{21}}$

f. $f(x) = \sin\left(7x + e^{x^2}\right)$

g. $f(x) = \tan(3x)\, e^{4x - \cos(3x)}$

Copyright 2021, Jennifer Hill, DivideandConquerMath.com

Examples: The Chain Rule Mixed with Quotient Rule or Product Rule (messy functions)

Find the following derivatives.

a. $y = \left(\frac{\sin 4x - e^{\tan x}}{\sec(3x-1) - x^7 + 9}\right)^8$

b. $f(x) = \sin(5x+1)\, e^{4x - \csc(x^3 - 2x)}$

c. $y = \left(\frac{\sin(4x) e^{5x} + x^3}{(x-7)^4 \tan^2 x}\right)^9$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step-by-Step Example: The Chain Rule and the Quotient Rule

Take the derivative: $y = \frac{3x}{(x^3+5)^6}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Trig Functions in the Chain Rule

a. $y = \sin\left(\frac{1}{\sqrt{x+7}}\right)$

b. $y = e^{\sin^2(7x+2)}$

c. $y = \frac{(x+3)^7}{\ln(\sin x)}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Chain Rule and Second Derivatives

Find the first and second derivatives of the following functions:

a. $y = \sin 6x^2$

b. $f(x) = xe^{-x}$

c. $y = x(3x + 5)^3$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Implicit Differentiation

Not all equations are modeled with y on one side and x's on the other. When you have an equation like:

$$x = y^2,$$

How do you take the derivative?

But there's another way we can take the derivative: implicit differentiation. Show how this works below.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Let's look at a few other examples of how to do this:

a. $2x + 3y - y^2 = 5$

b. $x^3y^5 = 5$

c. $\sqrt{x + 3y} = 8$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Implicit Differentiation

Use implicit differentiation to find $\frac{dy}{dx}$:

a. $x^2 = \frac{3-y}{4x+5y}$

b. $x^6 - \cos(xy) = x^2y^5$

c. $y\cos(3y) = 1 - 4xy$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step-by-Step Examples: Implicit Differentiation

Find $\frac{dy}{dx}$: $5x - 7y = 3x^2y^5$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Implicit Differentiation, Tangent Lines, and Normal Lines

Explain where we might encounter a normal line:

Show how to find the tangent and normal line to the curve at the following point:

$x^2 + y^2 = 100$ at $(8, -6)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Implicit Differentiation, Tangent Lines, and Normal Lines

Find the tangent and normal line to the curve at the following point:

a. $x^2y^2 = 25, (25, -1)$

b. $x^2 - x + y - y^2 = 4, (-2, 2)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Implicit Differentiation and Finding Tangent/Normal Lines With Different Characteristics

a. Find 2 points where $x^2 - xy + y^2 = 5$ crosses the x-axis. Are the tangent lines at these points parallel?

b. Find any normals to the curve $3xy + 4x - y = 0$ that are parallel to $y = -4x + 2$.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Second Derivatives with Implicit Differentiation

Find the first and second derivatives of the following:

a. $x^2 + y^2 = 9$

b. $y^2 = x + 3$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Second Derivatives with Implicit Differentiation

Find the first and second derivatives of the following:

a. $y^2 = e^{5x} - 5x$

b. $3 - \cos y = 4x + x^2$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Implicit Differentiation and Simplifying with a Fraction within a Fraction

Find $y'': 3x^2 - 2y^2 = 5$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Understanding Related Rates with an Example

A 12-ft ladder is leaning against a wall when the base starts to slide. By the time the base is 10 feet from the house, the base is moving at the rate of 4 ft/sec.

a. How fast is the top of the ladder sliding down the wall?

b. What is the rate that the area is changing for the triangle formed between the wall, ladder, and floor?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Related Rates – Volume

A conical tank is being filled with water at a rate of $45m^3$/min. The tank is 40 m tall and 10 m wide. How fast is the water level rising when the water is 15 ft deep?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Related Rates – Two Moving Entities

A hot air balloon is riding straight up from a point at a constant rate of 2 ft / sec. When the balloon is 50 ft about the ground, a motorcycle passes underneath is traveling at a constant speed of 30 ft / sec. How fast is the distance between the balloon and the motorcycle increasing 3 seconds later?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Related Rates – A Cone and a Cylinder

Este is making tea by pouring hot water into a conical filter with tea leaves and allowing them to drain into cylindrical pot. The water is draining into the pot at a rate of $11\frac{in^3}{min}$.

a. How fast is the level in the pot rising when the water in the cone is 5 inches deep?

b. How fast is the level in the cone falling?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Related Rates – Problems with Angles

a. A particle is moving along the curve $y = 3x^2$ in the first quadrant of a graph. The x-coordinate is increasing by a rate of 15 m/sec. How fast is the angle of inclination θ of the line joining the origin to the particle changing when x=1?

b. A sun is shining over a building creating a shadow. The building is 88 feet tall and the shadow behind the building is 66 feet long. The angle θ that the sun makes with the ground is changing at a rate of 0.00436 radians per min. What is the rate that the shadow is decreasing?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Extreme Values of Functions on Closed Intervals

The general idea for closed intervals:

To begin, we need to understand the following definitions

Let $f(x)$ be a function. We say that:

- $f(x)$ has an absolute **maximum value** at a point c if $f(x) \leq f(c)$ for all x in the domain
- $f(x)$ has an absolute **minimum value** at a point c if $f(x) \geq f(c)$ for all x in the domain

What are some other words for maximum and minimum values?

The Extreme Value Theorem

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Here are some ways that extreme values can occur on closed intervals:

<u>Example</u> Sketch each graph and determine whether the function has any absolute extrema. Does this contradict the extreme value theorem?

$$f(x) = |x + 1|, 0 < x < 2$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Sketch each graph and determine whether the function has any absolute extrema. Does this contradict the extreme value theorem?

a. $f(x) = \begin{cases} \frac{1}{x^2}, & -1 \leq x < 0 \\ x+3, & 0 \leq x \leq 2 \end{cases}$

b. $y = 2\cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

If we focus on maximum or minimum values that occur within the interior of the interval, what do we notice about the tangent lines at these points? How can we think of these points?

A function $f(x)$ has a **local maximum value** at a point c with its domain if:

A function $f(x)$ has a **local minimum value** at a point c with its domain if:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

What is another word for local maxima/minima?

Critical Points

How to Find Absolute Extrema of a Continuous Function $f(x)$ on a Finite Closed Interval

1.

2.

3.

<u>Example</u> Find the critical points for the function, and then determine the absolute maximum and minimum on the given interval.

$$f(x) = \frac{2}{5}x - 2, \qquad -5 \leq x < 0$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the critical points for the function, and then determine the absolute maximum and minimum on the given interval.

a. $f(x) = -\frac{1}{x}, 1 \leq x \leq 3$

b. $f(x) = \sec x, -\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$

c. $f(x) = x^{\frac{4}{3}}, -1 \leq x \leq 27$

d. $f(x) = 3x^{\frac{2}{3}},\ 1 \leq x \leq 8$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Absolute Extrema on Closed Intervals

Find the absolute extrema for the following functions on the given closed intervals.

a. $f(x) = 4 - x^2, -1 \leq x \leq 2$

b. $f(x) = \sin x\,, 0 \leq x \leq \frac{\pi}{2}$

c. $f(x) = \frac{1}{x} + \ln x\,, 2 \leq x \leq 4$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Review: Factoring with Rational Exponents (Trick for Critical Points)

Find the critical points and domain:

a. $y = x^{\frac{4}{5}}(x + 3)$

b. $y = x^{\frac{3}{4}}(x - 1)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Review: Factoring Out Fractions and Rational Exponents from Expressions

Factor the following out of each expression.

a. $\frac{1}{4}x^{-\frac{1}{3}}$ from $\frac{7}{4}x^{\frac{5}{3}} + \frac{3}{4}x^{-\frac{1}{3}}$

b. $\frac{2}{3}x^{-\frac{2}{5}}$ from $\frac{4}{3}x^{\frac{3}{5}} - \frac{2}{3}x^{-\frac{2}{5}}$

c. $\frac{1}{2}x^{\frac{4}{7}}$ from $6x^{\frac{11}{7}} + 5x^{\frac{4}{7}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Rolle's Theorem

State Rolle's Theorem

Justify why this makes sense:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Show that $x^3 + 4x + 2 = 0$ has exactly 1 real solution.

<u>Example</u> Show $f(x) = x^4 + 4x + 1$ has exactly one zero in the interval $[-2, -1]$.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

The Mean Value Theorem

State the Mean Value Theorem

Explain what this means:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Find the value or values of c that satisfy the equation:

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

a. $f(x)=\sqrt{x+1}, [0,3]$

b. $f(x)=3x^2+6x-5, [-2,1]$

Examples Determine whether the functions satisfy the hypotheses of the mean value theorem on the given interval, with reasoning.

a. $f(x)=x^{\frac{4}{5}}, [0,1]$

b. $f(x)=3x^{\frac{2}{3}}, [-1,1]$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Consequences of the Mean Value Theorem

Write the 2 corollaries associated with the mean value theorem

<u>Examples</u> Find a function with the following derivative that goes through the given point.

a. $f'(x) = 4x; f(0) = 5$

b. $f(x) = \frac{1}{x^2} + 2x; (-1, 1)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Given the velocity function and initial position of an object, determine the object's position at time t.

a. $v = 9.8t + 5; s(0) = 10$

b. $v = \sin t,\ s(0) = 1$

Examples Find all possible functions with the given derivatives:

a. $y' = \cos x$

b. $y' = \frac{1}{\sqrt{x}}$

c. $y' = \sin\frac{x}{2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Increasing and Decreasing Functions and the First Derivative Test
Part 1: Theory and An Example

Before we begin, let's first make sure we have an intuitive understanding of what increasing and decreasing functions look like:

There is a third corollary to the Mean Value Theorem regarding whether functions are increasing or decreasing:

What does monotonic mean?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Review: What is a critical point? How do you find critical points?

Now we can state the **First Derivative Test for Local Extrema:**
Let $f(x)$ be a continuous differentiable function at every point in some interval, except possibly at a critical point c itself. If we examine this interval from left to right:

1.

2.

3.

We can visualize this:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this first set of examples, you will be given the derivative:

$$f'(x) = (x-1)(x+2)(x-3)$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Increasing and Decreasing Functions and the First Derivative Test
Examples 1

<u>Examples</u> Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this first set of examples, you will be given the derivative:

a. $f'(x) = \frac{(x+1)(x-4)}{(x+3)(x+6)}$

b. $f'(x) = x^{-\frac{1}{3}}(x+3)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Increasing and Decreasing Functions and the First Derivative Test
Examples 2

<u>Examples</u> Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this set of examples, you will have to find the derivative:
a. $f(x) = 4x^2 + 8x + 13$

b. $f(x) = x^2 e^{-\frac{x}{3}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Increasing and Decreasing Functions and the First Derivative Test
Examples 3

<u>Examples</u> Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this set of examples, you will have to find the derivative:
a. $f(x) = x^2 - 4x - 8\ln x$

b. $f(x) = \sin x - \cos x, \quad 0 \le x \le 2\pi$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Increasing and Decreasing Functions and the First Derivative Test
Examples 2

<u>Examples</u> Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

$$f(x) = -6x^{\frac{6}{5}} + 3x^{\frac{2}{5}}$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Concavity and the Second Derivative Test for Concavity

Explain concavity in non-technical terms:

For a differentiable function $f(x)$, the function is:
1. **concave up** on an open interval if I if:

Another word for concave up is: _____

2. **concave down** on an open interval I if:

The Second Derivative Test for Concavity:
Let $f(x)$ be a twice-differentiable function on some interval. Then:

1. If $f''(x) > 0$ on that interval, the graph is concave up on that interval
2. If $f''(x) < 0$ on that interval, the graph is concave down on that interval

When concavity changes from up to down (or vice versa), what do we have? Draw a picture of this idea:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

How do you find points of inflection?

Examples Estimate the points of inflection from the graphs (draw your own graphs, sorry!)

a.

b.

Examples Determine i) where the graph is concave up or concave down and ii) any points of inflection

$$f(x) = 2x^3 - 3x^2 - 12x + 3$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Concavity and the Second Derivative Test for Concavity

<u>Examples</u> Determine i) where the graph is concave up or concave down and ii) any potential points of inflection

a. $f(x) = 1 + x^{\frac{1}{3}}$

b. $f(x) = 5 + \cos 2x,\ [0, \pi]$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Second Derivative Test for Local Extrema

How can concavity indicate maximums or minimums? Draw some graphs that illustrate this idea.

The Second Derivative Test for Local Extrema

Suppose $f''(x)$ is continuous on an open interval that contains $x = c$.

1.

2.

3.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Use the second derivative test for local extrema to find local maxima and minima. Then compare this to the first derivative test.

a. $f(x) = x^2 - 4x + 3$

b. $f(x) = x^4 - 8x^2$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Second Derivative Test for Local Extrema

<u>Examples</u> Use the second derivative test for local extrema to find local maxima and minima.

a. $f(x) = x^2(x-1)^2$

b. $f(x) = \cos^2 x, -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

c. $f(x) = (x^2 - x)e^{-x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Limits as x approaches infinity

a. $\lim_{x \to -\infty} \frac{2x^6+3x^2-1}{6x^3-2x+1}$

b. $\lim_{x \to \infty} \frac{5x^{\frac{8}{3}}-3x^{\frac{5}{3}}+2}{x^{\frac{13}{5}}+2x-x^{\frac{1}{2}}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Comparing the Conjugate Trick as $x \to \infty$

a. $\lim_{x\to\infty}\left(x^2 - \sqrt{x^4+3}\right)$

b. $\lim_{x\to\infty}(x^2 - \sqrt{x^6+3x^4})$

c. $\lim_{x\to-\infty}(5x + \sqrt{25x^2+5x-9})$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Horizontal Asymptotes and Limits as $x \to \infty$

Define a Horizontal Asymptote in terms of limits:

<u>Examples</u> Find the horizontal asymptotes of the following:

a. $f(x) = \frac{|x|^3+2}{x^3-1}$

b. $f(x) = 3^x$

c. $f(x) = \frac{3x+1}{5x+2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Sometimes we can have horizontal asymptotes in surprising places.

<u>Example</u> Find the horizontal asymptotes of $x - \sqrt{x^2 + 4}$

<u>Example</u> Use the Sandwich Theorem to find any horizontal asymptotes of $y = 3 + \frac{\sin x}{x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Infinite Limits

To begin this conversation, we investigate the graph of $f(x) = \frac{1}{x}$. Graph this below then evaluate the following:

a. $\lim_{x \to 0^+} f(x) =$

b. $\lim_{x \to 0^-} f(x) =$

c. $\lim_{x \to 0} f(x) =$

Show how to use a table to prove this result:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Now compare this to $g(x) = \frac{1}{x^2}$. Graph this below:

a. $\lim_{x \to 0^+} g(x) =$

b. $\lim_{x \to 0^-} g(x) =$

c. $\lim_{x \to 0} g(x) =$

Show how to use a table to prove this result:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Define vertical asymptotes using limits:

<u>Example</u> Evaluate the following and state any vertical asymptotes.

a. $\lim_{x \to 2^+} \frac{1}{x-2}$

b. $\lim_{x \to 4} \frac{3}{(x-4)^2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Infinite Limits

Evaluate the following:

a. $\lim_{x \to 0^+} \frac{1}{x^{\frac{1}{5}}}$

b. $\lim_{x \to 0^-} \csc x$

c. $\lim_{x \to 5} \frac{1}{x^2(x-5)}$

d. $\lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Using Substitution with Limits as $x \to \infty$

In this lesson, we are going to look at a few limits that end up using clever substitutions.

<u>Examples</u> Evaluate the following:

a. $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

b. $\lim_{x \to 0^-} e^{\frac{1}{x}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Horizontal and Vertical Asymptotes

Find any horizontal, vertical, or oblique asymptotes

a. $f(x) = \frac{x^2-5x+6}{x-2}$

b. $f(x) = \frac{\sqrt{25x^2+100}+x^2}{5x^2-25}$

c. $f(x) = \frac{x^2}{x-1}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Horizontal, Oblique, and Vertical Asymptotes

Find any horizontal, oblique, or vertical asymptotes.

a. $f(x) = \frac{\sqrt{x^2+1}}{x}$

b. $g(x) = \frac{x^2-4}{x-3}$

c. $y = \frac{3x}{x^2-1}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Curve Sketching

Demonstrate all the steps for sketching a curve while sketching $f(x) = \frac{x^2+2}{2x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Curve Sketching Examples 1

<u>Example</u> Sketch $f(x) = 6 + 8x^2 - x^4$, showing all the relevant steps

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Curve Sketching Examples 2

<u>Example</u> Sketch $f(x) = \cos x + \sqrt{2}\sin x,\ 0 \le x \le 2\pi$, showing all the relevant steps

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Curve Sketching Examples 3

<u>Example</u> Sketch $x^{\frac{2}{3}}(5-2x)$, showing all the relevant steps

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Curve Sketching Examples 4

<u>Example</u> Sketch $x^{\frac{2}{3}}(5-2x)$, showing all the relevant steps

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Curve Sketching when you only know y'

In another video, we discussed how to sketch curves. Given the following derivatives, what can you confidently sketch about the curves?

a. $f'(x) = x^2 + 5x + 6$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

b. $f'(x) = (x-2)^2(2x+1)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Optimization Problems

Example 1: A piece of sheet metal is rectangular, 3 ft wide and 6 ft long. Congruent squares are to be cut from its 4 corners. The resulting piece of metal is to be folded and welded to form an open-top box. How should this be done to get the maximum volume?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Optimization Problems

Example 2: What is the maximum possible area of a rectangle with a base that lies on the x-axis with two upper vertices that lie on the graph of the equation $y = 4 - x^2$?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Optimization Problems

Example 3: Find the dimensions of the largest right circular cylinder of greatest volume that can be inscribed in a right circular cone of radius R and height H.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Summary: Derivative Theorems About Min/Max/Inflection Points

In this summary, we will go through summarizing all of the theory and theorems about finding maximums and minimums with functions. We will then relate this to graphing and optimization problems.

First, we will review all of the vocabulary and theorems that usually come up in these concepts.

Absolute minimums/maximums versus local minimum/maximums

Critical point

Monotonic functions

Definition of an increasing function

Definition of a decreasing function

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Definition of concave up

Definition of concave down

Inflection point

Extreme Value Theorem

<u>Example</u> Does the extreme value theorem guarantee that $f(x) = \frac{1}{x}$ will have an absolute minimum or maximum point on the interval $[-3, 3]$?

<u>Example</u> Does the extreme value theorem guarantee that $f(x) = \frac{1}{x}$ will have an absolute minimum or maximum point on the interval $[1, 5]$?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

The First Derivative Test for Local Extrema

The Second Derivative Test for Local Extrema

<u>Examples</u> For each of the following functions, use a) the first derivative test for extrema to find all local/absolute extrema, and then compare that result with b) the second derivative test for extrema. Then c) find the intervals of increasing/decreasing.

a. $f(x) = x^3 - 3x + 3$

b. $f(x) = x^{\frac{2}{3}}(x - 2)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

What are the steps for graphing a function?

<u>Examples</u> For each of the following functions, use a) the second derivative test for concavity to determine the intervals of concavity. Then determine b) if there are any inflection points. Then c) complete the graph of the function.

$$f(x) = x^3 - 3x + 3$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples For each of the following functions, use a) the second derivative test for concavity to determine the intervals of concavity. Then determine b) if there are any inflection points. Then c) complete the graph of the function using the steps above.

a. $f(x) = x^{\frac{2}{3}}(x-2)$

b. $f(x) = x + \cos x, 0 \leq x \leq 2\pi$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

One other application with min/max theorems are optimization problems.

Example A rectangular garden next to a barn needs to be fenced in. Since the garden is next to the barn, the side on the barn does not need fencing. If there is 600 feet of fencing, what is the largest area that can be enclosed?

Example Determine the rectangle of largest area that can be inscribed in a semicircle of radius 4.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Newton's Method

What is the idea and derivation behind Newton's Method?

Write the iterative formula for Newton's Method here:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Use Newton's method to approximate a root for the following function. Stop when the first 5 decimal places do not change over 2 iterations.

$$f(x) = x^3 + x^2 + 1; x_0 = -1.5$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Use Newton's Method to find a solution to $4x - \sin x = 4$ on $[1, 2]$.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Linearization

Linearization is a way to estimate certain difficult calculations without a calculator. Explain the basic idea behind this below:

How do we define the linearization of a function f at a point a?

<u>Example</u> Find the linearization of an integer near the point a at which the given function and its derivative are easy to calculate. Then compare the actual value of the function at a to the approximated value.

$$f(x) = x^{-1}, a = -0.9$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the linearization of an integer near the point a at which the given function and its derivative are easy to calculate. Then compare the actual value of the function at a to the approximated value.

$$f(x) = \sqrt[3]{x}, a = 28.5$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Differentials

Leibniz notation $\frac{dy}{dx}$ represents the derivative and does not represent a ratio, however we will now introduce some very similar notation dx and dy, and when their ratio exists then it is equal to the derivative.

Define the differential below:

<u>Example</u> Find

a. dy if $y = x^3 - 5x^2$

b. the value of dy if $x = 2$ and $dx = 0.1$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Differentials are related to linearization and can also be used to do estimation.

<u>Example</u> Use differentials to estimate

a. $8.1^{\frac{1}{3}}$

b. $\sqrt{6}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Antiderivatives versus Indefinite Integrals versus Differential Equations

Define an antiderivative and any special notes about it:

Define an indefinite integral and any special notes about it:

Define a differential equation and any special notes about it:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Find the antiderivative to the function: $2x$

Example Find the indefinite integral: $\int 2x\, dx$

Example Find the general solution to: $\frac{dy}{dx} = 2x$

Example Solve the initial value problem: $\frac{dy}{dx} = 2x; y(0) = 7$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

List all of the relevant antiderivatives you know here:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples of Basic Antiderivatives / Indefinite Integrals

Find the antiderivatives of the following functions:

a. $\frac{1}{x^2}$

b. $3x^2$

c. $x^{-\frac{1}{2}}$

d. $\frac{2}{3}x^{-\frac{1}{3}}$

Find the indefinite integrals:

a. $\int 2\sin x \ dx$

b. $\int \sec^2 x \ dx$

c. $\int \frac{\pi}{2}\cos\frac{\pi}{2}x \, dx$

d. $\int -\csc 3x \cot 3x \ dx$

e. $\int e^{-2x} dx$

f. $\int \frac{2}{\sqrt{1-x^2}} dx$

g. $\int 3^x dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples of Basic Antiderivatives / Indefinite Integrals

Integrate the following:

a. $\int \frac{5}{x} dx$

b. $\int 5x^{\sqrt{5}} dx$

c. $\int \frac{1+\cos 3x}{2} dx$

d. $\int \cot^2 x \, dx$

e. $\int \cos x \, (\tan x + \sec x) dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples of Basic Differential Equations

Solve the initial value problems:

a. $\frac{dy}{dx} = 5 - 3x^2; y(0) = 4$

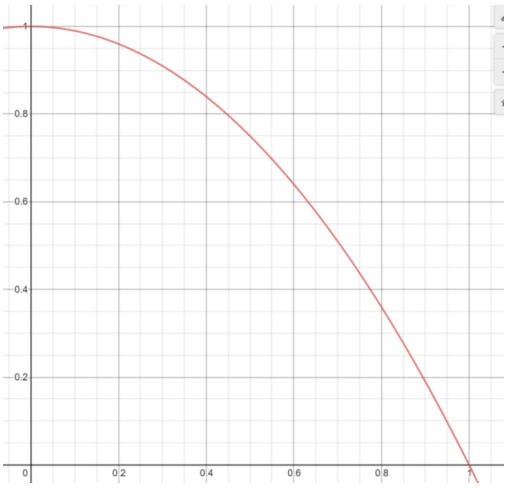
b. $\frac{dy}{dx} = 3x^{-\frac{2}{3}}; y(8) = 1$

c. $\frac{dy}{dx} = -\pi \sin \pi x\,; y(0) = 1$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

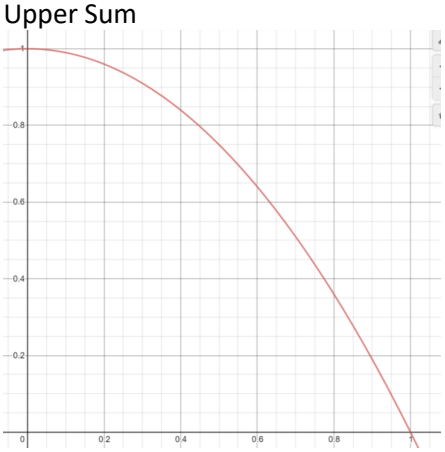
Area and Estimating with Finite Sums

First we will try to understand the idea behind this. Draw out one way to find the area of the curve $y = 1 - x^2$ below.

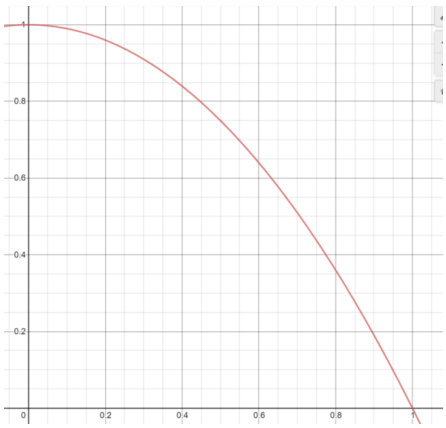


Now we need more formality around this idea. Show the idea of how to create sums using a more algorithmic approach. Detail what each sum means:

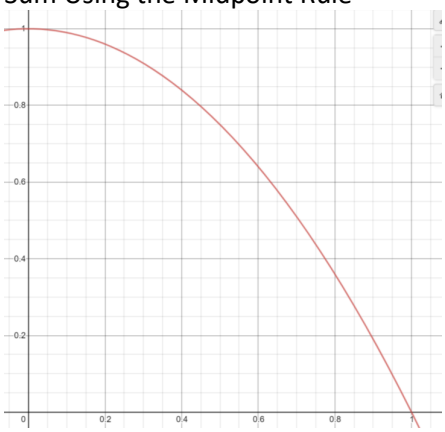
Upper Sum



Lower Sum

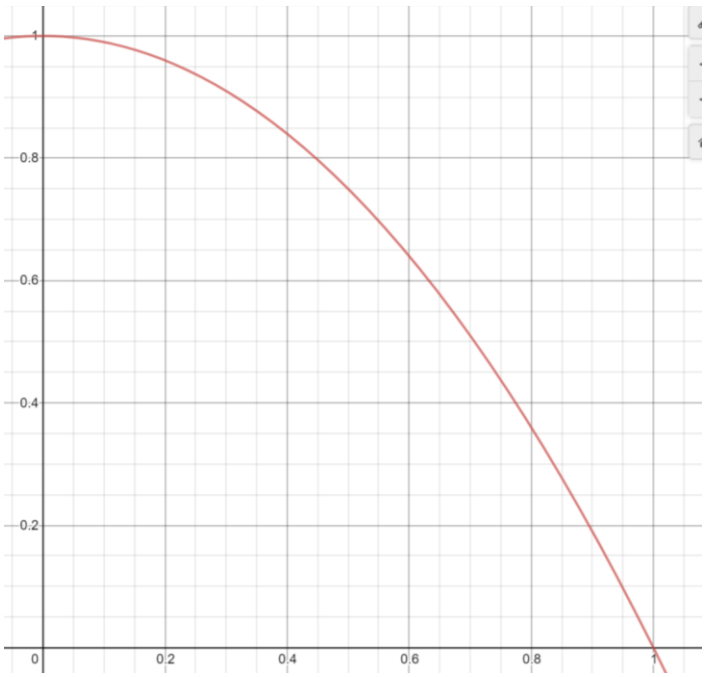


Sum Using the Midpoint Rule



Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Now we need some vocabulary. Separate the graph in 4 parts and then label it with the correct terminology:



Define a Riemann Sum:

What is a left Riemann sum?

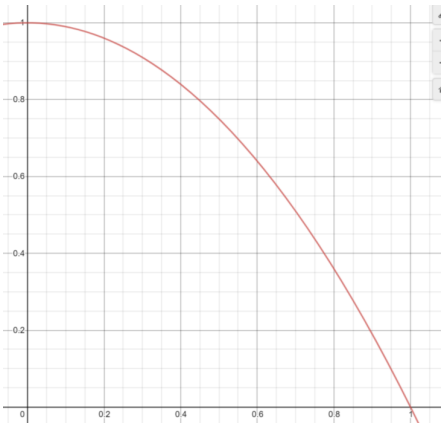
What is a right Riemann sum?

What is a midpoint Riemann sum?

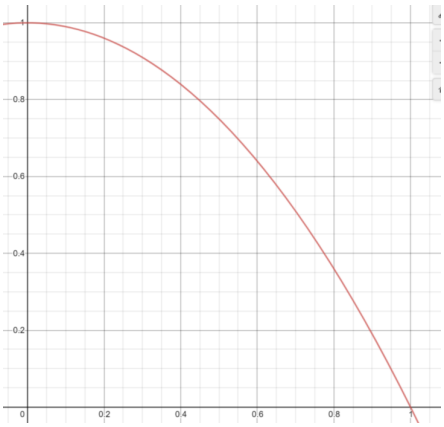
Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Show how you can make a more precise sum:

8 Subintervals

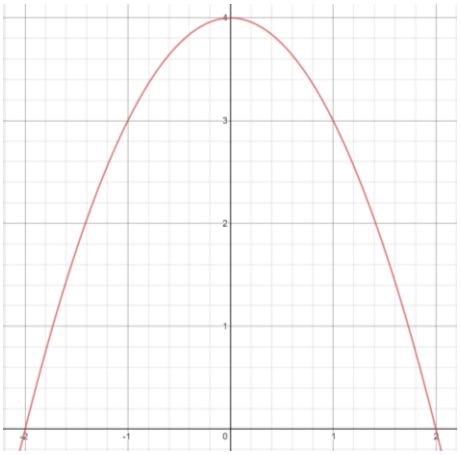


16 Subintervals

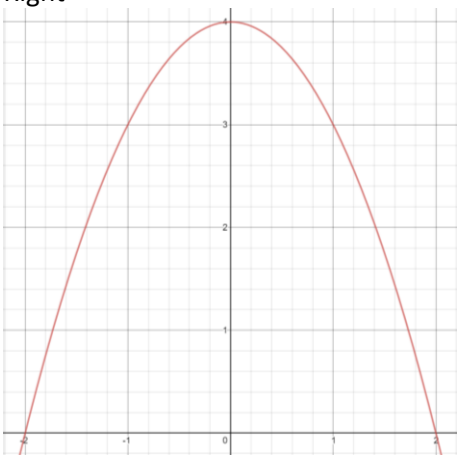


<u>Example</u> Compute the left, right, and midpoint Riemann sums for $f(x) = 4 - x^2$ between $[-2, 2]$ using 4 subintervals

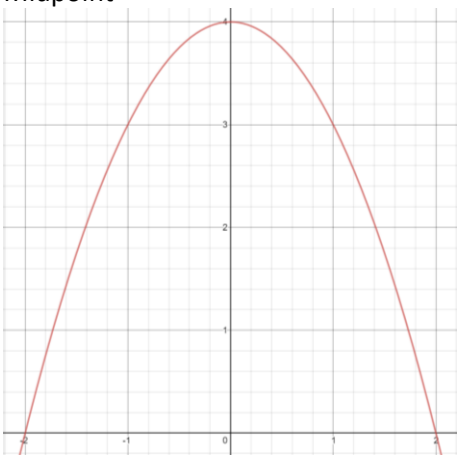
Left



Right



Midpoint



Graphs were created using Desmos.com. They have the greatest graphing software for free, check them out!

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples of Riemann Sums

Compute the left, right, and midpoint Riemann sum for $f(x) = 3 + x$ from $[1,3]$ using 4 subintervals.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples of Riemann Sums – When It Goes Negative?

Compute the left, right, and midpoint Riemann sum for $f(x) = x^3$ from $[-2,2]$ using 4 subintervals.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Different Ways to Compute Area Estimations

Using $f(x) = -x^2 + 2x + 3$, compute an area estimate using 4 subintervals from x=-1 to x=3.

Compute this estimate using an upper, lower, left-hand, and right-hand sum.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Basics of Sigma Notation

Explain what sigma notation looks like and what it means:

<u>Examples</u> Compute the following sums:

a. $\Sigma_{k=1}^{4} k$

b. $\Sigma_{i=3}^{7} (2i - 1)$

c. $\Sigma_{k=1}^{4} (-1)^{k+1} k$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

We have a few handy rules for sigma notation:

1. Sum Rule:

2. Difference Rule:

3. Constant Multiple Rule:

4. Constant Value Rule:

<u>Examples</u> Use the rules above to help calculate the following sums:

a. $\Sigma_{k=1}^{3}(k+5)$

b. $\Sigma_{k=1}^{4}(-3k-2)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Using Sigma Notation and Sum Formulas to Calculate Larger Sums

We've already discussed these basic algebraic rules for sums:

Sum Rule

$$\Sigma_{k=1}^{n}(a_k + b_k) = \Sigma_{k=1}^{n} a_k + \Sigma_{k=1}^{n} b_k$$

Difference Rule

$$\Sigma_{k=1}^{n}(a_k - b_k) = \Sigma_{k=1}^{n} a_k - \Sigma_{k=1}^{n} b_k$$

Constant Multiple Rule

$$\Sigma_{k=1}^{n} ca_k = c * \Sigma_{k=1}^{n} a_k$$

Constant Value Rule

$$\Sigma_{k=1}^{n} c = n * c$$

Now we will add on a few other handy formulas for sums:

The First n Terms

The First n Squares

The First n Cubes

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

These formulas are particularly handy for larger sums.

<u>Examples</u> Sum the following using the sum formulas:

a. $\Sigma_{k=1}^{15} 3k^2$

b. $\Sigma_{i=1}^{52} (i^3 + 4i - 2)$

c. $\Sigma_{k=1}^{35} k(2k + 5)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Sigma Notation with Riemann Sums

Our goal in this lesson is to figure out how to use Riemann Sums with Sigma Notation. To get you warmed up for this idea, we will practice figuring out how to write certain Riemann sums:

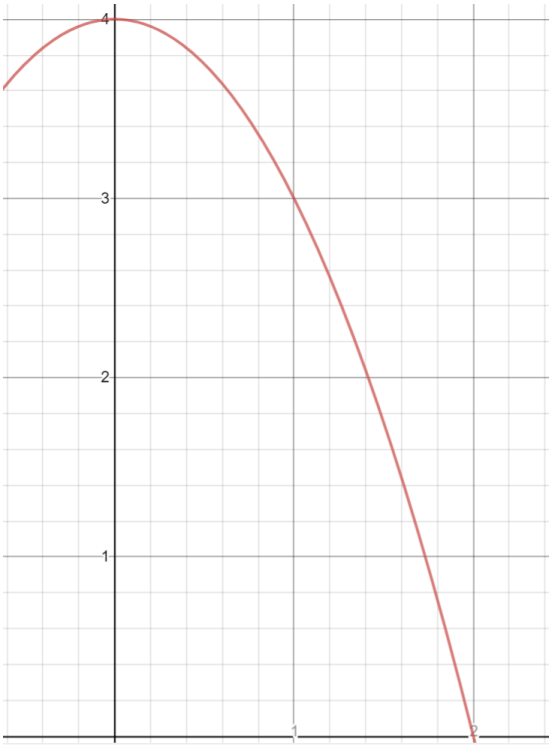
<u>Examples</u> Write the following sums using sigma notation.

a. $1 + 2 + 3 + 4 + 5$

b. $2 + 4 + 6 + 8 + 10 + 12$

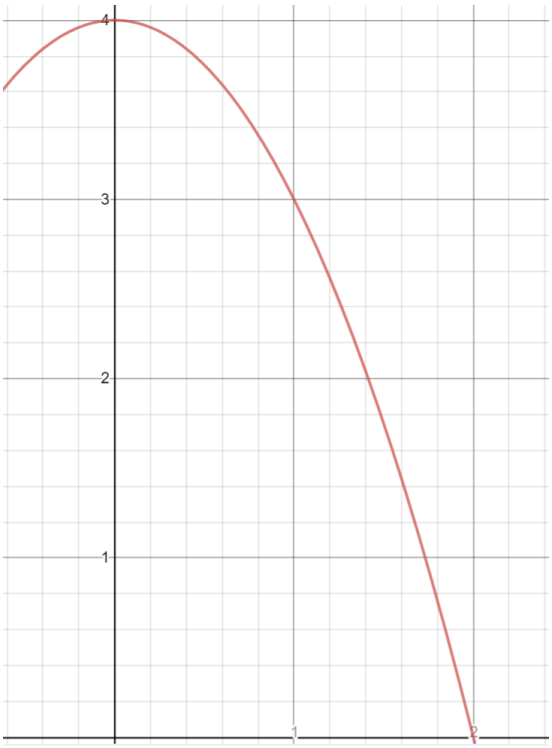
c. $1 - 3 + 5 - 7$

Previously we had discussed how to computer Riemann sums. Using $f(x) = 4 - x^2$ on $[0, 2]$, create 4 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?



Copyright 2020, Jennifer Hill, DivideandConquerMath.com

We will continue the pattern. Continuing to use $f(x) = 4 - x^2$ on $[0, 2]$, create 8 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?



Can you determine how you might generalize this for n subintervals?

Write out the Riemann sum formula for using right-hand endpoints:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

How can we manipulate this formula so that we use left-hand endpoints instead? Write this down:

<u>Example</u> Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^2 + 1$ on $[0,2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Evaluating Riemann Sums as $n \to \infty$

<u>Example</u> Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = 4 - x^2$ on $[0, 2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Evaluating Riemann Sums as $n \to \infty$

<u>Example</u> Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^3$ on $[0,1]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Basics of Definite Integrals

Before we begin, we need to define some terminology. Draw and label the following:

Let $f(x)$ be a function defined on $[a, b]$. We say that a number J is the **definite integral of f over** $[\boldsymbol{a}, \boldsymbol{b}]$ and that J is the limit of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\Delta x_k$ if the following is true:

> Given any $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for every partition P = $\{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$ we have:
>
> $$|\Sigma_{k=1}^{n} f(c_k)\Delta x_k - J| < \epsilon$$

This looks very similar to the definition of a limit in some ways, which I will state here:

> We have $\lim_{x \to c} f(x) = L$ if, for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that:
>
> $$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

However, if the limit exists and using equal width partitions, we can instead state this (label what you write down):

Write down and label the definite integral notation:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

There are a few rules of definite integrals that we will want to note. When $f(x)$ and $g(x)$ are integrable functions over $[a, b]$, then we can use the following rules:

1. Order of Integration

2. Zero Width Interval

3. Constant Multiple

4. Sum and Difference

5. Additivity

6. Max-Min Inequality

7. Domination

<u>Example</u> if $\int_3^5 f(x)\ dx = 5,\ \int_0^3 f(x)\ dx = 2,$ and $\int_0^5 g(x)\ dx = 9,\ \int_3^5 g(x)\ dx = 4$ find:

a. $\int_0^5 f(x)\ dx$

b. $\int_0^3 g(x)\ dx$

c. $\int_5^3 f(x)\ dx$

d. $\int_0^5 \big(3f(x) + g(x)\big)dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Finally, we have one final theorem to state that helps us determine when a function is integrable.

Theorem: Integrability of Continuous Functions

A question that usually arises from this is, “What is a function that isn’t integrable then?” This is the classic example:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Finding Definite Integrals Using Area Formulas

If $f(x)$ is nonnegative and integrable over $[a, b]$, how do we define the area under the curve $y = f(x)$?

This means that we can calculate definite integrals using known area formulas when possible, as opposed to using Riemann sums.

Example Sketch the following curves and use known area formulas to evaluate the definite integrals.

a. $\int_{-1}^{1}(3 - |x|)\ dx$

b. $\int_{0}^{3}(2x)\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

c. $\int_0^2 (2x+1)\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Average Value of a Function

If a function $f(x)$ is integrable on $[a, b]$, then its average value or mean is calculated by:

In this lesson, we will calculate integrals only using known area formulas.

<u>Examples</u> Find the average value of the following functions:

a. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$

b. $f(x) = |x| - 1$ on $[-1, 1]$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Fundamental Theorem of Calculus, part 1 proof

Define the initial function we are working with and the corresponding graph:

Compute F'(x), including any relevant graphs. Interpret what this means:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

State the first part of the Fundamental Theorem of Calculus

<u>Example</u> Find $\frac{dy}{dx}$ given $y = \int_3^x (t^5 + 1)\, dt$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Fundamental Theorem of Calculus, part 1

Differentiate the following in 2 ways:

i. evaluate the integral then differentiate

ii. differentiate the integral directly

a. $\frac{d}{dx}\int_3^x \sin t \ \ dt$

b. $\frac{d}{dx}\int_1^{\cos x} 2t \ dt$

c. $\frac{d}{dx}\int_4^{\sqrt{x}} (5t^3 - \cos t) \ dt$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Fundamental Theorem of Calculus, part 1

Compute $\frac{dy}{dx}$ for each of the following:

a. $y = \int_4^x \sqrt{3t+2}\, dt$

b. $y = \int_x^5 \sin t \;\; dt$

c. $y = \int_{\sin x}^4 3t^2 \;\; dt$

d. $y = \int_{-3}^{e^{4x}} \frac{1}{\sqrt{3t+1}} \; dt$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step by Step Example: Fundamental Theorem of Calculus, part 1

Find $\frac{dy}{dx}$ for $y = x \int_5^{x^2} \sin t^3 \ dt$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Fundamental Theorem of Calculus, part 2 proof

State the Mean Value Theorem for Definite Integrals

<u>Example</u> if $\int_a^b f(x)\,dx = 0$, must there be a point c in $[a, b]$ where $f(c) = 0$?

From the Fundamental Theorem of Calculus part 1, we know that the antiderivative of f exists in the form:

$$F(x) = \int_a^x f(t)\ dt$$

Now, we know that the most general form of an antiderivative for f would be $F(x) + c$. Now we want to make this more precise.

What is $F(b) - F(a)$?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

State the second part of the Fundamental Theorem of Calculus

<u>Examples</u> Compute the following:

a. $\int_1^3 2x\,dx$

b. $\int_0^{\frac{\pi}{2}} \cos x\,dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Fundamental Theorem of Calculus, part 2 – basic examples

Evaluate the following integrals:

a. $\int_1^4 (3x^2 - x)\ dx$

b. $\int_0^{\frac{\pi}{3}} 5\sec^2 x\ dx$

c. $\int_0^4 \left(4x^2 + \sqrt{x}\right) dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Fundamental Theorem of Calculus, part 2 – tricky trig examples

Evaluate the following integrals:

a. $\int_0^{\frac{\pi}{4}} \frac{1+\cos x}{2}\, dx$

b. $\int_0^{\frac{\pi}{3}} \frac{1+\cos 2x}{2}\, dx$

c. $\int_{\frac{\pi}{2}}^{\pi} \sin^2 x\ dx$

d. $\int_0^{\frac{\pi}{2}} (\sec x + \cos x)^2\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Fundamental Theorem of Calculus, part 2 – tricky non-trig examples

Evaluate the following integrals:

a. $\int_3^5 (x+1)^2 \ dx$

b. $\int_1^3 \left(\frac{x^6}{3} - \frac{1}{x}\right) dx$

c. $\int_0^4 \frac{x+\sqrt{x}}{x} \ dx$

d. $\int_{-2}^2 |x| \ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Which Formula Should I Use for Integrals?

In this video, we will compute the integral $\int_1^4 (2x+1)\,dx$ 3 ways:

Way 1: Area Formulas

Way 2: Riemann Sum

Way 3: Using the fundamental theorem of calculus / evaluation formula

Use the space below to write out each way.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Summary: Everything about Indefinite and Definite Integrals

Compare $\int 2x\, dx$, $\int_0^3 2x\ \ dx$, and $\int_3^0 2x\ \ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Indefinite Integrals and the Substitution Method

Before we begin, let's just review how indefinite integrals work:

$\int (3x^4 - \sin 5x + 3e^{3x})\, dx$

Now we can explain substitution using the following example: $\int (4x^3 + 2x)(x^4 + x^2)^{18}\, dx$

<u>Example</u> Integrate:

a. $\int (4x^3 - 6x^2)(x^4 - 2x^3)^8\, dx$

b. $\int 3\sin(3x)\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Sometimes we must manipulate the problem to get substitution to work.

Use the example $\int (6x+9)^{12}\, dx$ to demonstrate this idea:

<u>Examples</u> Integrate:

a. $\int 4\sec^2(3x+1)\ dx$

b. $\int \frac{1}{\sqrt{3x-5}}\ dx$

c. $\int \tan x\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Sometimes we must go nuclear with our manipulations. Below are a few examples of creative ways to manipulate the problems:

a. $\int \sin^5 x \cos^3 x \ dx$

b. $\int (x-4)(x+3)^{15} \ dx$

c. $\int x^3 \sqrt[3]{x^2+4} \, dx$

d. $\int \frac{dx}{1-e^x} \ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Indefinite Integrals and the Substitution Method to Solve Certain Trig Integrals

In this lesson, it is assumed that you already understand the basics of the substitution method. Here we will focus on how to use substitution to integrate the following integrals:

a. $\int \tan x \ dx$

b. $\int \cot x \ dx$

c. $\int \sec x \ dx$

d. $\int \csc x \ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Basic Examples of Indefinite Integrals and the Substitution Method

a. $\int 2x\sqrt{x^2+5}\ dx$

b. $\int \frac{1}{2\sqrt{x}\left(3+\sqrt{x}\right)^2}\ dx$

c. $\int 3\sin^3 3x\cos 3x\ dx$

d. $\int -\frac{1}{x^2}\sqrt{3+\frac{1}{x}}\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples of Indefinite Integrals and the Substitution Method Where Some Manipulation is Required

a. $\int 5x\sqrt{3x^2+2}\,dx$

b. $\int 5x^{\frac{1}{2}}\cos\left(3x^{\frac{3}{2}}+1\right)\,dx$

c. $\int \frac{\cos(5x+2)}{\sin^2(5x+2)}\,dx$

d. $\int \sqrt{\frac{x^8}{x^5+4}}\,dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Difficult Examples of Indefinite Integrals and the Substitution Method Where Creativity is Needed

a. $\int \sqrt{\frac{x^2-x}{x^6}}\ dx$

b. $\int 5x^5\sqrt{4x^3+1}\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

c. $\int \frac{7}{4+25x^2}\ dx$

d. $\int \frac{3}{x\sqrt{1-x^4}}\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Substitution with Definite Integrals

This lesson assumes you are already familiar with substitution for indefinite integrals. To get warmed up, evaluate the following:

a. $\int \frac{6x^2}{\sqrt{1-x^3}}\ dx$

b. $\int \frac{x}{\sqrt{1+x}}\ dx$

Using these examples, we can demonstrate 2 different ways to evaluate definite integrals with substitution.

$\int_1^3 \frac{6x^2}{\sqrt{1-x^3}}\ dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Now you try using $\int_0^3 \frac{x}{\sqrt{1+x}}\, dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Substitution with Definite Integrals

Evaluate the following:

a. $\int_1^4 \frac{1}{2x\left(1+\sqrt{x}\right)^2}\ dx$

b. $\int_0^{\frac{\pi}{3}} \cos^{-2} x \sin x\ \ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

c. $\int_1^4 x\sqrt{1+3x}\,dx$

d. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1+e^{\tan x})\sec^2 x\ \ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example: Definite Integrals and Substitution

$\int_2^4 \frac{dx}{x(\ln x)^6}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Step by Step Example: Substitution and Definite Integrals – Trig Problem!

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{x}\sin^2\left(x^{\frac{3}{2}}\right)\,dx$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Definite Integrals of Symmetric Functions (Proof and Examples)

In this lesson, we will look at how substitution can be used to prove the following theorem:

Theorem Suppose that $f(x)$ is a continuous function on $[-a, a]$. Then:

a. if $f(x)$ is even, then $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{2} f(x)\ dx$

b. if $f(x)$ is odd, then $\int_{-a}^{a} f(x)\ dx = 0$

Prove this below:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Evaluate the following integrals:

a. $\int_{-4}^{4}(x^3 + x)\, dx$

b. $\int_{-5}^{5}(|x| - x^4 + x^2)\, dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: The Trapezoidal Rule

With Riemann sums, we use rectangles for making area estimates. However, there are other shapes we can use.

First we need to know the area of a trapezoid formula:

$$A = \frac{a+b}{2}h$$

You may want to draw a reference picture for that.

Now we can discuss how to use trapezoids for area estimates below:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

State the Trapezoidal Rule

<u>Example</u> Estimate the area of $f(x) = 2x + 1$ on $[0, 2]$ using four subintervals and the trapezoidal rule.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Derivatives of Inverse Functions and Logarithms

Find the inverse of each function:

a. $y = x^2, x > 0$

b. $y = x^3 + 2$

Select a random point for each function. How does that point relate to each function's inverse? Write down your choice of point here and how it would appear in the inverse below:

a. Choice:

b. Choice:

Now take the derivative of each function and the derivative of its inverse below:

a. Derivatives:

b. Derivatives:

What is the relationship between the points you found when you plug them into the derivatives?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Write down the derivative rule for inverses below:

Explain how to leverage this rule to find the derivative of $f(x) = \ln x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: How to Find the Derivatives of a^x and $log_a x$

Explain how to find the derivative of a^x:

Explain how to find the derivative of $\log_a x$:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the derivatives of the following:

a. $f(x) = 3^{x^2}$

b. $y = \log_5(3x + 2)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Derivatives with $e^x, \ln x, \log_a x, a^x$

Find the derivatives of the following functions:

a. $f(x) = \ln(x^2)$

b. $g(x) = e^{\sin x + \ln x}$

c. $y = 5^{\ln x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

d. $f(x) = \log_3(x) + \log_3(x^4)$

e. $g(x) = \ln\,(\ln(\ln(\ln(4x))))$

f. $y = 7^{\log_2 x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Logarithm Defined as an Integral

Evaluate the following integrals.

a. $\int_{-4}^{-2} \frac{dx}{x}$

b. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

c. $\int 3x^2 e^{-x^3} dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

d. $\int_{\ln\left(\frac{\pi}{3}\right)}^{\ln\left(\frac{\pi}{2}\right)} 2e^x \sin e^x \, dx$

e. $\int_1^3 x^{3x}(1 + \ln x)dx$

f. $\int_1^9 \frac{\log_3 x}{x} dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Logarithm Defined as an Integral, Initial Value Problems

Solve the initial value problems.

a. $\frac{dy}{dx} = e^{-x}\csc^2(\pi e^{-x}),\ y(\ln 4) = \frac{2}{\pi}$

b. $\frac{d^2y}{dx^2} = 3e^{-x}, y(0) = 1, y'(0) = 0$

c. $\frac{d^2y}{dx^2} = \sec^2 x,\ y(0) = 0, y'(0) = 1$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Logarithm Defined as an Integral, Initial Value Problems

Solve the initial value problems.

a. $\frac{dy}{dx} = e^{-x}\csc^2(\pi e^{-x}),\ y(\ln 4) = \frac{2}{\pi}$

b. $\frac{d^2y}{dx^2} = 3e^{-x}, y(0) = 1, y'(0) = 0$

c. $\frac{d^2y}{dx^2} = \sec^2 x,\ y(0) = 0, y'(0) = 1$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Determining if Functions are One-to-One

Before we begin, you should already be familiar with the concept of functions and the vertical line test. First let's do some review.

<u>Review</u> Determine whether the following relations are functions, then determine the domain and range.

a. $\{(2,3),(5,7),(2,4)\}$

b.

Now we will discuss a specific type of function: a one-to-one function. Define this below:

How can we visualize this?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Determine if any of the following are i) functions and ii) one-to-one functions

a. $\{(2,4),(-1,7),(3,7)\}$

b. $\{(3,6),(-1,2),(-3,-4)\}$

Similar to how we have a test to determine if a graph represents a function, we have another test to determine if a function is one-to-one.

Describe the horizontal line test below:

Examples Determine whether the following graphs represent i) functions and ii) one-to-one functions (draw your own):

a. b. c.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Finding the Inverse of Functions

One-to-one functions have inverses. Describe what the inverse of a function is:

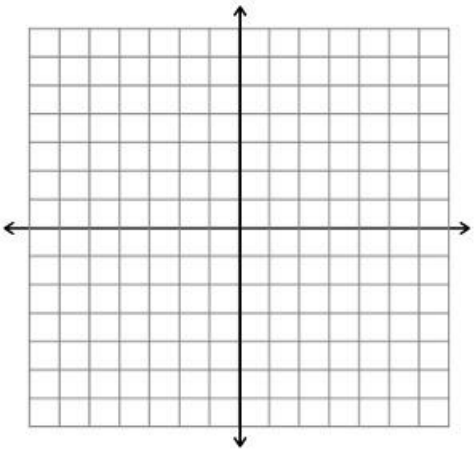
<u>Example</u> Find the inverse of the following function: $\{(1,4),(7,8),(3,-2)\}$

Describe the steps to find the inverse of a function:

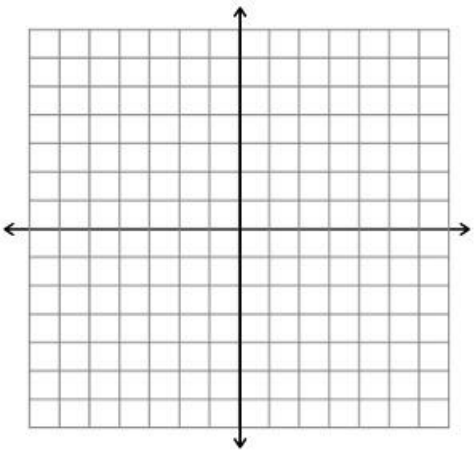
Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Find the inverse of the following one-to-one functions. Then plot the original function on the graph with its inverse.

a. $f(x) = 3x + 1$



b. $g(x) = -\frac{1}{2}x + 4$



What is the relationship of a graph with its inverse?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Find the inverse of the following one-to-one functions.

a. $h(x) = \sqrt[3]{x}$

b. $f(x) = x^2, x \geq 0$

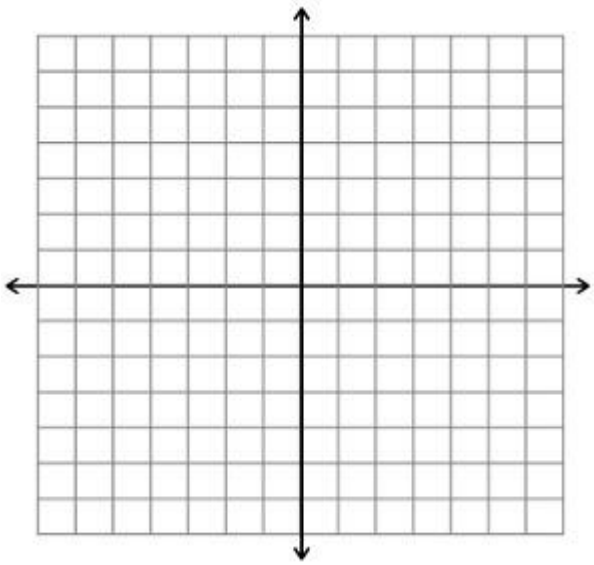
Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Exponential Functions

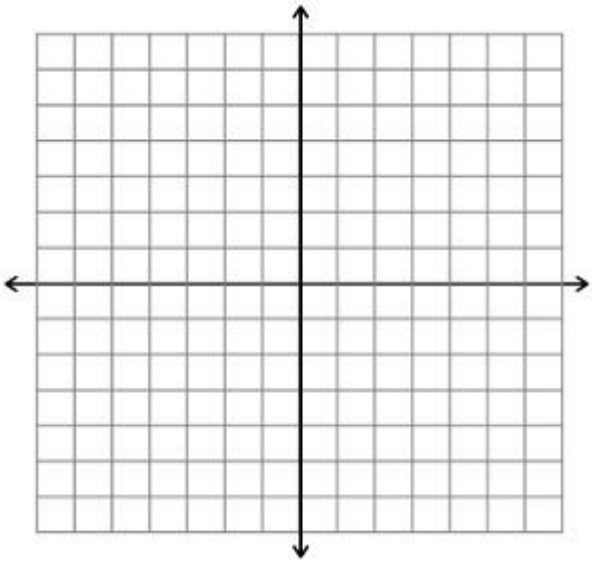
An exponential function is a function of the form _____ where $a > 0, a \neq 1$.

<u>Examples</u> Graph the following functions, then determine their domain and range.

a. $f(x) = 2^x$



b. $g(x) = \left(\frac{1}{2}\right)^x$

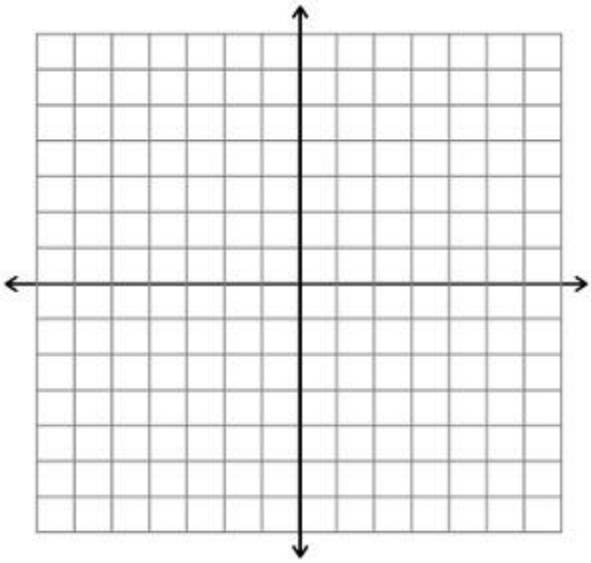


Using the graphs above, summarize some of the characteristics of these graphs:

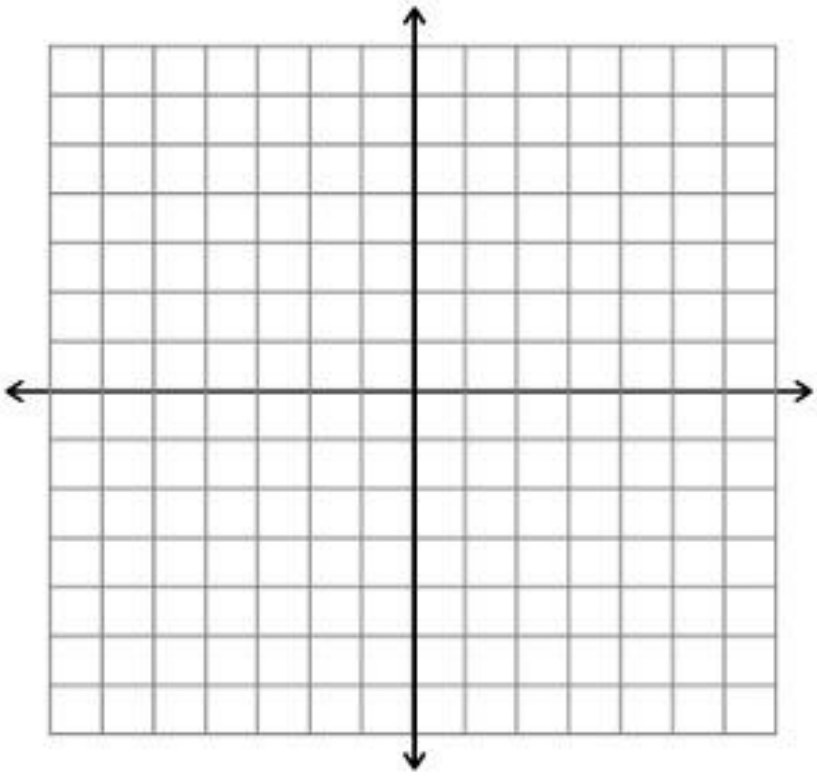
The number e is an irrational number that is frequently used in math. What is an approximation of e?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Graph $f(x) = 2^x, g(x) = e^x, h(x) = 3^x$ all on the same graph below for comparison:



Translations work similarly to other graphs we have discussed. Using the graph of $f(x) = 2^x$ as a "base graph", draw the following graphs together on the same graph: $f(x) = 2^x, g(x) = 2^x + 3, h(x) = 2^{x-1}$



Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Solving Exponential Equations

In this lesson, we will solve equations that look like this:

$$3^x = 27 \qquad \text{or} \qquad 4^{x+3} = 8^{x-2}$$

To do this, we will write directions around the following example: $2^{3x} = 8^{x+1}$

<u>Examples</u> Solve:

a. $49^{x-3} = 7^{2x}$

b. $9^{4x} = 27^{x+4}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Solve:

a. $5^{x-2} = \frac{1}{25}$

b. $\left(\frac{3}{5}\right)^{2x} = \left(\frac{27}{125}\right)^{x+2}$

c. $27^{3x+1} = 9^{4x+2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Review: Exponents for Calc/Pre-calc Students

The 3 Main Rules:

<u>Examples</u>

a. $(3x^4y^7)^2 * (x^7y^3)$

b. $\frac{(-2x^3y^6)^3}{(3xy^2)^4}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples

a. $(4x^3y^7z^{10})^{-3}$

b. $\left(\frac{3x^{-4}y^{-7}}{x^3y^{-10}}\right)^{-2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples

a. $\sqrt[3]{x^2} * \sqrt{x}$

b. $\frac{x^{\frac{1}{2}}x^{\frac{1}{3}}}{x^{\frac{1}{4}}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Using Properties of Logarithms to Simplify Derivatives

Take the derivatives of the following:

a. $y = \log_2 \left(\frac{3x+1}{x-1}\right)^{\ln 2}$

b. $y = \log_7 \sqrt[3]{\left(\frac{x+1}{4x-6}\right)^{\ln 7}}$

c. $y = x \log_4 e^{(\tan 2x)(\ln 4)}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Logarithmic Differentiation

Logarithmic differentiation is another way to take derivatives.

Show how to use logarithmic differentiation to take the following derivatives:

a. $y = \frac{x\sqrt{5x+1}}{(x+9)^7}$

b. $f(x) = (x+1)^x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Separable Differential Equations, Initial Value Problems

<u>Examples</u> Solve the initial value problems

a. $\frac{dy}{dx} = \sqrt{y}; y(0) = 4$

b. $\frac{dy}{dx} = \frac{x}{y}; y(2) = 6$

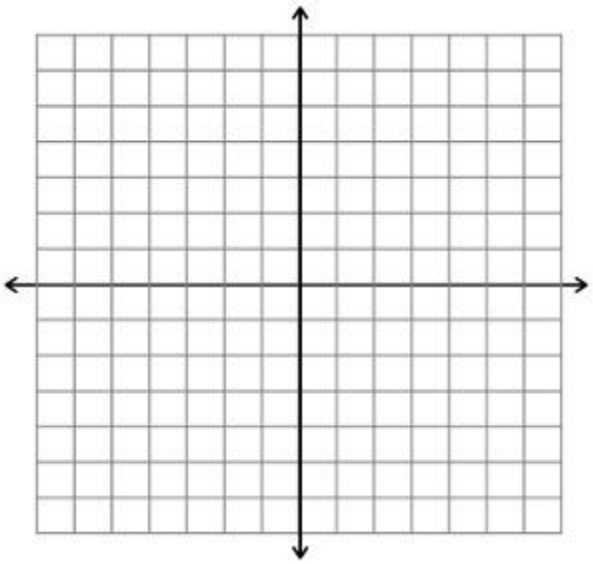
c. $\frac{dy}{dx} = 3x^2y^2 - y^2; y(0) = \frac{1}{2}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Logarithmic Functions

First, we want to review how to function inverses of functions. Find the inverse of: $f(x) = 2x + 8$

Graph the exponential function $f(x) = 3^x$ below:



Recall the exponential functions are one-to-one, which means that they have inverses. Attempt to find the inverse of $f(x) = 3^x$ below:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

As we can see, there is an issue with finding the inverse – we have no way to solve for y! As such, we need new mathematics to help solve this.

Write down the definition of a logarithm:

Examples Write the following in exponential form:

a. $\log_3 81 = 4$

b. $\log_2 \frac{1}{8} = -3$

c. $\log_5 125 = 3$

d. $\log_9 9 = 1$

Examples Write the following in logarithmic form:

a. $2^4 = 16$

b. $3^0 = 1$

c. $6^{-2} = \frac{1}{36}$

d. $8^{\frac{1}{3}} = 2$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Now we can finally determine what is the inverse of $f(x) = 3^x$:

There are a few common logarithms:

- $\log x$ means:

- $\ln x$ means:

There are also 2 key properties of logarithms:

1. $\log_a a = 1$: explain why this makes sense

2. $\log_a 1 = 0$: explain why this makes sense:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Basics of Solving Logarithmic Equations and Evaluating Logarithms

We want to solve equations that look like:

$$\log_3 x = 4 \qquad \text{or} \qquad \log_x 36 = 2$$

Before you begin, you should already be able to do the following:

a. Put 3^5 in logarithmic form:

b. Put $\log_5 5 = 1$ in exponential form:

Now we can explain how to solve logarithmic equations using the example $\log_3(2x + 1) = 3$

<u>Example</u> Solve: $\log_x 16 = 4$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Solve:

a. $\log_9 x = 2$

b. $\log_6(2x + 4) = 2$

Now we can also discuss how to evaluate logarithms. Explain how to evaluate: $\log_3 27$:

Examples Evaluate:

a. $\log_5 \frac{1}{25}$

b. $\log 100$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Evaluate:

a. $\ln e$

b. $\log_3 27$

c. $\log_{25} 5$

d. $\log \frac{1}{10}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: How to Graph a Logarithm

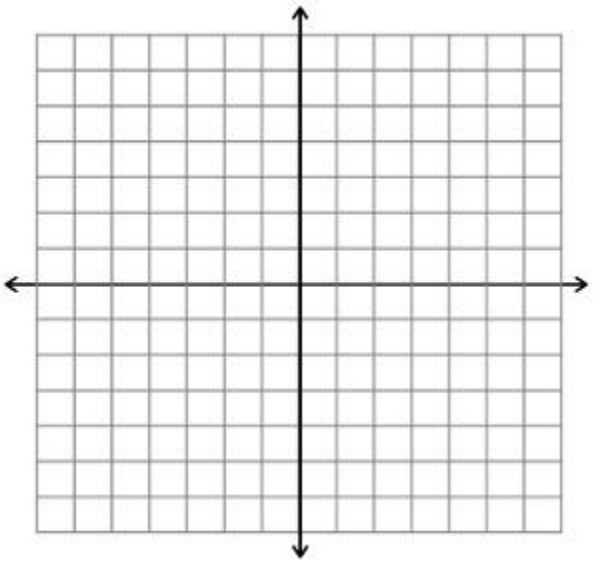
Before we begin, you should be able to do the following:

a. Convert to logarithmic form: $25 = 5^2$

b. Convert to exponential form: $\log_2 8 = 3$

c. Find the inverse of $y = 2^{\mathrm{x}}$

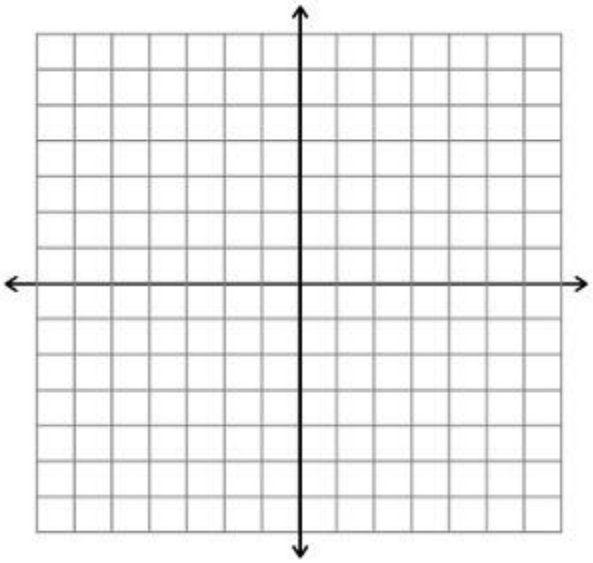
Now we can explain how to graph $y = \log_3 x$



For comparison, also graph $f(x) = 3^x$ on the same graph above.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Graph $y = \log_{\frac{1}{2}} x$ below:



Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Review: How to Solve Logarithmic Equations

What are the 5 main properties of logarithms?

When solving logarithmic equations, what else should you remember about logarithms?

<u>Examples</u>

a. $\log x + \log x^2 = 3$

b. $\log_5(3x + 5) + \log_5(x + 1) = 1$

Copyright 2024, Dr. Jennifer Hill, DivideandConquerMath.com

c. $\log(5x-6)-\log(x+1)=\log\ 3$

d. $\log_2(5+4x)+\log_2(x+3)=\log_2 3$

Copyright 2024, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Logarithmic Differentiation

Use logarithmic differentiation to find the following derivatives:

a. $y = x(x-4)(x+5)(x+2)^7$

b. $f(x) = \sqrt[3]{\frac{\sin x}{(x^2+5)^3}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

c. $x^y = y^x$

d. $y = x^{\sin x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Understanding the Derivatives of Inverse Trig Functions

Show how to find the derivative of $y = \sin^{-1} x$

Show how to find the derivative of $y = \tan^{-1} x$

Show how to find the derivative of $y = \sec^{-1} x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

State the Inverse Function-Inverse Cofunction Identities

Explain how these relate to finding the derivatives of $y = \cos^{-1} x,\ y = \cot^{-1} x,\ y = \csc^{-1} x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: The Derivatives of Inverse Trig Functions

Find the derivatives of the following:

a. $y = \cos^{-1}(x^5)$

b. $f(x) = \cot^{-1}\sqrt{3x+1}$

c. $y = \ln\left(\sec^{-1}(2x-1)\right)$

d. $f(x) = \sin^{-1}(e^{-x})$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Hyperbolic Functions

The hyperbolic sin and hyperbolic cosine functions are defined by the equations:

How do we say “sinh x” ?

What about “cosh x” ?

Using these definitions, we get some interesting identities. Determine the value of $2\sinh x\ \cosh x$:

Now determine the value of $\cosh^2 x - \sinh^2 x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

List all of the relevant identities for hyperbolic functions:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Derivatives of Hyperbolic Functions – A Quick Overview

Find the derivative of $\sinh x$

Find the derivative of $\operatorname{csch} x$

Find the derivative of $\tanh x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Write the derivative rules for all of the hyperbolic functions below:

Write the integral formulas for all of the hyperbolic functions:

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Population Growth

We can study how diseases die when properly treated by using the following logic let $\frac{dy}{dt}$ represent the rate at which the number of infected people changes is proportional to the number y. The number of people who are cured is proportional to y, the number who are infected with the disease. As of April 23, 2020, the number of coronavirus cases was 2,659,557. For simplicity, let's round that number to 2,660,000. Let's suppose that we can reduce the number of cases of the disease by 50% in a year. If there are 2,660,00 cases, then how many years will it take to the reduce the number to 2,000 cases?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Population Growth

We can study how diseases die when property treated by using the following logic let $\frac{dy}{dt}$ represent the rate at which the number of infected people changes is proportional to the number y. The number of people who are cured is proportional to y, the number who are infected with the disease. As of April 23, 2020, the number of coronavirus cases was 2,659,557. For simplicity, let's round that number to 2,660,000. Let's suppose that we can reduce the number of cases of the disease by 50% in a year. If there are 2,660,00 cases, then how many years will it take to the reduce the number to 2,000 cases?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Separable Differential Equations, General Solutions Only

<u>Examples</u> Find general solutions to the following:

a. $\frac{dy}{dx} = 2x\sqrt{y}$

b. $\frac{dy}{dx} = (xy)^{\frac{3}{2}}$

c. $\frac{dy}{dx} = \frac{(x^5-1)y^3}{x^2(3y^2-2)}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Radioactivity and Exponential Decay

Some atoms are unstable and spontaneously emit mass, which is called radioactive decay. An element that goes through this process is called radioactive. The decay of a radioactive element is described by the differential equation:

$$\frac{dy}{dy} = -ky, k > 0$$

If y_0 is the number of initial nuclei present, then the number at a later time is:

$$y = y_0 e^{-kt}, k > 0$$

An interesting consequence of this is that half-life can be calculated by the equation:

$$\frac{\ln 2}{k}$$

The half-life of the plutonium isotope is 24,360 years. If 15 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 75% of the isotope to decay?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: Newton's Law of Cooling

A metal beam was brought from the outside cold into a machine shop where the temperature was held at $75°\mathrm{F}$. After 10 min, the beam warmed to $45°\mathrm{F}$ and after another 15 min it was $55°\mathrm{F}$. Use Newton's Law of Cooling to estimate the beam's initial temperature.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Calculating the Area Between 2 Curves With Integration

An integral is calculating what area?

Draw out an explanation of how we can understand using integration to find the area between 2 curves. Watch the explanation, and then rewrite it in terms that make sense to you.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example Find the area enclosed between $y = x^2 - 1$ and $y = 3$.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

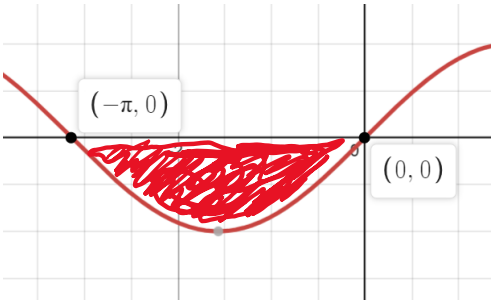
<u>Example</u> Find the area between $x = 2y^2$, $x = 0$, and $y = 3$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Total Area vs Area

What's the difference between area and total area? Demonstrate this concept with a graph.

<u>Example</u> Find the total area for the curve $y = \sin x$ in the shaded region (graph from Desmos.com)



Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: The Area Between 2 Curves With Respect to x

<u>Example</u> Find the area between $y = x^2 - 2x$ and $y = -x^2 + 4x$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: The Area Between 2 Curves With Respect to y

<u>Example</u> Find the area between $x = 2y$ and $x = y^2 - 3$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the area enclosed by $x + 4y^2 = 4$ and $x + y^4 = 1$ for $x \geq 0$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Area Between Hard to Visualize Curves

a. Find the area between the curves enclosed $y = x^4 - 9x^2 + 1$ and $y = x^2 - 8$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

b. Find the area enclosed by the curves $9x^2 + y = 9$ and $x^8 - y = 1$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

c. Find the area of the region enclosed by the curve $x = 3\sin y\ \sqrt{\cos y}$ and the line $x = 0, 0 \leq y \leq \frac{\pi}{2}$

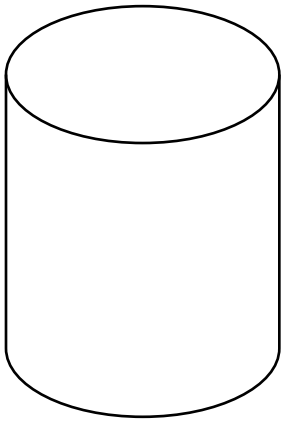
Copyright 2024, Jennifer Hill, DivideandConquerMath.com

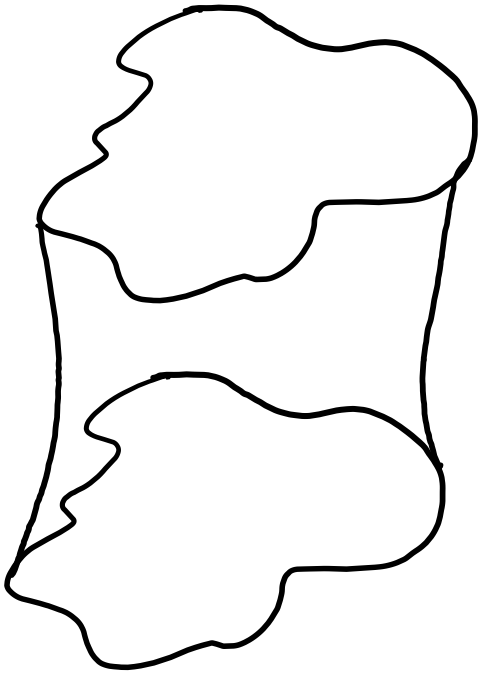
Lesson: Understanding Volume by Slicing with Parallel Planes

We're going to discuss how integration relates to volume. One way is using cross-sections.

What is a cross-section?

Below we have a few different kinds of cylinders. What would the cross-sections look like?

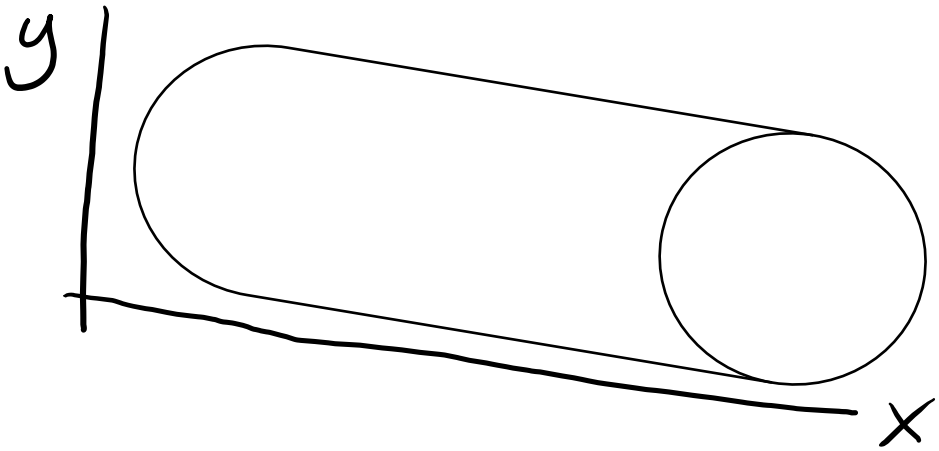




Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Which way do you want these slices to be oriented?

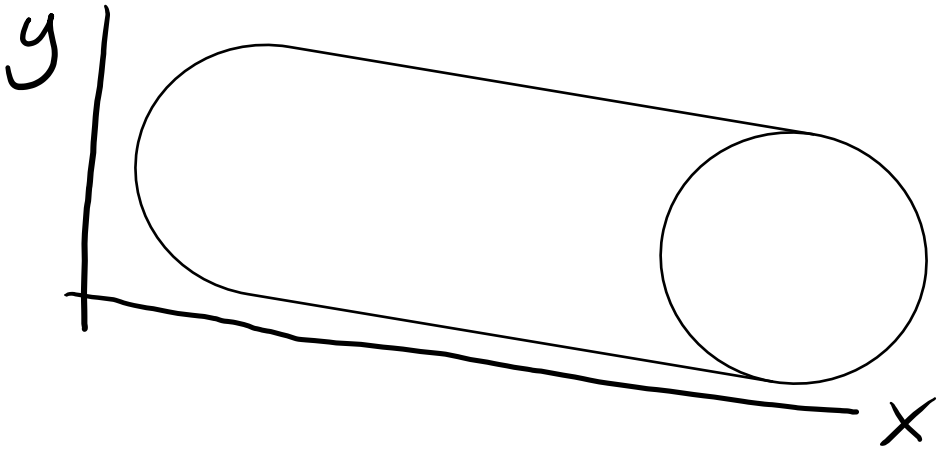
Below is how we would orient the circular cylinder.



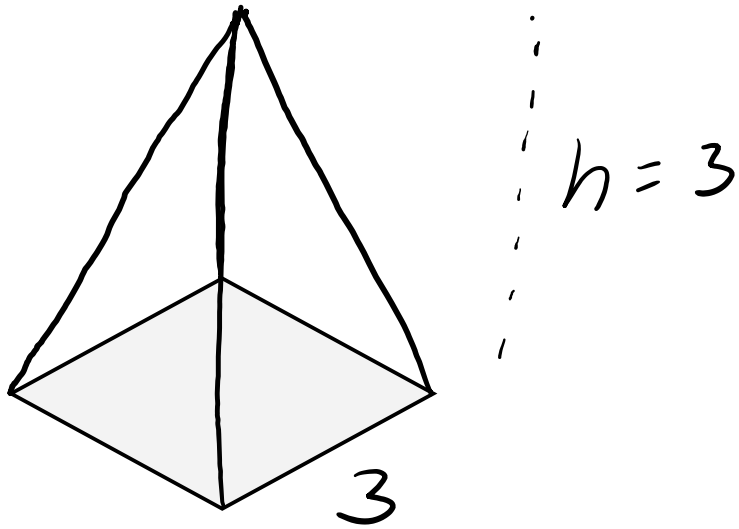
Draw a sketch showing how we would orient the second (odd looking) cylinder on the x- and y-axis.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Explain how we derive volume using cross-sections for the cylinder.



Using the following pyramid with a square base, draw how we might opt to orient this object to make it easier to analyze the cross-sections.



Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Summarize the explanation of how we would opt to take the volume of cross-sections in the case of the pyramid. In your explanation, explain how Riemann sums fit into this.

Write down both formulas for finding volume with cross-sections.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

What are the steps to calculating the volume of a solid with cross-sections?

Example Find the volume of a solid where the base is the region between the curve $y = 4\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis, where the cross-sections are equilateral triangles with the bases running from the x-axis to the curve y.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Volumes Using Cross-Sections

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 4 - x^2$. Find the volume.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x$. Find the volume when the cross-sections perpendicular to the x-axis are:

a. isosceles triangles of height 4

b. semicircles with diameters running across the base of the solid

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Volume by Slicing, Harder Examples

A square of side length 2x lies in a plane perpendicular to a line L. One vertex of the square lies on L. As the square moves a distance h along L, the square turns one revolution about L to generate a corkscrew shaped column with square cross sections. What is the volume?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Find the volume of the tetrahedron. Please refer to the video for the reference picture.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

The base of the solid is the disk $x^2 + y^2 = 4$. The cross-sections by planes perpendicular to the y-axis between $y = -2$ and $y = 2$ are isosceles right triangles with one leg in the disk. Find the volume.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Solids of Revolution via the Disk Method

Before you do this lesson, you need to make sure you understand the lesson about volume by cross-sections!

The disk-method is basically an application of volume by cross-sections.

What is a solid of a revolution?

What are the cross-sections in this case?

How can we represent the area of these disk?

Formula for Finding Volume by Disks for Rotation About the x-axis

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Formula for Finding Volume by Disks for Rotation About the y-axis

<u>Example</u> Find the volume of the solid generated by revolving the bounded region about the x-axis:

$$y = x^3, y = 0, x = 1$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example Find the volume of the solid generated by revolving the bounded region about the y-axis:

$$x = \sqrt{4\sin y}, 0 \leq y \leq \pi, x = 0$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the volume of the solid generated by revolving the bounded region about the line $y = 1$.

$$y = \sqrt{2x}, y = 1, x = 2$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: The Disk-Method, Revolved around the x-axis or y-axis

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = \sec x$, $y = 0, x = -\frac{\pi}{4}, x = \frac{\pi}{4}$ around the x-axis

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

B. $x = \frac{2}{\sqrt{y+1}}, x = 0, y = 0, y = 3$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: The Disk-Method, Revolved Around a Line (not an axis)

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 4x, y = 0, x = 1$ about the line $x = 1$

B. $y = \sqrt{x}, y = 2, x = 0$ about the line $y = 2$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Solids of Revolution via the Washer Method

You should understand the idea of volume by cross-sections and volume by the disk method before you try this lesson.

The washer method is used with solids of revolution, but what characteristic do these solids have?

What do we need to know about the washers?

Formula for Volume by Washers for Rotation about the x-axis

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example Find the volumes of the solid generated by revolving the enclosed region about the x-axis.

$$y = x, y = 2, x = 0$$

Example Find the volumes of the solid generated by revolving the enclosed region about the y-axis.

$$x = \tan y\,, y = 0, x = 1, y \leq 0 \leq \frac{\pi}{4}$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: The Washer Method

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 2\sqrt{x}, y = 2, x = 0$ about the x-axis

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

B. $y = \sec x,\ y = \sqrt{2}, -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Volume by Disks and Washers Around Different Axes

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, y = 0, x = 2$ about:

a. The x-axis

b. The line $y = -2$

c. The line $y = 6$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

d. The y-axis

e. The line $x = 3$

f. The line $x = -3$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: When to Use Subtraction with Finding the Radius

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, x = 0, y = 4$ about:

a. The x-axis

b. The y-axis

c. The line $y = 4$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Help! I Feel Like I'm Bad At Sketching

Sketch the region bounded by:

$y = 3x, x = 2, y = 0$

a. Show the solid of a revolution obtained by revolving this region around the x-axis

b. Show the solid of a revolution obtained by revolving this region around the y-axis

c. Show the solid of a revolution obtained by revolving this region around the line $x = 2$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

Sketch the region bounded by:

$y = \sqrt{x}, y = 2x - 1, x = 0$

a. Show the solid of a revolution obtained by revolving this region around the y-axis

b. Show the solid of a revolution obtained by revolving this region around the line $x = 4$

c. Show the solid of a revolution obtained by revolving this region around the line $y = 3$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

Lesson: Volume by Cylindrical Shells

Sometimes cross-sections can be awkward.

Demonstrate why cross-sections might be awkward using the example $y = 2x - x^2$ revolved around $x = -1$. Focus on the area enclosed by the x-axis.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Another way to calculate volume is by using a cylindrical shell. Draw how a cylindrical shell is constructed. How do we compute the volume of a shell?

Now draw how a cylindrical shell might fit into your sketch. Your sketch does not have to be perfect – you just need to use it as a way to wrap your head around the idea. Draw multiple sketches if needed. Label the pieces of your sketch.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

One of the keys to using cylindrical shells is what?

Explain how the volume formula will relate to integration:

Shell Formula for A Revolution Around Any Vertical Line x=L

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

What assumptions do we make about calculating volume in this manner?

<u>Example</u> Find the volume generated by revolving the bounded region about the y-axis:

$$y = x, y = -\frac{1}{2}x, x = 4$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the volume generated by revolving the bounded region about the line $y = -1$

$$x = \frac{1}{2}y, y = 0, x = 4$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Volume by Using Cylindrical Shells with a Piecewise Function

Let $g(x) = \begin{cases} \frac{\sin x}{x}, & 0 < x \leq \pi \\ 1, & x = 0 \end{cases}$

a. Show the $x * g(x) = \sin x,\ 0 \leq x \leq \pi$

b. Find the volume of the solid generated by revolving the region enclosed by the curve and the x-axis about the y-axis.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: The Shell Method Where The Integral is Split Up

Using the region enclosed by $y = x^2$ and $y = x + 6$, use the shell method to find the volume of solids generated by revolving the regions about the given lines:

a. $x = 3$

b. $x = -4$

c. the x-axis

d. $y = -2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Volume by Using Cylindrical Shells About Different Lines

For the region bounded by the curves:

$$y = 2x, y = 0, x = 3$$

Find the volume generated by revolving the region about:

a. The line $x = -1$

b. The line $y = -2$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Volume by the Washer Method vs Volume by Cylindrical Shells

Compute the volume of the solid bounded by the curves:

$$y = \frac{1}{2}x + 1, y = x, x = 0$$

In the following ways:

a. Revolved about the x-axis using the washer method

b. Revolved about the y-axis using the shell method

c. Revolved about the line x=3 using the shell method

d. Revolved about the line y=4 using the washer method

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Summary: How to Set Up Volume by Integration Problems

There are a few keys to setting up volume by integration problems:

1. Always draw the region **and** draw a sketch of the solid – you should always have 2 drawings
2. Decide what formula is appropriate for the situation and with respect to which variable
3. Label your solid to go with the formula
4. Fill in the formula and integrate

Let's review a few concepts.

Write out the formulas for the disk method.

Write out the formulas for the washer method.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

When do you use the washer method?

How can you determine if your revolution is a solid or has a hole in it?

With the disk/washer method, when do you integrate with respect to x?

With the disk/washer method, when do you integrate with respect to y?

In what circumstances do you have to “adjust” the radius? How does this adjustment look? What does this mean about rotating about the x-axis or y-axis?

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Write out the formulas for the shell method:

With the shell method, when do you integrate with respect to x?

With the shell method, when do you integrate with respect to y?

If you rotate around the follow axis, what is the radius in that situation?

a. x-axis

b. y-axis

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

When do you need to subtraction with finding the shell radius?

If you need to use subtraction to find the shell radius, how do you determine the radius?

In the disk, washer, or shell methods: if you are integrating with respect to x, how can you determine the limits of integration?

In the disk, washer, or shell methods: if you are integrating with respect to y, how can you determine the limits of integration?

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

If you struggle to sketch your solids, I have an entire video just focusing on this. *You absolutely need to be able to quickly sketch a solid, and it is worth your time to do this video if you have anxiety about doing this.*

<u>Example</u> Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

First, sketch the region:

a. Now we will find the volume generated by revolving this region about the x-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

b. Now we will find the volume generated by revolving this region about the y-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

c. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

d. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

e. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

f. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

g. Now we will find the volume generated by revolving this region about the x-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

h. Now we will find the volume generated by revolving this region about the y-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

i. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

j. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

k. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

l. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Arc Length and How the Formula is Derived

Suppose we want to find the length of the curve $y = \sqrt{x-2}$ from $x = 2$ to $x = 6$. Draw a sketch of the curve.

To understand how we can use integration to find the length of a curve, we need to recall how we find the length of a line segment. Draw out a line segment from (x_1, y_1) to (x_2, y_2), and write out how we would find the formula for this line segment.

How can we apply this idea to a Riemann sum with a curve? Draw an illustration to accompany this explanation.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Show how we can take the ideas from algebra and mold them to using integration to finding the length of the curve.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

State the arc length formula.

<u>Example</u> Find the length of the curve $y = 4x^{\frac{3}{2}}$ from $x = 0$ to $x = 1$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Arc Length with Discontinuous Derivative Values

There is an issue if we try to from the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$. Explain what the issue is.

How can we resolve this issue?

What is the formula for the length of the curve $x = g(y), c \leq y \leq d$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Show how to find the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Finding the Area of a Surface of Revolution

How do you create a solid of revolution?

How to find the area of surface from revolving the graph of f(x) around the x-axis:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

How to find the area of surface from revolving around the y-axis:

<u>Examples</u> Find the areas of surface generated by revolving the curve around the given access.

a. $f(x) = \sqrt{x+1}, 1 \leq x \leq 5$, x-axis

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

b. $x = \left(\frac{1}{3}\right) y^{\frac{3}{2}} - y^{\frac{1}{2}}, 1 \leq y \leq 3$, y-axis

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Surface Areas of Revolution

Find the surface area of the revolution generated by revolving the given curve about the given axis.

a. $x = \sqrt{4y - 3}, 1 \leq y \leq 3$, y-axis

b. $y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}, 1 \leq x \leq 2$, x-axis

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $x = \frac{1}{2}(e^{y} + e^{-y}), 0 \leq y \leq \ln 3$, y-axis

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: A Guide to Approaching Work Problems

Constant force work formula:

Variable force work formula:

A note on calculating force:

The units are very important because they will likely guide you on how to do the problem.

Let's note units too for the following:

Measure	SI Units	US Customary
Force		
Work		
Mass		
Length		
Acceleration due to gravity		

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

What are some important notes regarding how we think of force in different applications?

With problems involving objects:

With pumps/volume problems:

If you have completely forgotten how to do dimensional analysis, *please watch my review video on this topic.* This will really help you!

Going back to our integral, rewrite the integral to explain how we should be watching out for units.

A guideline for how to get to these units:

With objects

With pumps/volume

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

Why is it obnoxious to say “weighs” here? What’s a better way to think about this?

What elements go into how this formula is set up?

When we set up our integral, what ultimately needs to happen?

Show how to finish the problem and show how the units guide us that we are right.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

What if I wanted to use (25-x) instead of x to model the length left of the rope?

What does this mean?

Note the following fact about integration:

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example You want to haul a 15-m length of hanging rope. How much work will it take if the rope weighs 0.08 kg/m.

Example A bag of sand weighing 160lbs was lifted at a constant rate. As it rose, sand leaked out at a constant rate. One-quarter of the sand was gone when the bag had been lifted to 20ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment).

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example A vertical right-circular cylindrical tank measures 20 ft high and 10 ft in diameter. It is full of milk weighting 64 $\frac{lb}{ft^3}$. How much work does it take to pump the milk to the level of the top of the tank?

Example Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 5 ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to the rim of the tank? (See picture)

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 8ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to 5ft above the top of the tank?

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Hooke's Law for Springs

What is Hooke's law?

What are the relevant units to pay attention to?

<u>Example</u> It took 1500 J of work to stretch a spring from its natural length of 2m to a length of 4m. Find the spring's force constant.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example If a force of 80 N stretch a spring 2 m beyond its natural length, how much work does it take to stretch the spring 6 m beyond its natural length?

Example A spring has a natural length of 8 in. A 500-lb force stretches the spring to 13 in.

a. Find the force constant

b. How much work is done in stretching the spring from 8 in to 13 in?

c. How far beyond its natural length will a 800-lb force stretch the spring?

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Fluid Forces

Fluid Force on a Constant-Depth Surface

Fluid Force Against a Vertical Flat Plate (Integration Formula)

Example Refer to the video for the sketch. Calculate the fluid force on one of the triangular plate using the given coordinates.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example In a pool fill with water to a depth of 12 ft, calculate the fluid force on one side of a 4ft x 6ft rectangular plate if the plate rests vertically at the bottom of the pool on its 4-ft edge.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Fluid Forces

Calculate the fluid force on one side of a semicircular plate of radius 4 ft that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 7 feet.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

A metal cube that is 8ft on each side has a parabolic gate on one side at the bottom. See the picture in the video. It is designed to withstand a fluid force 500 lb without rupturing. The liquid you plan to store has a weight-density of 62.4 lb/ft^3.

a. What is the fluid force on the gate when the liquid is 6 feet deep?

b. What is the maximum height to which the container can be filled without exceeding the gate's design limitation?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: A Review of "Basic" and Creative Integration Techniques

Write down all of the "basic" integration formulas we should know at this point:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Now we will review some more creative “basic” integration techniques.

I. More Creative Substitution Ideas with Completing the Square

$\int \frac{dx}{\sqrt{2x-x^2}}$

II. Manipulating an Expression to Use Substitution

$\int \frac{dx}{\sqrt{e^{2x}-1}}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

III. Multiply by the Conjugate

$\int \frac{dx}{1-\sec x}$

IV. Multiply by an Expression Equivalent to 1

$\int \frac{1}{e^x+e^{-x}}\, dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

V. Split Into 2 Parts

$\int \frac{1+\sin x}{\cos^2 x}\, dx$

VI. Use Long Division

$\int \frac{4x^2-7}{2x+1} dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Completing the Square to Integrate

$\int \frac{dx}{\sqrt{2x-x^2}}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Review: How to Complete the Square (A Few Ways)

Complete the square on the polynomial:

a. $x^2 - 4x$

b. $x^2 + 6x + 9$

c. $x^2 + 3x$

d. $5x^2 - 20x - 10$

e. $6 - 2x - x^2$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Multiplying by a Strategic Fraction to Integrate

a. $\int \frac{dx}{1-\sin x}$

b. $\int \frac{dx}{\sec x+\tan x}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Improper Fractions and Division

What makes an improper fraction?

a. $\int \frac{x^2}{x^2+1} dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $\int \frac{4x^3-x^2+16x}{x^2+4}\ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Separating Fractions to Integrate

$\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Using Substitution More Creatively

a. $\int \frac{dx}{\cos^2 x \tan x}$

b. $\int \frac{2^{\ln x^3}}{16x} dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Integration by Parts

State the formula for **integration by parts**:

We will look at a few basic examples in this video.

<u>Basic Example</u> $\int x \cos \pi x \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Basic Example $\int \ln x \ dx$

Example of Using Integration by Parts Multiple Times $\int x^3 e^x \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example Where the Integral Cycles</u> $\int e^x \sin x \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Integration by Parts with Definite Integrals

Let's look at an example of how to use integration by parts with definite integrals.

$\int_0^4 x\, e^{-x}\ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Integration by Parts, Examples Where You Have to Use Integrate Multiple Times

a. $\int x^2 \cos x \ dx$

b. $\int (x^2 - 2x + 1)e^x \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Integration by Parts, Examples Where The Integral Cycles Back to Itself

a. $\int e^{4x} \cos 3x \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $\int e^{-x} \cos x \, dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Using Substitution AND Integration by Parts Together

a. $\int e^{\sqrt{4x+1}}dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $\int x\,(\ln x)^2\,dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\int \cos\sqrt{x}\, dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Trigonometric Integrals with Different Powers of Sine and Cosine

I. Integrals of the Form $\int \boldsymbol{sin^m\, x\, cos^n\, x\ \ dx}$

A. When m is even, what should you do?

<u>Example</u> $\int \sin^3 4x \cos^2 4x\ \ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

B. When n is odd, what should you do?

<u>Example</u> $\int \cos^3 2x \ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

C. When both m and n are odd, what should you do?

<u>Example</u> $\int \sin^4 x \cos^2 x \ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Integrals with Powers of Sines and Cosines

a. $\int \sin^4 3x \cos 3x \ \ dx$

b. $\int \cos^3 4x \sin^5 4x \ \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\int 4\, sin^4\, 2\pi x\ \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Trigonometric Integrals with Square Roots or Different Products of Sine and Cosine

II. Integrals with Square Roots

When an integral has a square root with sine or cosine, what identity should we try to leverage?

<u>Example</u> $\int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}\ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> $\int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\, dx$

<u>Example</u> $\int_0^{\frac{\pi}{6}} \sqrt{1 + \sin x}\, dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

III. Integrals with Different Products of Sine and Cosine

For integrals of the form:

- $\int \sin mx \sin nx \, dx$
- $\int \cos mx \cos nx \, dx$
- $\int \sin mx \cos nx \, dx$

What trig identities can be used to help integrate these types of integrals?

<u>Example</u> $\int \sin 7x \cos 3x \;\; dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Integrals and Square Roots with Trig Functions

a. $\int_0^{\pi} \sqrt{1-\cos^2 x}\ dx$

b. $\int_0^{\frac{\pi}{2}} x\sqrt{1-\cos 2x}\, dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Integrals and Products of Sines and Cosines

a. $\int \sin 3x \cos 2x \, dx$

b. $\int \cos^2 2x \sin x \;\; dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Trigonometric Integrals with Tangent, Cotangent, Secant, and Cosecant

For integrals with $\tan x$ and $\sec x$, what identities should be used to help with integration?

For integrals with $\cot x$ and $\csc x$, what identities should be used to help with integration?

<u>Examples</u> Integrate the following.

a. $\int \tan^3 x \ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

b. $\int \sec^4 x \ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

c. $\int \cot^2 x \csc^4 x \ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Integrals and Powers of Tangent and Secant

a. $\int \sec^2 x \tan^2 x \ \ dx$

b. $\int \sec^3 x \tan x \ \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\int \csc^4 x \ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Trigonometric Substitution

Trig substitution is a special way to use substitution for integrals of a special form.

Reference Guide

If the integral involves…	Then substitute:	And use:

<u>Example</u> Write out instructions for how to approach these types of problems: $\int \frac{dx}{\sqrt{9+x^2}}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

What is another way to integrate $\int \frac{dx}{\sqrt{9+x^2}}$?

$\sinh^{-1}\frac{x}{a}$ is equivalent to:

<u>Example</u> Evaluate $\int \frac{x^2}{\sqrt{25-x^2}}\,dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> $\int_0^2 \frac{dx}{50+2x^2}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Use substitution before using trig substitution: $\int_{\frac{1}{12}}^{\frac{1}{4}} \frac{3}{\sqrt{x}+4x\sqrt{x}}\ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Basic Trig Substitution Examples

a. $\int \frac{dx}{\sqrt{25+x^2}}$

b. $\int \sqrt{16-x^2}dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: More Complicated Trig Substitution Examples

a. $\int \sqrt{\frac{x+1}{1-x}}\, dx$

b. $\int \frac{dx}{(x^2-4)^{\frac{3}{2}}}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\int \frac{6}{(4x^2+1)^2}\ dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Integrals That Use Both Substitution and Trig Sub

$\int \frac{e^x}{1+e^{2x}}\, dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Review: Completing the Square for Advanced Integration Techinques

First, let's review a few examples of completing the square in general. Complete the square on the following:

a. $5x^2 + 10x - 20$

b. $4 - 2x - x^2$

c. $x^2 + 3x - 7$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Now that we have thoroughly reviewed this skill, let's use it for 2 different integrals.

Integrate the following.

a. $\int \frac{dx}{\sqrt{2x+x^2}}$

b. $\int \sqrt{5-4x-x^2}\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Integration by Partial Fractions

What are the keys to writing an expression $\frac{f(x)}{g(x)}$ with partial fractions?

I. Nonrepeated Linear Factors

$\int \frac{3x+11}{x^2-x-6}\, dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

II. Repeated Linear Factors

$\int \frac{3x-2}{x^2-2x+1}\,dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

III. Irreducible Quadratic Factors

$\int \frac{x}{x^4-x^2-2}\,dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

IV. Improper Fractions

$\int \frac{x^3+1}{x^3-x}\, dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Summary of How to Solve Partial Fractions

Factor in Denominator	Form for Partial Fractions
$ax + b$ (unique factor)	
$(ax + b)^k$	
$ax^2 + bx + c$	
$(ax^2 + bx + c)^k$	

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Integration with Partial Fractions Where the Denominator Has Linear Factors Only

a. $\int \frac{dx}{x^3+x^2-2x}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\int \frac{2x+1}{x^2-7x+12}\, dx$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Integration with Partial Fractions Where the Denominator Has Quadratic Factors

a. $\int \frac{x^2-x+2}{x^3-1}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $\int \frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Integration with Partial Fractions With an Improper Fractions

$$\int \frac{16x^3}{4x^2 - 4x + 1} \, dx$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Some Hard Integrals that Involve Partial Fractions and Creativity

a. $\int \frac{\cos x}{\sin^2 x - \sin x - 6} dx$

b. $\int \frac{1}{x^{\frac{3}{2}} - x^{\frac{1}{2}}} \, dx$

Copyright 2025, Jennifer Hill, DivideandConquerMath.com

c. $\int \frac{1}{\cos x + \sin 2x}\, dx$

Copyright 2025, Jennifer Hill, DivideandConquerMath.com

Lesson: L'Hopitals Rule

Does $\frac{0}{0} = 1$?

List all of the indeterminate forms:

Write down L'Hopital's Rule

<u>Examples</u> Determine the following limits.

a. $\lim_{x \to 0} \frac{5x + \sin x}{4x}$

b. $\lim_{x \to 0} \frac{\sqrt{4+x} - 2}{x}$

c. $\lim_{x \to 0} \frac{x - \sin x}{2x^3}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

What must be true to use L'Hopitals Rule?

<u>Example</u> Evaluate

$$\lim_{x \to 0} \frac{(x+\cos x)}{x+x^3}$$

You can also combine L'Hopital's Rule with other limit knowledge

$$\lim_{x \to 0} \frac{3 \sin x}{x^2}$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: L'Hopital's Rule for $\frac{\infty}{\infty}, \infty \times 0, \infty - \infty$ Forms

L'Hopitals Rule can also be applied when we find these types of indeterminate forms: $\frac{\infty}{\infty}, \infty \times 0, \infty - \infty$

a. $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}$

b. $\lim_{x \to \infty} \frac{e^x}{x^3}$

c. $\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

a. $\lim_{x \to 0^+} \sqrt{x} \ln x$

b. $\lim_{x \to \infty} (\ln 2x - \ln(x+2))$

c. $\lim_{x \to 0^+} \left(\frac{2x+1}{x} - \frac{1}{\sin x} \right)$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: L'Hopital's Rule for Indeterminate Powers

L'Hopitals Rule can also be applied to indeterminate powers of these forms:

In this case, L'Hopital's Rule works a little differently. We use logarithms to help evaluate. Write down how this works below

<u>Examples</u> Evaluate.

a. $\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}}$

b. $\lim_{x \to \infty} x^{\frac{1}{x}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples

a. $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$

b. $\lim_{x \to \infty} \left(\frac{x^2+2}{x+1}\right)^{\frac{1}{x}}$

c. $\lim_{x \to e^+} (\ln x)^{\frac{1}{x-e}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: What Happens when L'Hopitals Rule Doesn't Work?

In some cases, L'Hopitals Rule may not work and you will have to think of other ways to evaluate the limit. Let's look at a few cases.

a. $\lim_{x\to\infty} \frac{\sqrt{4x+1}}{\sqrt{x+3}}$

b. $\lim_{x\to\infty} \frac{2^x-5^x}{4^x+3^x}$

c. $\lim_{x\to 0^+} \frac{x}{e^{-\frac{1}{x}}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples: L'Hopitals Rule

Evaluate the following limits using L'Hopitals Rule, if possible.

a. $\lim_{x \to 0} \frac{\sin x^2}{x}$

b. $\lim_{x \to \frac{\pi}{3}} \frac{3x+\pi}{\sin\left(x+\frac{\pi}{3}\right)}$

c. $\lim_{x \to 0} \frac{\left(\left(\frac{1}{3}\right)^x - 1\right)}{x}$

d. $\lim_{x \to \infty} x^2 e^{-x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

e. $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right)$

f. $\lim_{x \to \infty} (\ln x)^{\frac{1}{x}}$

g. $\lim_{x \to 0^+} x^x$

h. $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Summary: L'Hopital's Rule

State L'Hopital's Rule

What are all the indeterminate forms that apply to L'Hopital's Rule?

For which type of indeterminate forms can you just take the derivative of the top and bottom?

<u>Examples</u> Find the following limits.

a. $\lim_{x \to 0} \frac{4x^2}{2\cos x - 2}$

b. $\lim_{x \to \infty} \frac{\ln x}{3\sqrt[3]{x}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

For the forms $0 * \infty$ or $\infty - \infty$, what must you try?

<u>Examples</u> Find the following limits.

a. $\lim_{x \to \infty} 3x\, e^{-x}$

b. $\lim_{x \to 0^+} (\ln 5x - \ln(\sin x))$

What must you do with indeterminate powers?

<u>Example</u> Find the limit $\lim_{x \to 0^+} x^{\frac{-1}{\ln x}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

What happens if L'hopitals rule continues to cycle?

<u>Example</u> Find the limit $\lim_{x \to 0^+} \frac{\cot x}{\csc x}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Improper Integrals Type I – Integrals Containing Infinity

In this lesson we will consider type 1 improper integrals. What are some examples of these types of integrals?

We will use limits to help us integrate these types of integrals. Explain the different ways to solve integrals of this type:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

If the integral has an answer, what do we say about that integral?

What if the integral does not have a finite answer?

Find the area under the curve $y = \frac{1}{x^2+1}$ from $x = 0$ to $x = \infty$ if possible.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Improper Integrals Type II – Integrals Involving Vertical Asymptote

Explain what type of integrals are type II integrals:

How do you evaluate these types of integrals?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Show how to integrate $\int_0^3 \frac{dx}{\sqrt{3-x}}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example $\int_{-8}^{1} \frac{1}{x^{\frac{1}{3}}} \ dx$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Comparison Tests for Integrals

Why might we want to use a comparison test for an integral?

Explain the Direct Comparison Test

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples Use the direct comparison to evaluate the following integrals.

a. $\int_1^\infty e^{-x^2}\, dx$

b. $\int_1^\infty \frac{1}{\sqrt{x^2-0.01}}\, dx$

c. $\int_0^\pi \frac{dx}{\sqrt{x}+\sin x}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Explain the Limit Comparison Test

Example Use the limit comparison test to evaluate the integrals.

a. $\int_1^\infty \frac{1}{1+x^2}\, dx$

b. $\int_0^1 \frac{dx}{x-\sin x}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Summary: How to Study for the Integration Exam

Obviously you are told over and over again that you need to practice for an integration exam. If you go through each section of your notes / book / homework, what techniques are used?

You should be able to do a few problems of each of these types. In this video, we will take a look at one type of common integral form that can sprawl into many different types of problems.

Analyze the following, determine appropriate techniques for integration, and then integrate.

a. $\int \sqrt{4-x^2}\, dx$

b. $\int_0^1 \frac{5x+5}{\sqrt{x^2+2x}} dx$

c. $\int \frac{dx}{x^2+4x}$

d. $\int \frac{dx}{\sqrt{x^2-2x}}$

e. $\int \frac{x^3}{x^2+2x+1}\, dx$

f. $\int \frac{1}{\sqrt{x^2-4x+5}}\, dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

a. $\int \sqrt{4 - x^2}\, dx$

b. $\int_0^1 \frac{5x+5}{\sqrt{x^2+2x}} dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

c. $\int \frac{dx}{x^2+4x}$

d. $\int \frac{dx}{\sqrt{x^2-2x}}$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

e. $\int \frac{x^3}{x^2+2x+1}\ dx$

f. $\int \frac{1}{\sqrt{x^2-4x+5}}\ dx$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Introduction to Sequences

What is a sequence?

What is the index of a sequence?

How do we describe the length of a sequence?

Write does a few examples of how sequences can be presented:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Write down the first 5 terms of each sequence.

a. $a_n = \frac{1}{n!}$

b. $a_n = 2 + (-1)^n$

What is a recursive sequence?

Find the first six terms of the recursive sequence if:

$a_1 = 2, a_n = \frac{na_{n-1}}{n-1}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Introduction to Convergence of Sequences

When does a sequence converge?

What a sequence does not converge, what do we say about it?

What notation do we use to denote a sequence has converged?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

How do we define sequence divergence?

<u>Example</u> Use the definition of sequence convergence to prove that $\lim_{n\to\infty} \frac{1}{n} = 0$

<u>Example</u> Use the definition of sequence convergence to prove that $\lim_{n\to\infty} c = c$ for any constant c.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Rules for Limits of Sequences

Let a_n and b_n be sequences. Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then the following rules hold:

<u>Example</u> Determine if the following converge or diverge. If they converge, then what is the limit?

a. $a_n = \frac{1-3n}{1+3n}$

b. $a_n = \sqrt{n}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Determine if the following converge or diverge. If they converge, then what is the limit?

a. $a_n = \frac{3+(-1)^n}{n}$

b. $a_n = \sqrt{\frac{2n}{n+1}}$

Explain the Sandwich Theorem for Sequences

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Determine if and where the sequence converge to: $a_n = \frac{\sin n}{n}$

State the Continuous Function Theorem for Sequences

Example Determine if and where the sequence converge to: $a_n = \sqrt{\frac{n+1}{n}}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Explain the theorem that connects functions to sequences:

What is the cool result of this theorem?

<u>Example</u> Explain why $\lim_{n\to\infty} \frac{\ln n}{n^2} = 0$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Determine if and where the sequence converge to: $a_n = \left(\frac{4n+1}{4n-1}\right)^n$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Commonly Occurring Limits for Sequences

State the 6 commonly occurring limits for sequences:

<u>Example</u> Determine whether the following sequences converge and to what:

a. $a_n = \frac{(-5)^n}{n!}$

b. $a_n = \sqrt[n]{5^{2n+1}}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Determine whether the following sequences converge and to what:

a. $a_n = \frac{\ln n}{n^{\frac{1}{n}}}$

b. $a_n = \left(1 - \frac{1}{n}\right)^n$

c. $a_n = \frac{7^n}{n!}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Boundedness and Monotonicity for Sequences

Explain the concepts of being bounded from above versus bounded from below:

What does it mean to be a bounded versus an unbounded sequence?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Describe a sequence with a lower bound but no upper bound:

Example Describe a sequence with an upper bound but no lower bound:

Example Explain a bounded sequence that does not converge:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Explain the difference between nonincreasing vs nondecreasing:

What does it mean for a sequence to be monotonic?

Example Write an example of each type of sequence:

a. A nondecreasing sequence

b. A nonincreasing sequence:

c. A sequence that is not monotonic:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

State the Monotonic Sequence Theorem:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Determining the Convergence of Sequences

Determine if the following sequences converge or diverge.

a. $a_n = \frac{3^n}{n^3}$

b. $a_n = \frac{(2n+2)!}{(2n-1)!}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $a_n = (\ln n - \ln(n-1))$

d. $a_n = \tan^{-1} n$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Determining the Convergence of Recursively Defined Sequences

Determine if the following sequences converge or diverge.

a. $a_1 = -1, a_{n+1} = \frac{a_n+6}{a_n+2}$

b. $a_1 = 3, a_{n+1} = 8 - \sqrt{a_n}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Infinite Series

Write down an example of a sequence:

Using that sequence, show how we can turn a sequence into a series:

What is the difference between a finite series and an infinite series?

Show how we can denote summing up only n terms:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

In some cases, a pattern can emerge with infinite series. Let's investigate the sum of $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}+\cdots$

Partial Sum	Value of Sum	Another way to think of this
$S_1 = 1$	1	
$S_2 = 1+\frac{1}{2}$	$\frac{3}{2}$	
$S_3 = 1+\frac{1}{2}+\frac{1}{4}$	$\frac{7}{4}$	
$S_n = 1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}$	$\frac{2^n-1}{2^{n-1}}$	

How do we denote a sequence of partial sums?

How do we denote an infinite series?

What happens if the partial sums of an infinite series converge to a limit L?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Use the sum $a_n = 1 + \frac{1}{n}$ show how we can visualize the sums of a series with rectangles:

Use the sum $a_n = \frac{1}{n^2}$ show how we can visualize the sums of a series with rectangles:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Geometric Series

What is the general form of a geometric series?

Show an example of a geometric series:

What's are two ways to write a geometric series?

What is the ratio of a geometric series?

True or false: The ratio of a geometric series can only be positive

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Explain how we can evaluate the value of a geometric series.

Explain when a geometric series converges versus diverges:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Write out the first 4 terms of the series, then determine if the series converges or diverges.

$$\sum_{n=0}^{\infty}(\frac{(-1)^n}{5^n}$$

Example Determine if the geometric series converges or diverges. If it converges, find its sum.

$$\frac{9}{4}-\frac{27}{8}+\frac{81}{16}-\frac{243}{32}+\cdots$$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

If $\Sigma a_n = A$ and $\Sigma b_n = B$ are convergent series, then what other properties hold?

<u>Example</u> Write out the first 4 terms of the series, then determine if the series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{5}{2^n}+\frac{1}{3^n}\right)$$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Express the number as the ratio of two integers: $0.\overline{231} = 0.231\ 231\ 231\ 231\ \ldots$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Nth Term Test for a Divergent Series

If $\Sigma_{n=1}^{\infty} a_n$ converges, what can be said about a_n?

What is the nth-Term Test for Divergence

<u>Examples</u> Why do the following series diverge?

a. $\Sigma_{n=1}^{\infty} n$

b. $\Sigma_{n=1}^{\infty} \left(\frac{2n+3}{n}\right)$

c. $\Sigma_{n=1}^{\infty} (-1)^{n+1}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Combing Series, Reindexing Terms in a Series, and Adding or Deleting Terms in a Series

How do you find the value of the series $\Sigma_{n=5}^{\infty}\left(\frac{1}{4^n}\right)$? Note the index.

How can we reindex the series $\Sigma_{n=1}^{\infty}\left(\frac{1}{7^{n-1}}\right)$?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Working with Geometric Series

Determine if the following series converge or diverge.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$

b. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Repeating Decimals and Geometric Series

Express each number as the ratio of two integers.

a. $0.\overline{27} = 0.27\ 27\ 27\ 27\ \ldots.$

b. $1.35\overline{127} = 1.35\ 127\ 127\ 127\ \ldots$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: The nth Term Test for Divergence

Use the nth-term test for divergence to show that the series is divergent OR that the test is inconclusive.

a. $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$

b. $\sum_{n=0}^{\infty} \frac{n}{n^2+7}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Partial Sums with Telescoping Series

Find a formula for the nth partial sum, then use it to determine if the series converges or diverges.

a. $\sum_{n=1}^{\infty}(\tan n - \tan(n-1))$

b. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: The Integral Test for Infinite Series

When does a series of nonnegative terms converge?

Explain how this idea applies to the harmonic series $\Sigma_{n=1}^{\infty} \frac{1}{n}$

State the Integral Test

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Does the following series converge: $\Sigma_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$

For what values of p does the p-series converge?

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Determine if the following series converge by the Integral Test

a. $\Sigma_{n=1}^{\infty}\left(\frac{n}{n^2+5}\right)$

b. $\Sigma_{n=2}^{\infty}\left(\frac{1}{n(\ln n)^2}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Using the Integral Test for Series

Determine which series converge or diverge using the integral test.

a. $\sum_{n=1}^{\infty} e^{-2n}$

b. $\sum_{n=1}^{\infty} \frac{n^2}{e^{\frac{1}{3}n}}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Comparison Tests for Series

State the direct comparison test:

<u>Example</u> Show how to use the direct comparison test to determine if the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

State the Limit Comparison Test:

<u>Example</u> Use the limit comparison test to determine if the series converges or diverges:

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Example Use the limit comparison test to determine if the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Comparison Tests for Series

Use any methods to determine if the following converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

b. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$

d. $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{2.3}}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: The Alternating Series Test

State the Alternating Series Test

<u>Example</u> Does the alternating harmonic series converge? $\Sigma_{n=1}^{\infty}\left((-1)^{n+1}\left(\frac{1}{n}\right)\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Determine if the alternating series converge or diverge. Not all series satisfy the conditions of the Alternating Series Test.

a. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}1}{n5^n}\right)$

b. $\Sigma_{n=1}^{\infty}((-1)^{n+1}\left(\frac{n}{5}\right)^n$

c. $\Sigma_{n=1}^{\infty}\left((-1)^n(\frac{n}{n^2+1})\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Conditional Convergence

Explain what conditional convergence is:

Explain how conditional convergence applies to the alternating p-series: $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{n^p}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

What is the Rearrangement Theorem for Absolutely Convergent Series?

Example What happens if we multiply the alternating harmonic series by 2?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Summary of Tests to Determine Convergence or Divergence

State the 6 different tests we've discussed so far for infinite series.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Using each test at most once, determine the convergence of the following series.

a. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^n n}{n+5}\right)$

b. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}(0.5)^n}{n}\right)$

c. $\Sigma_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n}}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

d. $\Sigma_{n=1}^{\infty}(-6)^{-n}$

e. $\Sigma_{n=1}^{\infty}\left(\frac{(-105)^{n}}{n!}\right)$

f. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}(\sqrt{n}+1)}{n+1}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Alternating Series Test

Determine if the following satisfy the conditions of the alternating series test, and then determine if the series converges.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+5}{n^2+4}$

b. $\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{n}\right)$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Conditional Convergence

Determine if the series converges absolutely, if it converges, or if it diverges.

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+2}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+3}$

c. $\sum_{n=1}^{\infty}(-1)^{n+1}\sqrt[n]{13}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

d. $\sum_{n=1}^{\infty}(-1)^n \frac{(-2)^{n+1}}{n+5^n}$

e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \tan^{-1} n}{n^2+1}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Absolute Convergence for Infinite Series

What does it mean for a series to converge absolutely?

What is the Absolute Convergence Test?

<u>Examples</u> Determine if the following series converge or diverge absolutely.

a. $\Sigma_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^3}\right)$

b. $\Sigma_{n=1}^{\infty}\left(\frac{\cos n}{n^2}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: The Ratio Test

State the Ratio Test

<u>Examples</u> Use the Ratio Test to determine the convergence of the series.

a. $\Sigma_{n=1}^{\infty}\left(\frac{(3n)!}{n!n!}\right)$

b. $\Sigma_{n=1}^{\infty}\left(\frac{n!n!}{(3n)!}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: The Root Test

What is the Root Test?

<u>Example</u> Use the Root Test to determine if the series converge absolutely or diverge.

a. $\Sigma_{n=1}^{\infty}\left(\frac{5n-3}{4n+7}\right)^{n}$

b. $\Sigma_{n=1}^{\infty}\left(\sin^{n}\left(\frac{1}{\sqrt[3]{n}}\right)\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Review: Factorials within a Series

Back to basics: use $f(x) = x^2 - 2x + 1$ to compute the following

a. $f(3)$

b. $f(n+1)$

Now let's compute $\frac{u_{n+1}}{u_n}$ for the following:

a. $\frac{3^n}{(n+1)!}$

b. $\frac{(2n)!}{(3n+1)!}$

Copyright 2025, Jennifer Hill, DivideandConquerMath.com

Compute the ratio test for the following:

$$\sum_{n=1}^{\infty}\left(\frac{(n!)^2}{(2n)!}\right)$$

Copyright 2025, Jennifer Hill, DivideandConquerMath.com

Examples: The Ratio Test

Determine if the series converges absolutely or if it diverges.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{n+5}{5^n}$

b. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+4)^2}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: The Root Test

Determine if the series converges absolutely or if it diverges.

a. $\sum_{n=1}^{\infty} \frac{5^n}{(4n)^n}$

b. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Series with Recursively Defined Terms

Determine if the series $\sum_{n=1}^{\infty} a_n$ converge or diverge.

a. $a_1 = 2$ $a_{n+1} = \frac{1+\cos n}{n}$ a_n

b. $a_1 = 5, a_{n+1} = \frac{\sqrt[n]{n}}{2}$ a_n

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: The Binomial Series for Powers and Roots

Determine the Taylor series generated by: $f(x) = (1 + x)^n$

What is this series called?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

When does the binomial series converge?

What is another way we can define the binomial series?

<u>Example</u> Determine the first 4 terms of the binomial series for the function $f(x) = \left(1 + \frac{x}{2}\right)^{-2}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Evaluating Nonelementary Integrals

We can use a familiar Taylor series to find the sum of given power series. For instance, how could we express:

$\int \cos x^2 \ dx$ as a power series?

How can we estimate $\int_0^1 \cos x^2 \ dx$ with an error less than 0.001.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find a polynomial that will approximate $F(x)$ in the given interval with an error of magnitude less than 10^{-3}:

$F(x) = \int_0^x t^2 e^{-t^2} dt, [0, 1]$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Evaluating Indeterminate Forms

We can evaluate indeterminate forms by expressing the functions involved as Taylor series.

<u>Example</u> Evaluate the limits:

a. $\lim_{x \to 0} \frac{\sin x - x + (\frac{x^3}{6})}{x^5}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

b. $\lim_{x \to 2} \left(\frac{x^2-4}{\ln(x-1)} \right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Euler's Identity

Review the powers of i:

Explain what $e^{i\theta}$ is expanded:

What result does this give us?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

What is Euler's identity?

<u>Example</u> Write the power in the form $a + bi$: $e^{\frac{i\pi}{4}}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Approximations and Nonelementary Integrals

a. Use series to estimate the integrals' value with an error of magnitude less than 10^{-5}:

$$\int_0^{0.4} \frac{e^{-x} - 1}{x} dx$$

b. Use series to approximate the values of the integral with an error of magnitude less than 10^{-8}:

$$\int_0^{0.1} e^{-x^2} dx$$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Indeterminate Forms

Use a series to evaluate the limits.

a. $\lim_{x \to 0} \frac{(e^x - e^{-x})}{x}$

b. $\lim_{x \to \infty} (x+1) \cos \frac{1}{x+1}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Power Series

What is a power series?

What is one power series that we are already familiar with?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Describe the different terms in the power series:

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

Describe some polynomial approximations of this power series, and then try to find some way to explain what these mean in the greater idea of a power series. You have some extra space here to draw our any graphs if you'd like.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> For what values of x do the power series converge?

a. $\Sigma_{n=1}^{\infty}(-1)^{n-1}\left(\frac{x^n}{n}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

b. $\Sigma_{n=0}^{\infty} n!\, x^{2n}$

c. $\Sigma_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

State the Convergence Theorem for Power Series

What is a corollary to this theorem?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Explain the steps for how to test a power series for convergence:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> For each of the following, i) find the series interval of convergence, ii) determine for what values the series converges absolutely, and iii) determine for what values the series converges conditionally.

a. $\Sigma_{n=0}^{\infty}(x+8)^n$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

b. $\Sigma_{n=1}^{\infty}\left(\frac{(x-3)^n}{n^4 4^n}\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Operations on Power Series

Explain series multiplication for power series

Give an example of how this works:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

State the next theorem in this video

Explain term by term differentiation for power series:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find $f'(x)$ and $f''(x)$ if $f(x) = \frac{1}{x-1} = \Sigma_{n=0}^{\infty} x^n, -1 < x < 1$

Explain term by term integration for power series:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Show how to differentiate and integrate term by term with the function: $\Sigma_{n=0}^{\infty}\left((-1)^n\left(\frac{x^{2n+1}}{3n+1}\right)\right)$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Intervals of Convergence

Find the series' radius and interval of convergence.

a. $\sum_{n=1}^{\infty} \frac{x^n}{n*5^n*\sqrt{n}}$

b. $\sum_{n=1}^{\infty} x^n(\ln n)$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\sum_{n=2}^{\infty} \frac{x^n}{n\,(\ln n)^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{2*4*6*\ldots*2n}\ x^n$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Convergence of Taylor Series

State **Taylor's Theorem**

State **Taylor's Formula**

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

What is another way to think of Taylor's formula?

What happens if $R_n(x) \to 0$ as $n \to \infty$?

<u>Example</u> Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for all values of x.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

State **The Remainder Estimation Theorem**

Example Show that the Taylor Series for $\sin x$ at $x = 0$ converges for every value of x.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

In the next examples, we will see how to use addition, subtraction, and multiplication with Taylor series. All of these operations are valid on the intersection of their intervals of convergence.

Example Use known series to find the first few terms of the Taylor series for the given function.

a. $\frac{1}{2}(3x + x\cos x)$

b. $x^2 \sin x$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ and obtain an error who magnitude is no greater than $3 - 10^{-4}$?

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Some Common Taylor Series

List the common Taylor series that you should have memorized:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Taylor Series with Substitution

Use substitution to find the Taylor series at $x = 0$ of the functions below.

a. e^{-3x}

b. $\cos 7x^2$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Taylor Series with Substitution

Use power series operations to find the Taylor series at $x = 0$ of the functions below.

a. $\sin x - x + \frac{x^3}{3!}$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $\sin x * \cos x$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $\sin^2 x$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Introduction to Taylor and Maclaurin Series

Recall our power series:

$\Sigma_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$

Consider term-by-term differentiation for this series:

It we plug in x=a, what is the result of our derivatives?

From here, what result do we get?

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Define a Taylor series generated by f(x) at x=a:

Explain what a Maclaurin series is:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $x = 2$.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Define a Taylor polynomial of order n:

<u>Example</u> Find the Taylor polynomials of orders 0, 1, 2, and 3 by: $f(x) = \frac{1}{x+1}$ at $x = 0$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the Taylor series and Taylor polynomials generated by $f(x) = e^x$ at $x = 0$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Taylor Polynomials

Find the Taylor polynomials of orders 0, 1, 2, and 3 for $f(x) = \ln(1 + x)$ at $a = 0$.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Taylor Series at x=0 (Maclaurin Series)

Find the Maclaurin series for the following:

a. xe^x

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $\sinh x$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

c. $x^4 - x^3 - 2x + 3$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Finding Taylor Series

Find the Taylor series generated by each function at $x = a$

a. $f(x) = x^4 + 2x^2 + 1, a = -2$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

b. $f(x) = \frac{1}{x^2}, a = 1$

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Examples: Maclaurin Series and Absolute Convergence

Find the first three nonzero terms of the Maclaurin series for the function $f(x) = (\sin x)(\ln(1 + x))$ and the values of x for which the series converges absolutely.

Copyright 2023, Jennifer Hill, DivideandConquerMath.com

Lesson: Parametrization of Plane Curves

Write out the vocabulary for parametrization:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Sketch the curve defined by the parametric equations.

a. $x = \cos\frac{\pi t}{4}, y = t, 0 \leq t \leq 8$

b. $x = t^2, y = t - 2, -\infty < t < \infty$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> The position $P(x, y)$ of a particle moving in the xy-plane is given by $x = 2t - 5, y = 2t, 0 \leq t \leq 1$. Identify the Cartesian equation for its path and graph the equation. Indicate the portion of the graph traced by the particle and the direction of motion.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> The position $P(x, y)$ of a particle moving in the xy-plane is given by $x = \frac{t}{t-1}, y = \frac{t-2}{t+1}, -1 < t < 1$. Identify the Cartesian equation for its path and graph the equation. Indicate the portion of the graph traced by the particle and the direction of motion.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

How can any function be parametrized? Given an example.

<u>Example</u> Find a parametrization for the line through the point (a, b) with slope m.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Tangents and Areas

When is a parametrized curve differentiable?

State the parametric formula for $\frac{dy}{dx}$:

State the parametric formula for $\frac{d^2y}{dx^2}$:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the tangent to the curve at the given value of t:

$x = 2\sin 2\pi t,\ y = \cos 2\pi t,\ t = -\frac{1}{6}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the value of $\frac{d^2y}{dx^2}$ at the point t:

$x = 2\sin 2\pi t,\ y = \cos 2\pi t,\ t = -\frac{1}{6}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Assuming that the equations define x and y implicitly as differentiable functions, find the slope of the curve at the given value of t:

$x = t^3 + t, y + 2t^3 = 2x + t^2, t = 1$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the area enclosed by the y-axis and the curve:

$x = t - t^2, y = 1 + e^{-t}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Summary: Derivative Rules

For a derivatives exam, you will obviously need to know many formulas. To begin, we will summarize all of these formulas.

The formula for a derivative according to the definition:

The alternative formula for a derivative:

Compare using the formulas by taking the derivative of this function: $f(x) = x^2 - 3x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Next we will review all of the derivative rules.

Basic Rules

Notes: c stands for a constant, n is any number, g(x) and h(x) are functions

For each function form, state what the derivative would be.

Derivative of a constant function: $f(x) = c; \frac{d}{dx}(c) =$

Power rule: $f(x) = x^n; \frac{d}{dx}(x^n) =$

Derivative constant multiple rule: $f(x) = cg(x); \frac{d}{dx}\big(cg(x)\big) =$

Derivative sum rule: $f(x) = g(x) + h(x); \frac{d}{dx}\big(g(x) + h(x)\big) =$

Derivative of the natural exponential function: $f(x) = e^x, \frac{d}{dx}(e^x) =$

Derivative product rule: $f(x) = g(x) * h(x), \frac{d}{dx}\big(g(x) * h(x)\big) =$

Derivative quotient rule $f(x) = \frac{g(x)}{h(x)}\ (h(x) \neq 0); \frac{d}{dx}\left(\frac{g(x)}{h(x)}\right) =$

Examples Find the derivatives of the functions.

a. $f(x) = \frac{e^x}{\sqrt{x} - 4x^2}$

b. $y = 5x^{\pi}e^x$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Trig Rules and the Chain Rule

Derivative of sine: $f(x) = \sin x$; $\frac{d}{dx}(\sin x) =$

Derivative of cosine: $f(x) = \cos x$, $\frac{d}{dx}(\cos x) =$

Derivative of tangent: $f(x) = \tan x$, $\frac{d}{dx}(\tan x) =$

Derivative of cotangent: $f(x) = \cot x$, $\frac{d}{dx}(\cot x) =$

Derivative of secant: $f(x) = \sec x$, $\frac{d}{dx}(\sec x) =$

Derivative of cosecant: $f(x) = \csc x$, $\frac{d}{dx}(\csc x) =$

Chain rule: $f(x) = g(h(x))$; $\frac{d}{dx}\left(g(h(x))\right) =$

Examples Find the derivatives of the functions.

a. $f(x) = \sin x\, e^x$

b. $g(x) = \frac{e^x}{\tan x}$

c. $y = \sin(e^{3x+1})$

d. $h(x) = \sec(sin\, x + (3x^2 + 5)^{18})$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Derivatives of Logarithms and Exponentials

Derivative of natural logarithm: $f(x) = \ln x; \frac{d}{dx}(\ln x) =$

Derivative of general exponential function: $f(x) = a^x, a > 0, a \neq 1; \frac{d}{dx}(a^x) =$

Derivative of general logarithm: $f(x) = \log_a x,\ a > 0, a \neq 1; \frac{d}{dx}(\log_a x) =$

Derivatives of Inverse Trig Functions

Derivative of inverse sine: $f(x) = \sin^{-1} x\,; \frac{d}{dx}(\sin^{-1} x) =$

Derivative of inverse cosine: $f(x) = \cos^{-1} x; \frac{d}{dx}(\cos^{-1} x) =$

Derivative of inverse tangent: $f(x) = \tan^{-1} x; \frac{d}{dx}(\tan^{-1} x) =$

Derivative of inverse cotangent: $f(x) = \cot^{-1} x; \frac{d}{dx}(\cot^{-1} x) =$

Derivative of inverse secant: $f(x) = \sec^{-1} x; \frac{d}{dx}(\sec^{-1} x) =$

Derivative of inverse cosecant: $f(x) = \csc^{-1} x; \frac{d}{dx}(\csc^{-1} x) =$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Find the derivatives of the functions.

a. $f(x) = \log_3 x^5$

b. $y = \ln(sin(4^x))$

c. $g(x) = \sin^{-1}(\sqrt{x})$

d. $y = \csc^{-1}(\sin x)$

Suggestion: for each rule, make 2 of your own exercises and take the derivatives. Use a derivative calculator on Google to check your work!

Implicit Differentiation

Examples Find the derivatives of the functions.

a. $x^2 - xy + y^2 = 4$

b. $x^2 + y \sin x = x \cos 4y$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Logarithmic Differentiation

Why would you want to use this?

Examples Let's review a few examples.

a. $y = \frac{x^2\sqrt{(4x^2-2)^5}}{\sqrt[3]{x^3-1}}$

b. $y^{lnx} = x^y$

Logarithms and Differentiation

Sometimes we will want to use properties of logarithms to help use simplify functions so that we can take the derivatives of them. This is not logarithmic differentiation, this is just leveraging properties of logarithms

Examples Take the derivatives of the following functions.

a. $f(x) = \ln \frac{x\sqrt{x+1}}{x-4}$

b. $y = \log_4 x - \log_4 x^3$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

What are all of the different types of notation for the following?

- First derivatives

- Second derivatives

- Third derivatives

- Fourth derivatives and higher derivatives

<u>Examples</u> Find the fourth derivative of the following function:

$$f(x) = \sin 3x + x^3 - e^{-2x} + 7$$

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Examples Find the second derivative of the following function:

$$x^3 - y^4 = 5$$

What are 4 different interpretations for the derivative at a point?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Find the equation of the line tangent to $f(x) = x^2 - 3x + 1$ at the point $(1, -1)$.

Example A function has the derivative $f'(x) = 3x^2 - 2x$.

a. Do you have enough information to determine the equation of the tangent line at the point $x = 2$? If so, find the equation of the line.

b. What is the slope of the tangent line at $x = -3$?

c. What is a potential form of the original function?

d. If you know that $(-1, -4)$ is a point on $f(x)$, can you determine the original function? If so, find it.

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Example Find the equation of the line that is tangent to $f(x) = 3x^2 - 4x$ and parallel to $y = 14x - 2$

Example At time t, the position of a body moving along the s-axis is $s = t^3 + 6t^2 + 9t$ ft.

a. What is the velocity of the object at a $t = 3$ seconds?

b. When will the object reach a velocity of zero?

d. Find the body's acceleration at a time t.

e. What is the speed of the object at $t = 3$ seconds?

Copyright 2020, Jennifer Hill, DivideandConquerMath.com

Lesson: Length of Parametrically Defined Curves

How can we find the length of a curve C with parametric equations?

<u>Example</u> Find the length of the curve defined by: $x = \frac{t^2}{2}, y = \frac{(2t+1)^{\frac{3}{2}}}{3}, 0 \leq t \leq 4$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

How does the formula apply for a curve defined by $y = f(x)$? Give an example.

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Explain the arc length differential:

<u>Example</u> Find the coordinates of the centroid of the curve:

$$x = \cos t + t \sin t \, , y = \sin t - t \cos t \, , 0 \leq t \leq \frac{\pi}{2}$$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Areas of Surfaces of Revolution from Parametric Curves

State the area of surface of revolution for parametrized curves:

<u>Example</u> Find the areas of the surfaces generated by revolving the curves:

$$x = \left(\frac{2}{3}\right) t^{\frac{3}{2}}, y = 2\sqrt{t}, 0 \leq t \leq \sqrt{3}$$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Polar Coordinates

How do you define polar coordinates?

<u>Example</u> Find all the polar coordinates of the point $\left(3, \frac{\pi}{4}\right)$

<u>Example</u> Graph the following:

a. $2 \leq r \leq 3$ and $\theta = \frac{\pi}{4}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Graph the following:

b. $-1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{4}$

c. $\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{6}$

d. $r \geq 1$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

How can you relate polar and Cartesian coordinates?

<u>Example</u> Rewrite as an equivalent Cartesian equation:

a. $r\cos\theta + r\sin\theta = 1$

b. $\cos^2\theta = \sin^2\theta$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Rewrite as an equivalent polar equation:

a. $x - y = 5$

b. $xy = 2$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Graphing Polar Coordinate Equations

Tests for Symmetry for Polar Graphs in the Cartesian xy-Plane

<u>Example</u> Identify any symmetries of the curves.

a. $r = 2 + \sin\theta$

b. $r^2 = \sin\theta$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

How to find the slope of the curve $r = f(\theta)$

<u>Example</u> Find the slopes of the curve at the given points.

a. $r = -1 + \sin\theta$ at $\theta = 0, \pi$

b. $r = \cos 2\theta\,, \theta = 0, \pm\frac{\pi}{2}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

How can you graph a polar equation in the Cartesian xy-plane?

<u>Example</u> Graph the following in the Cartesian xy-plane

a. $r = 2 + \sin\theta$

b. $r^2 = \sin\theta$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Areas in Polar Coordinates

Explain how to find areas with polar equations:

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Example Find the areas of the regions:

a. Bounded by the circle $r = 3\sin\theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

d. Inside the circle $r = 4$ and above the line $r = 2\csc\theta$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Lengths of a Polar Curve

How do you find the length of a polar curve?

<u>Example</u> Find the length of the curves:

a. The spiral $r = \frac{e^{\theta}}{\sqrt{2}}, 0 \leq \theta \leq \pi$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

b. The parabolic segment $r = \frac{4}{1+\cos\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$

Copyright 2024, Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Vectors

What is the definition of a vector?

How do we describe and represent a vector with a line segment?

How do we represent vectors?

When are two vectors equal?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

How can we represent a vector created by a line segment between two points?

Define the magnitude of a vector:

What vector(s) have a length of 0?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the component form and length of the vector with initial point $P(1, 2, 3)$ and terminal point $Q(-2, 1, 5)$.

Physics often discuss forces, which intuitively means the push or pull on a object and how that object moves. Forces are just vector quantities. Here is an example: a small cart is being pulled along a smooth horizontal floor with a 10-lb force $\vec{F}$**,** making a $45°$ angle to the floor. How can we describe the effective force moving the cart forward?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Vector Operations

Suppose we have three vectors: $\vec{u} =< u_1, u_2, u_3 >$, $\vec{v} =< v_1, v_2, v_3 >$, and $\vec{w} =< w_1, w_2, w_3 >$. Let a and b be scalars.

Describe vector addition:

Describe scalar multiplication:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Write out the **Properties of Vector Operations**

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Let $\boldsymbol{u} =< 1, 2, 3 >$ and $\boldsymbol{v} =< -4, 3, -2 >$. Find the components of the following:

a. $2\boldsymbol{u} + 4\boldsymbol{v}$

b. $\boldsymbol{v} - \boldsymbol{u}$

c. $|\frac{1}{2}\boldsymbol{v}|$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Unit Vectors

What are the standard unit vectors?

How can any vector be written as a linear combination of the unit vectors?

How can we take any vector and rewrite it as a unit vector?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Example Find a unit vector $\boldsymbol{u}$ in the direction of a vector from $P(0,1,2)$ to $Q(3,-1,1)$.

In physics, speed in relation to a vector just refers to the vector's magnitude.

Example If $\boldsymbol{v} = 2\mathrm{i} - \mathbf{3}\mathrm{j}$ is a velocity vector, express $\boldsymbol{v}$ as a product of its speed times its direction of motion.

Summarize the results of this lesson:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Components and Magnitudes of a Vector

Let $\boldsymbol{u} =< 2, 5 >$ and $\boldsymbol{v} =< 4, -1 >$. Find the component form and magnitude of the following vectors.

a. $4\boldsymbol{v}$

b. $\boldsymbol{u} - \boldsymbol{v}$

c. $\frac{3}{5}\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding the Component Form of a Vector

Find the component form of each vector.

a. The vector $\overrightarrow{PQ}$ when $P = (3, 4)$ and $Q = (7, -1)$

b. The vector $\overrightarrow{OP}$ where O is the origin, P is the midpoint of the line segment AB, and $A = (-3, 1)$ and $B = (-1, 5)$

c. The unit vector obtained by rotating the vector $< 1, 0 > 135°$ counterclockwise about the origin.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding the Magnitude of Vector Forces

Refer to the drawing in the video. A 50-N weight is suspended by two wires. IF the magnitude of vector $\overrightarrow{F_1}$ is 35N, find the angle α and the magnitude of the vector $\overrightarrow{F_2}$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Geometric Representations of Vectors

Copy the vector diagram in the video, and the perform the following operations geometrically.

a. $\boldsymbol{u} + \boldsymbol{v}$

b. $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}$

c. $2\boldsymbol{u} - \boldsymbol{v}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Expressing Vectors as a Product of Length and Direction

Express each vector as a product of its length and direction.

a. $3\boldsymbol{i} - 2\mathbf{j} + 5\mathbf{k}$

b. $\frac{1}{2}\boldsymbol{i} + \frac{1}{4}\boldsymbol{k}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Three-Dimensional Coordinate Systems

Show how we can draw the three-dimensional coordinate system using the right-hand rule.

How can we consider graphing $f(x, y) = x^2 + y^2$?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Using the (x, y, z) coordinate, what do we have if we set …

- $x = 0$?

- $y = 0$?

- $z = 0$?

How do these 3 coordinate plans divide up the coordinate system?

<u>Example</u> What part of the Cartesian coordinates is each referring to?

a. $x \geq 0$

b. $y = 3$

c. $x \leq 0, y \leq 0, z \leq 0$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> What points satisfy $x^2 + z^2 = 9$ and $y = 2$?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: The Standard Form of a Sphere

What is the standard equation for the sphere of radius a and center (x_0, y_0, z_0)?

<u>Example</u> Find the center and radius for the sphere: $4x^2 + 4y^2 + 4z^2 - 8x - 3y = 8$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: The Distance Between Two Points in 3 Dimensions

State the distance formula.

<u>Example</u> Find the distance between $P_1(4, 5, 6)$ and $P_2(5, 7, 8)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Describing the Regions Created by Different Equations

Describe what each set of points look like that fit the following conditions:

a. $x = 4, y = 6$

b. $x^2 + y^2 = 9, z = x$

c. $x^2 + y^2 + z^2 = 9, y = z$

d. $z = y^2, x = 2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Describing the Regions Created by Different Equations and Inequalities

Describe what each set of points look like that fit the following conditions:

a. $x \geq 0, y \geq 0, z \geq 0$

b. $x \geq 0, y \geq 0, z = 0$

c. $x^2 + y^2 \leq 4, z = 0$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

d. $x^2 + y^2 \leq 4$, no restriction on z

f. $1 \leq x^2 + y^2 + z^2 \leq 9$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: The Standard Form of a Sphere

Find the center C and the radius a for the spheres:

a. $x^2 + y^2 + z^2 + 4x - 6y - 4z = 0$

b. $2x^2 + 2y^2 + 2z^2 - 8y + 6z = 0$

c. $(x-2)^2 + (y-1)^2 + (z+1)^2 = 103 + 4x + 2y - 2z$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding the Equation that Fulfills Given Criteria

a. Find the equation that describes the plane perpendicular to the x-axis at $(5, 0, 0)$

b. Find the equation that describes the plane through the point $(2, -1, 1)$ perpendicular to the y-axis.

c. Find the equation of a circle of radius 3 centered at $(2, -2, 1)$ and lying in a plane parallel to the yz-plane.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: The Dot Product

How can we find the angle between two vectors?

Part 1: when we can utilize a right triangle

Part 2: when we cannot utilize a right triangle

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Define the Dot Product:

Summarize the angle between vectors:

What is another way to define the dot product?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Example Find the dot product.

a. $\langle 3, -2, 5\rangle \cdot \langle -1, 3, 1\rangle$

b. $(\frac{1}{2}\boldsymbol{i} - \frac{1}{3}\boldsymbol{j} + \frac{1}{4}\boldsymbol{k}) \cdot (6\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k})$

Example Find the angle between the vectors $\boldsymbol{u} = \langle 2, 1, 3\rangle$ and $\boldsymbol{v} = \langle -1, 0, 3\rangle$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Define what it means for two vectors to be orthogonal.

<u>Example</u> Determine if the following vectors are orthogonal: $\boldsymbol{u} = \langle 2, 3 \rangle$ and $\boldsymbol{v} = \langle -6, 4 \rangle$

What vector is orthogonal to every nonzero vector?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Vector Projections

Explain how vector projections work:

Summarize the vector projection of $\vec{u}$ onto $\vec{v}$:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Summarize the scalar component of $\vec{u}$ in the direction of $\vec{v}$L

<u>Example</u> Let $\boldsymbol{u} = 3\boldsymbol{i} - 2\mathbf{j} + \mathbf{k}$ and $\boldsymbol{v} = 2\boldsymbol{i} + \mathbf{j} - 3\mathbf{k}$.

a. Find $\boldsymbol{u} \cdot \boldsymbol{v}$, $|\boldsymbol{u}|$, $|\boldsymbol{v}|$

b. Find the cosine of the angle between the vectors.

c. Find the scalar component of $\boldsymbol{u}$ in the direction $\boldsymbol{v}$**.**

d. Find the vector $proj_{\vec{v}}\vec{u}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Work and Vectors

With vectors, we can study work done by a constant force when a force is applied in the direction other than the direction in motion.

What is the equation for work done by a constant force applied in the direction of motion? Explain this equation.

Explain how we can understand a force moving on an object through a displacement vector.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> If $|\boldsymbol{F}| = 20N$ and $|\boldsymbol{D}| = 2m$ and $\theta = 60°$, find the work done by $\boldsymbol{F}$**.**

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Dot Products and Projections

For the following vectors, find
i) $\boldsymbol{u} \cdot \boldsymbol{v}, |\boldsymbol{u}|, |\boldsymbol{v}|$
ii) the angle between the vectors
iii) the scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$
iv) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

a. $\boldsymbol{u} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

b. $\boldsymbol{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}} \rangle, \boldsymbol{v} = \langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \rangle$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Working with Diagonals of Shapes

Show that the diagonals of a rhombus are perpendicular.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Work

How much work does it take to slide a crate 10m along a loading dock by pulling on with a 100N force at an angle of $30°$ from the horizontal?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: The Cross Product

Define the cross product and where it comes from:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

When are two vectors parallel?

List the properties of the cross product:

Explain the significance of $|\vec{u} \times \vec{v}|$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

How can you use the determinant formula for calculate the cross product?

<u>Example</u> Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ If $\vec{u} = \langle 2, 1, 3 \rangle$ and $\vec{v} = \langle -1, 3, 1 \rangle$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Example Find a vector perpendicular to the plane of $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

Example Find the area of the triangle with vertices $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

Example Find a unit vector perpendicular to the plane of $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Explain how torque relates to vectors:

<u>Example</u> Find the magnitude of the force **F** at the pivot point P in the picture.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Explain the triple scalar or box product:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Summarize how to calculate the triple scalar product as a determinant:

<u>Example</u> Find the volume of the box determined by $\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = 5\boldsymbol{i} + 2\boldsymbol{k}, \boldsymbol{w} = -2\boldsymbol{j} + \boldsymbol{k}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Review: Determinants

Find the following determinants.

a. $\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}$

b. $\begin{vmatrix} -1 & 6 \\ 3 & 5 \end{vmatrix}$

c. $\begin{vmatrix} 3 & 1 & -1 \\ 2 & 1 & 2 \\ 1 & 4 & 2 \end{vmatrix}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Compute $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{vmatrix}$ two ways.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Triple Scalar or Box Product

Explain the Triple Scalar or Box Product

Summarize how to calculate the triple scalar product as a determinant:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the volume of the box determined by $\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = 5\boldsymbol{i} + 2\boldsymbol{k}, \boldsymbol{w} = -2\boldsymbol{j} + \boldsymbol{k}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Area of Triangles with the Cross Product

Find i) the area of the triangle determined by the points P, Q, and R. ii) Find a unit vector perpendicular to plane PQR.

$P(1,2,1), Q(1,0,-1), R(3,-1,1)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Lines and Line Segments in Space

Explain the vector equation for a line.

What are the parametric equations for a line?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the parametric equations for the line through $(-1, 0, 2)$ parallel to $\boldsymbol{v} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

<u>Example</u> Find the parametric equations for the line through $P(-2, 0, 1)$ and $Q(-1, 1, 2)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Parametrize the line segment joining the points $P(-2, 0, 1)$ and $Q(-1, 1, 2)$.

How do you find the distance from a point S to a line through P parallel to a vector $\boldsymbol{v}$**?**

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Planes in Space

How can we find the equation for a plane in space?

<u>Example</u> Find an equation for the plane through $P_0(-1, 2, 3)$ perpendicular to $\boldsymbol{n} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Example Find an equation for the plane through $A(0,0,2), B(3,0,0), C(0,1,0)$

Example Find a vector parallel to the line of intersection of the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Example Find parametric equations for the line in which the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$ intersect.

Example Find the point where the line:

$$x = \frac{3}{2} + 2t, y = 2t, z = 1 - t$$

Intersects the plane $2x - 3y - 6z = 12$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Describe the distance from a point S to a plane with normal **n** at point P.

<u>Example</u> Find the distance from $S(1, 2, 1)$ to the plane $2x - 3y + 6z = 12$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the angle between the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Parametric Equations for Lines

Find parametric equations for the lines.

a. The line through $P(1, 1, 2)$ and $Q(-1, 0, 1)$

b. The line through $(2, 4, 6)$ perpendicular to the plane $x - y + 4z = 8$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Equations for Planes

Find the equations for the planes.

a. The plane through $(1, 1, 2), (3, 0, 3), (0, -1, 1)$

b. The plane through $A(1, -1, 0)$ perpendicular to the vector from the origin to A.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Planes Containing Intersecting Lines

Find the plane containing the intersecting lines:

$$L_1: x = 1 - t, \qquad y = 3 + t, \qquad z = 1 + t, \qquad -\infty < t < \infty$$
$$L_2: x = 1 - 2s, \qquad y = 1 + 4s, \qquad z = 3 - 3s, \qquad -\infty < t < \infty$$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Distance from a Point to a Line

Find the distance from the point $(0, 1, 1)$ to the line $x = 2 + 3t, \;\; y = 1 - t, \;\; z = -2 + 2t$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Sketching Quadric Surfraces

Sketch the following.

a. $x^2 + z^2 = 9$

b. $z = 4 + x^2 + y^2$

c. $z = 1 - y^2 + x^2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Sketching Graphs in Cylindrical Coordinates

Sketch the graph describe by the cylindrical coordinates.

a. $z = r$

b. $r = \theta$

c. $0 \leq r \leq 2\sin\theta, 1 \leq z \leq 2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Sketching Graphs in Spherical Coordinates

Sketch the graph describe by the spherical coordinates.

a. $\rho = 4$

b. $\rho \cos \theta = 2$

c. $0 \leq \rho \leq 1, \frac{\pi}{2} \leq \phi \leq \pi, 0 \leq \theta \leq \pi$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Setting Up Triple Integrals Across Coordinate Systems

Set up the triple integrals for the volume of the sphere $\rho = 4$ in:

a. Spherical coordinates

b. Cylindrical coordinates

c. Rectangular coordinates

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Now let D be a solid ball of radius 4 units that is cut by a plane that is 2 units from the center of the sphere. Express the volume of this situation in:

a. Cylindrical coordinates

b. Spherical coordinates

c. Rectangular coordinates

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Intro to Vector-Valued Functions

What are component functions?

What is a vector-valued function?

Draw an example of a vector-valued function:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Determining a Space Curve from Its Function

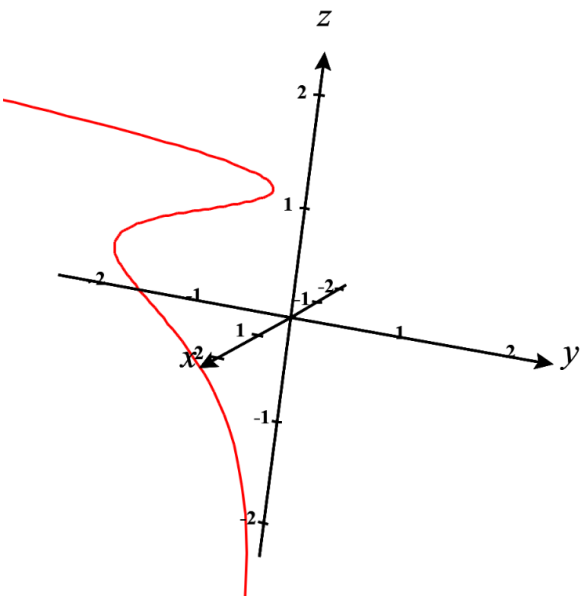
Determine which $\vec{r}(t)$ matches the space curves below.

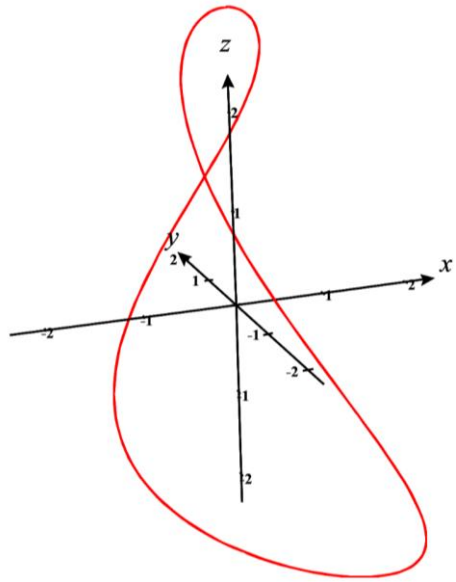
a. $\vec{r}(t) = t\vec{\imath} - (\sin t)\vec{\jmath} + (\ln t)\vec{k}$

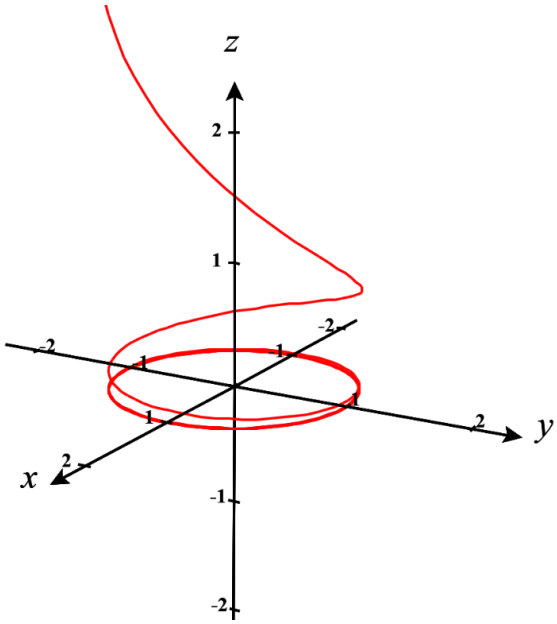
b. $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\vec{\jmath} + (e^t)\vec{k}$

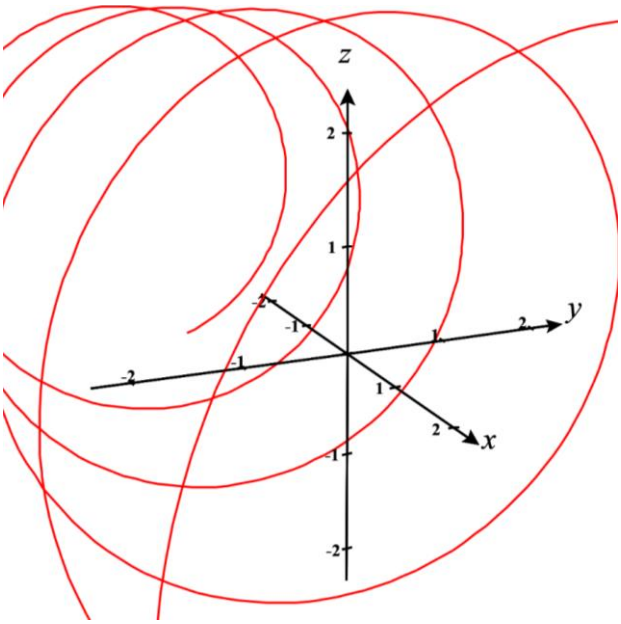
c. $\vec{r}(t) = \left(\frac{1}{2}t\right)\vec{\imath} + (3\cos t)\vec{\jmath} + (3\sin t)\vec{k}$

d. $\vec{r}(t) = (\sin 2t)\vec{\imath} + (3\cos t)\vec{\jmath} + (3\sin t)\vec{k}$









Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

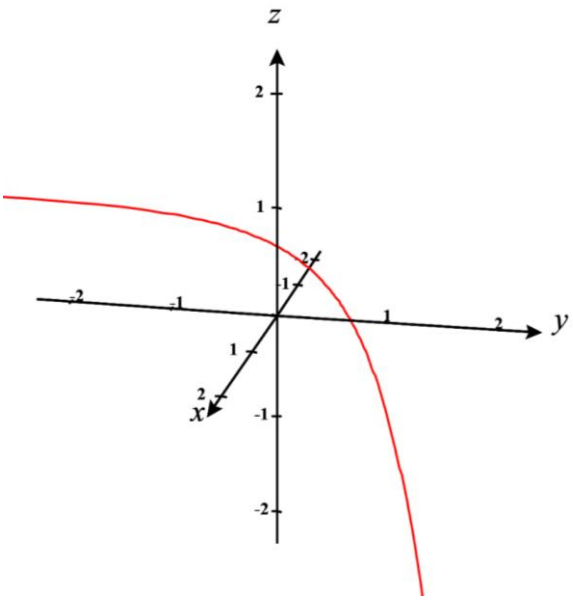
Determine which $\vec{r}(t)$ matches the space curves below.

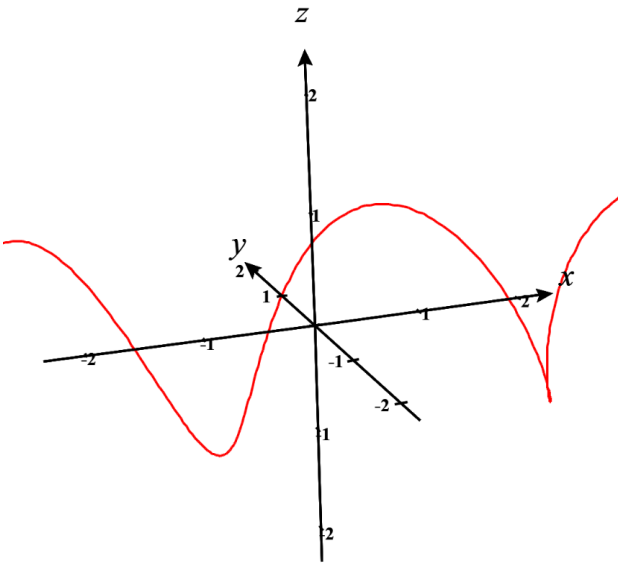
a. $\vec{r}(t) = (\sin t)\,\vec{\imath} + (\cos t)\vec{\jmath} + (\frac{1}{2}t)\vec{k}$

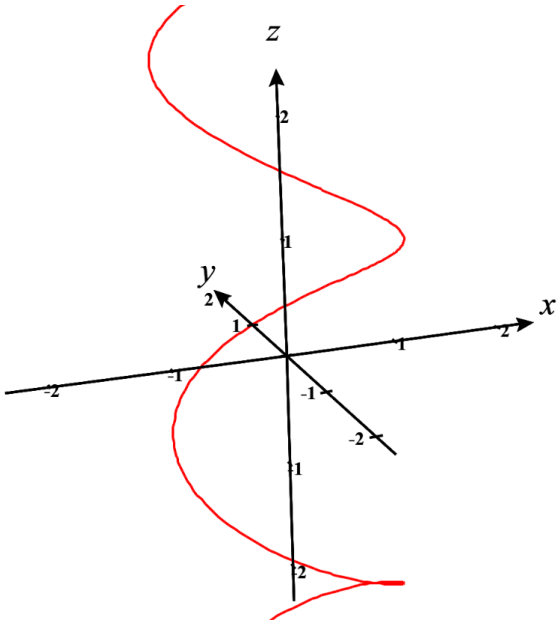
b. $\vec{r}(t) = \left(\frac{1}{2}t\right)\vec{\imath} + (\cos t)\vec{\jmath} + (\sin t)\vec{k}$

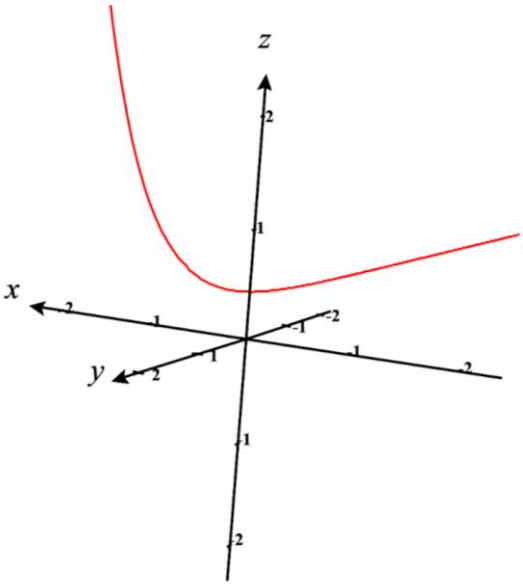
c. $\vec{r}(t) = (t)\vec{\imath} + (\ln t)\vec{\jmath} + (1)\vec{k}$

d. $\vec{r}(t) = (1)\vec{\imath} + (t)\vec{\jmath} + (e^t)\vec{k}$









Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Limits with Vectors

Find the following limits.

a. $\lim\limits_{t\to\frac{\pi}{2}}[(\cos t)\vec{\imath} + (\sin 2t)\vec{\jmath} + \tan(4t)\vec{k}]$

b. $\lim\limits_{t\to 1}\left[\frac{t-1}{\sqrt{t}-1}\vec{\imath} + \frac{t^3-1}{\ln t}\vec{\jmath} - (\tan^{-1} t)\vec{k}\right]$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: The Basics of Tangent Lines for Space Curves

Write out what is reviewed in the beginning of the video.

Show how to find the tangent line for $\vec{r}(t) = (\sin t)\vec{\imath} + (\cos t)\vec{\jmath} + (t)\vec{k}$ at $t = 0$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Differentiating Vector-Valued Functions

Below $\vec{r}(t)$ represents the position of a particle in space at time time. Find i) the particle's velocity, ii) the particle's acceleration, iii) the speed of the particle at any moment, and iv) find a way to write the particle's velocity as a product of its speed and direction

a. $\vec{r}(t) = (t^2)\vec{\imath} + (\cos t)\vec{\jmath} - (e^{-t})\vec{k}; t = 0$

b. $\vec{r}(t) = (\sec t)\vec{\imath} - (\ln 4t)\vec{\jmath} + (\tan t)\vec{k}; t = \frac{\pi}{4}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Derivative Rules Challenge for Calculus III

Name	Function Form	Derivative
Product Rule	$f(x) * g(x)$	
Quotient Rule	$\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$	
Sine	$\sin x$	
Cosine	$\cos x$	
Tangent	$\tan x$	
Cotangent	$\cot x$	
Secant	$\sec x$	
Cosecant	$\csc x$	
Chain Rule	$f\big(g(x)\big)$	
Natural Log	$\ln x$	
Natural Exponent	e^x	
Hyperbolic sine	$\sinh x$	
Hyperbolic cosine	$\cosh x$	

Find the derivatives:

a. $y = \cos 5t$

b. $y = e^{-t}$

c. $y = t \sin t$

d. $y = e^{t} \cos t$

e. $y = \frac{t^2-1}{\ln t}$

f. $y = \cos^3 t$

g. Find the second derivative of $y = \cos^3 t$

Examples: Finding Tangent Lines of Vector-Valued Functions

Find the tangent lines to the smooth curves at the given moment t_0. Write the line with parametric equations.

a. $\vec{r}(t) = (t^2)\vec{\imath} + (4t + 1)\vec{\jmath} + (t^4)\vec{k}$, $t_0 = 2$

b. $\vec{r}(t) = (t \ln t)\vec{\imath} - (\ln 2t)\vec{\jmath} + \left(\frac{t-1}{t+3}\right)\vec{k}$, $t_0 = 1$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Tangent Lines Given a Point

Find the value(s) of t so that the tangent line to the given curve contains the given point.

$\vec{r}(t) = (t)\vec{\imath} - \left(\frac{2}{3}t^{\frac{3}{2}}\right)\vec{\jmath} + (5)\vec{k}$, $(0, -\frac{8}{3}, 5)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Integrating Vector-Valued Functions

Integrate the following.

a. $\int_1^2 \left[(4-4t)\vec{\imath} + \left(6\sqrt{t}\right)\vec{\jmath} + \left(\frac{4}{t}\right)\vec{k} \right] dt$

b. $\int_0^{\frac{\pi}{2}} \left[(\cos t)\vec{\imath} + (t \sin t)\vec{\jmath} + (\sin^2 t)\vec{k} \right] dt$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Review: Some Common Integration Techniques and Tricks for Calc III

Trig Identities

Pythagorean Identities

Half Angle Identities + Modifications

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

a. $\int \tan^2 x \ dx$

b. $\int \sin^2 x \ dx$

Substitution

a. $\int 2x\sqrt{x^2+1}dx$

b. $\int t\ e^{t^2} dt$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Integration by Parts

Formula:

a. $\int x \sin x \ dx$

b. $\int \ln x \ dx$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Solving Initial Value Problems with Vector-Valued Functions

Solve the initial value problem.

$$\frac{d\vec{r}}{dt} = t\vec{\imath} + \frac{1}{2}t\vec{\jmath} + t^3\vec{k}$$

$$\vec{r}(0) = \vec{\imath} - \vec{\jmath}$$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Writing Acceleration with Tangential and Normal Components

Write $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$ for

$$\vec{r}(t) = (t\cos t)\vec{\imath} + (t\sin t)\vec{\jmath} + t^2\vec{k}$$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding $\vec{B}$ and τ

In section 13.4, you found $\vec{T}$, κ, and $\vec{N}$ for the space curves. Now find $\vec{B}$ and τ.

a. $\vec{r}(t) = (e^t \cos t)\vec{\imath} + (e^t \sin t)\vec{\jmath} + 5\vec{k}$

b. $\vec{r}(t) = (\cos^3 t)\vec{\imath} + (\sin^3 t)\vec{\jmath}, 0 < t < \frac{\pi}{2}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Curvature and Normal Vectors

How do we define curvature?

Explain the idea of curvature conceptually:

What is one alternative way to derive an easier formula for curvature?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Example Find the curvature κ for the curve $\vec{r}(t) = (\cos t + t\sin t)\vec{\imath} + (\sin t - t\cos t)\vec{\jmath}, t > 0$

Example Find the curvature of a circle $\vec{r}(t) = (a\cos t)\vec{\imath} + (a\sin t)\vec{\jmath}$ for a circle with radius a

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Define a principal unit normal vector.

Explain the principal normal vector conceptually.

Show how to derive a simpler formula for calculating $\vec{N}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Example</u> Find the normal vector $\vec{N}$ for the curve $\vec{r}(t) = (\cos t + t\sin t)\vec{\imath} + (\sin t - t\cos t)\vec{\jmath}, t > 0$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding $\vec{T}$, κ, and $\vec{N}$ for Space Curves

Find $\vec{T}$, κ, and $\vec{N}$ for the space curves:

a. $\vec{r}(t) = (e^t \cos t)\vec{\imath} + (e^t \sin t)\vec{\jmath} + 5\vec{k}$

b. $\vec{r}(t) = (\cos^3 t)\vec{\imath} + (\sin^3 t)\vec{\jmath}, 0 < t < \frac{\pi}{2}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Arc Length Parameter

We are going to take interest in different mathematical features of curves that will describe how they turn or twist.

First we will discuss this idea conceptually. Do your best to summarize the difference between arc length versus arc length parameter.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Write the arc length formula for a smooth curve $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$, $a \leq t \leq b$ that is traced only once as t increases from a to b:

What else does this formula represent?

How could we rewrite this formula in short hand?

<u>Example</u> Find the length of the path from $t = 0$ to $t = \pi$ for $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\,\vec{\jmath} + (t)\vec{k}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Now write out how to compute arc length parameter:

<u>Example</u> Find the arc length parameter for the previous example from $t = 0$ to $t = \pi$ for $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\,\vec{\jmath} + (t)\vec{k}$

What are the pros and cons of arc length versus arc length parameter?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: The Unit Tangent Vector

How do we define the unit tangent vector?

Explain the derivation for an easier way to compute this formula.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Arc Length Parameter

Find the arc length parameter along the curve from the point where t=0 by evaluating:

$$s = \int_0^t \{\vec{v}(\tau)d\tau$$

Then find the length of the indicated portion of the curve.

$$\vec{r}(t) = (1+2t)\vec{\imath} + (1+4t)\vec{\jmath} + (1+8t)\vec{k}, -1 \leq t \leq 0$$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Tangent Vectors and Lengths

Find the curve's unit tangent vector, then find the length of the indicated portion of the curve.

a. $\vec{r}(t) = (4\cos t)\vec{\imath} - (4\sin t)\vec{\jmath} + (3t)\vec{k}$, $0 \le t \le \pi$

b. $\vec{r}(t) = (t\sin t)\vec{\imath} - (t\cos t)\vec{\jmath}$, $0 \le t \le \pi$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Maximum Curvature

Find the maximum curvature for the function $f(x) = \ln x$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Find and Sketch the Domain for Multivariable Functions

Find and sketch the domain.

a. $f(x) = \ln(x^2 + y^2 - 4)$

b. $f(x) = \sqrt{(x^2 - 1)(x^2 - 4)}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Sketching Level Curves and Level Surfaces

Find the level curves or level surfaces for the following. Do your best to sketch or at least describe the region for surfaces.

a. $f(x, y) = \sqrt{x^2 + y^2}$

b. $f(x, y) = 1 - |x| - |y|$

c. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$

d. $f(x, y, z) = z - x^2 - y^2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Review: Algebra Tricks for Limits in 1 Variable vs Multivariables

1. Factoring to Cancel

$\lim_{x \to 1} \frac{x-1}{x^2-1}$

Crashing Course Factoring Review:

Factor by the GCF: $\frac{3x^2-5x^3}{x^5-2x^4+3x^3+2x^2}$

Factor by Grouping:

a. $ax + ay + bx + by$

b. $\frac{x-1}{xy-y-2x+2}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Factor Trinomials:

a. $x^2 + 5x + 6$

b. $2x^2 + 5x + 3$

c. $15x^2 + 14x - 8$

Difference of Cubes Formulas

Difference of Cubes with Polynomial Long Division: $(x^3 + 1) \div (x + 1)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

2. Using the Conjugate

What is the conjugate?

Show how to multiply by the conjugate in this one-variable example:

$\lim_{x \to 1} \frac{x-1}{\sqrt{x+3}-2}$

3. Two Common Trig Tricks

Write the two common limits for trig:

$\lim_{x \to 0} \frac{\tan 3x}{x}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Limits of Multivariable Functions Using Algebra and Trig Tricks

a. $\lim\limits_{(x,y)\to(0,0)} \frac{e^x \sin y}{y}$

b. $\lim\limits_{(x,y)\to(2,-5)} \frac{y+5}{x^2y-xy+5x^2-5x}$

c. $\lim\limits_{(x,y)\to(3,0)} \frac{\sqrt{3x-y}-3}{3x-y-6}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Limits that Do Not Exist

Show that the limits do not exist.

a. $\lim_{(x,y)\to(0,0)} \frac{x+\sin y}{y+\sin x}$

b. $\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Second-Order Partial Derivatives

Find all the second order partial derivatives.

a. $f(x, y) = xy^2 + \cos xy + y \sin x$

b. $f(x, y) = \ln(x + y)$

c. $z = xe^{\frac{y}{x^2}}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Tangent Lines to Intersecting Surfaces

Find parametric equations for the line tangent to the curve of intersection of the surface at the given point:

Surfaces: $x^2 + 2y + 2z = 4; y = 1; P\left(1,1,\frac{1}{2}\right)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: The Chain Rule for Multivariable Functions

Find the derivatives, and then determine the value at the given point.

a. $\frac{dw}{dt}$ for $w = \ln(x^2 + y^2 + z^2)$ where $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$ at the point $t = 3$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

b. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ for $u = e^{qp} \sin^{-1} r$ where $p = \frac{1}{z}, q = y^2 \ln z, r = \sin x$ at point $(x, y, z) = \left(\frac{\pi}{4}, 1, 1\right)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Implicit Differentiation, Again

Find $\frac{dy}{dx}$ at the given point: $ye^x + \sin xy + x - \ln 5 = 0, (\ln 5\,, 0)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Direction Derivatives

Find the derivative of the function P in the direction of $\vec{u}$

a. $f(x,y) = \frac{x-y}{xy+2}, P(1,-1), \vec{u} = 12\vec{\imath} + 5\vec{\jmath}$

b. $f(x,y,z) = \cos xy + e^{yz} + \ln zx\,, P\left(1, 0, \frac{1}{2}\right), \vec{u} = \vec{\imath} + 2\vec{\jmath} + 2\vec{k}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Directions Where a Function Increases or Decreases Most Rapidly

Where does $f(x, y, z) = xe^y + z^2$ increase or decrease most rapidly at $P(4, 1, 1)$? Find the derivative at these directions.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Tangent Lines to Level Curves

Sketch the curve with the gradient and the tangent line at the given point. Write the equation for the tangent line.

$$x^2 + y^2 = 9, \left(\sqrt{3}, \sqrt{3}\right)$$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Tangent Planes and Normal Lines to Surfaces

Find equations for the tangent plane and normal line at the point P on the surface:

$$x^2 + y^2 - 2xy - x + 3y - z = -4, P(2, -3, 18)$$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Lesson: Extreme Values and Saddle Points

Let $f(x, y)$ be defined on a range R containing the point (a, b). Then:

- $f(a, b)$ is a **local maximum** if:

- $f(a, b)$ is a **local minimum** if:

State the First Derivative Test for Local Extreme Values:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Define a **critical point** for $f(x, y)$

What is a saddle point?

What is the **discriminant** or **Hessian** of $f(x, y)$?

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

The Second Derivative Test for Local Extreme Values

Let $f(x, y)$ be a function where the first and second derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then:

i. f has a local maximum at (a, b) if:

ii. i. f has a local minimum at (a, b) if:

iii. f has a saddle point at (a, b) if:

iv. the test is inconclusive if:

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

<u>Examples</u> Find the local extreme values of the following functions.

a. $f(x, y) = -5x^2 - 2y^2 + 2xy + 4x + 4y - 4$

b. $f(x, y) = e^x - xe^y$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

c. $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Local Extrema

Find all the local maxima, local minima, and saddle points

a. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

b. $f(x, y) = \ln(x + y) + x^2 - y$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding The Critical Points for Interesting Systems

Find *only* the critical points for the following.

a. $f(x, y) = x^3 - y^3 + 3xy - 6$

b. $f(x, y) = 8xy - x^4 - y^4$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

c. $f(x, y) = e^{x^2+y^2-8x}$

d. $f(x, y) = \frac{1}{x^2+y^2-4}$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Finding Absolute Extrema

Find the absolute maxima and minima for:

$f(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines:

$x = 0, y = 4, y = x$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Lagrange Multipliers with Two Independent Variables and One Constraint

a. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

b. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Lagrange Multipliers with Three Independent Variables and One Constraint

Find the point on the sphere $x^2 + y^2 + z^2 = 9$ farthest from the point $(1, -1,\ 1)$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

b. $f(x, y) = \ln(x + y) + x^2 - y$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Evaluating Iterated Integrals

Evaluate the following integrals.

a. $\int_0^2 \int_0^4 \left(\frac{x}{4} + \sqrt{y}\right) dx\, dy$

b. $\int_1^3 \int_1^{e^2} \frac{\ln x}{xy}\, dx\, dy$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

c. $\int_1^2 \int_1^2 x \ln y \ dy \ dx$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Evaluating Double Integrals Over Rectangles

Evaluate the following integrals over the region R.

a. $\int \int_R y \sin(x+y)\, dA,\ R: 0 \le x \le \pi, -\pi \le y \le 0$

b. $\iint_R \frac{xy^3}{x^2+1} dA\,, R: 0 \le x \le 1, 0 \le y \le 1$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Area by Double Integrals

Sketch the region bounded by the given lines and curves, Then express the region's area as a double integral, and integrate.

a. The parabola $x = y - y^2$ and the line $y = x$

b. The lines $y = x, y = \frac{x}{2}$,and $y = 2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Sketching Regions of Integration

Sketch the describe regions of integration.

a. $0 \leq x \leq 4, 0 \leq y \leq 3x$

b. $0 \leq x \leq 1, e^x \leq y \leq e$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Write an Integral Over a Region

Write an iterated integral for the region R using vertical cross sections AND horizontal cross sections.

a. $y = x^2, y = 2x$

b. $y = 3x, x = 2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Sketch the Region of Integration and Evaluate the Integral

Sketch the region of integration and evaluate the integral.

a. $\int_0^{\pi} \int_0^{\cos x} y \, dy \, dx$

b. $\int_0^{1} \int_0^{x^2} 3x^3 e^{xy} \, dy \, dx$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Sketch the Region of Integration and Reverse the Order of Integration

Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

a. $\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$

b. $\int_0^3 \int_1^{e^y} (x+y) dx\, dy$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Volume Beneath a Surface

Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Integrating with Polar Coordinates

Change the Cartesian integral into an equivalent polar integral, then integrate.

a. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx\, dy$

b. $\int_0^1 \int_x^{\sqrt{2-x^2}} (x + 2y)\ dx\ dy$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Evaluating Triple Integrated Integrals

Integrate the following.

a. $\int_0^1 \int_0^{1-x^2} \int_3^{9-x^2-y^2} x \, dz \, dy \, dx$

b. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e x \, e^x \ln y \frac{(\ln z)^2}{z} \, dz \, dy \, dx$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Reversing the Order for Triple Integrals

Using the integral: $\int_{-1}^{1}\int_{x}^{2}\int_{0}^{1-x} dz\, dy\, dx$

Rewrite an equivalent integral in the following order:

a. $dy\, dz\, dx$

b. $dy\, dx\, dz$

c. $dz\, dx\, dy$

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Volume with Triple Integrals

Find the volume in the following situations.

a. The region between the cylinder $z = x^2$ and the xy-plane that is bounded by the planes $x = -1, x = 1, y = 0, y = 1$.

b. The region between the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder

$x = 4 - y$^2

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Examples: Double Integral Substitutions

Evaluate $\int_0^2 \int_{\frac{y}{2}}^{\frac{y+4}{2}} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$ using the transformation $x = u + \frac{v}{2}, y = v$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Evaluate $\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\, dx$ using the transformation $x = u^2 - v^2, y = 2uv$.

Copyright 2025, Dr. Jennifer Hill, DivideandConquerMath.com

Not included (no notes file):

112 (Lesson Introducing Where Calculus Starts)

823 (Field Trip on CalcPlot3D for Space Curves)