**Review: Piecewise Functions for Calc Students**

Piecewise functions look like this:

Explain how to graph the following examples. It would be best to actually write out directions for yourself in case you ever need to reference this again.

a. $f(x) = \begin{cases} 2, & x \geq 3 \\ -3, & x < 3 \end{cases}$

b. $f(x) = \begin{cases} 4, & x \geq 2 \\ x + 2, & -1 \leq x < 2 \\ -2x + 1, & x < -1 \end{cases}$

c. $f(x) = \begin{cases} 3, & x > 4 \\ 2, & -1 < x \leq 1 \\ x^2, & x < -1 \end{cases}$

d. $f(x) = \begin{cases} 3x + 1, & x \geq 3 \\ x + 1, & x < -2 \\ 3, & x = 0 \end{cases}$



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**Review: Basic Graphs and Translations**

Sketch the basic graphs from the video.



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Explain the basic translations.

**Examples** State what translations have occurred and sketch the graph.

a. $f(x) = (x + 2)^2 - 3$

b. $g(x) = -|x - 2| + 1$

c. $f(x) = \sqrt{x + 3}$



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**Review: Zeros vs Factors vs Intercepts**

Use the following functions. Find the zeros, factors, and x-intercepts of each function.

a. $f(x) = x^2 + 6x + 8$

b. $g(x) = x^2 - 9$

c. $f(x) = x^2 - 7x + 12$



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**Lesson: A Quick Review of Some Things About Graphing**

Remind yourself how to graph the following points. Draw your own graph.

Point 1: $(-3, 4)$ Point 2: $(0, -2)$

Draw another rectangular coordinate system below and label the quadrants and origin.

Often when graphing, we use equations in two variables.

**Example** Find the ordered pair associated with the equation $3x + y^2 = 7$ when $x = 1$



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**Example** Find at least 3 ordered pairs for the following equations.

a. $y = \sqrt{x + 2}$

b. $2x + 3y = 6$

Finding 3 points can be a strategy for graphing.

**Example** Sketch a graph for the previous 2 examples.

What is the x-intercept? And how do you find it?

What is the y-intercept? And how do you find it?



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**Examples** Find the x- and y-intercepts for the following equations.

a. $y = 3x - 1$

b. $x = \sqrt{y - 1}$

c. $x^2 - 3x + 2 = y$



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**Lesson: The Distance Formula**

Graph the points $(2, 4)$ and $(-4, -6)$. Explain how you can find the distance between these points by summarizing the explanation given in the video (make a decent effort to explain this – you will have an easier time remembering the formula if you understand WHY it works).

State the distance formula:



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**Example** Find the distance between the following points: $(4, -1)$ and $(7, 3)$

There are 2 ways we can use the distance formula.

**Example** Show how to use the distance formula to determine whether the following points are the vertices of a right triangle.

a. $(0, -2), (-4, 3)$ and $(5, 1)$

b. $(-2, -8), (0, -4)$ and $(-4, -7)$



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What does it mean for 3 points to be collinear?

**Example** Show how to use the distance formula to determine if the 3 points are collinear.

a. $(-1, 4), (-2, -1), (1,14)$

b. $(-2, -4), (-6, 11), (0, -12)$



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**Lesson: The Midpoint Formula**

What is the midpoint formula?

So what is the midpoint formula effectively doing?

**Example** Find the midpoint between the points $(-5, -6)$ and $(-3, 2)$

**Example** The midpoint of a line segment is $(5, 8)$ and the endpoint of that line segment is $(11, 10)$. What is the OTHER endpoint on this line segment?



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**Example** The midpoint of a line segment is $(3, 1)$ and the endpoint of that line segment is $(6, 8)$. What is the OTHER endpoint on this line segment?

**Example** The midpoint of a line segment is $(2a, 5b)$ and the endpoint of that line segment is $(-1a, 9b)$. What is the OTHER endpoint on this line segment?



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**Lesson: Review of Functions**

How do you represent a function?

What is a function often interchangeable with? How can you rewrite $f(x) = 3x + 8$?

So what IS a function? What does it actually represent?

What's another handy feature of functions?



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What are the domain and range?

Does domain and range only work for functions?

**Examples** Are the following functions, and what is the domain and range?

a. $\{(2,3), (5, -1), (-3, 7)\}$

b. $\{(-1, 5), (-8, 3), (-1, 6)\}$



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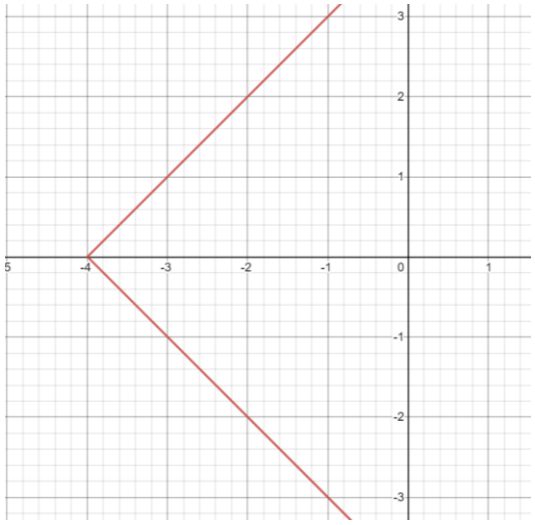
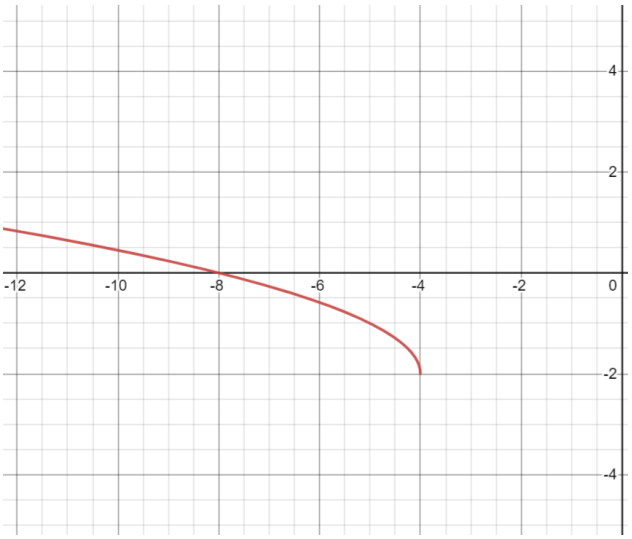
What is the Vertical Line Test, and how does it help you determine if you have a function?

**Example** State the domain and range of the following, and use the vertical line test to determine if the graph represents a function.

a.




b.



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**Examples** Determine if the following represent functions.

a. $y = 3x^2$

b. $x = y^2$

c. $y = \frac{5}{x^2}$



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**Example** Write the following in function notation: $3x + 2y = 6$

**Example** Suppose $f(x) = 3x - 5$ and $g(x) = x^2 + 4x - 1$ and $h(x) = \{(2, 5), (6, -3), (-1, -4)\}$. Find the following:

a. $f(7) =$

b. $g(3) =$

c. $f(k) =$

d. $g(x + h) =$

e. $h(6) =$



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**Lesson: Increasing, Decreasing, and Constant Functions**

Below are the "formal math" definitions of whether a function is increasing, decreasing, or constant. Let I be an open interval and let $x_1$ and $x_2$ be two points in that interval.

- A function is **increasing** over I if when $x_1 < x_2, f(x)_1 < f(x_2)$
- A function is **decreasing** over I if when $x_1 < x_2, f(x)_1 > f(x_2)$
- A function is **constant** over I if when $x_1 < x_2, f(x)_1 = f(x_2)$

While these are the formal definitions, what we really want to know are the intuitive meanings of these definition.

Sketch a graph that is increasing:

Decreasing:

And constant:



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**Example** Sketch the graph in the video, and then identify where the graph is increasing, decreasing, or constant.

a.

b.



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**Lesson: Graphs of Basic Functions**

In this section, we will review some basic graphs that are vital to know for this class and calculus. Knowing the "basic shapes" of certain graphs will make your life much easier and give you greater intuition when graphing.

Show hot to graph the following and highlight the **key points** and **key patterns.** This will come in handy for the next lesson. Take your time to sketch your own NEAT CLEAN CLEAR graphs.

$f(x) = x$

$f(x) = x^2$



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$f(x) = x^3$

$f(x) = \sqrt{x}$



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$f(x) = \sqrt[3]{x}$

$f(x) = |x|$



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**Lesson: Piecewise Functions**

What is a piecewise function? Draw an example of what they look like.

**Example** Given the function: $f(x) = \begin{cases} 3x - 1, & x \geq 2 \\ \frac{1}{2}x - 4, & x < 2 \end{cases}$, evaluate the following:

a. $f(3) =$

b. $f(-1) =$

c. $f(2) =$

**Example** Given the function: $f(x) = \begin{cases} 4, & x > 3 \\ 4x - 2, & -2 < x \leq 3 \\ x^2 - 5, & x \leq -2 \end{cases}$ evaluate the following:

a. $f(4) =$

b. $f(-5) =$

c. $f(-2) =$

d. $f(3) =$



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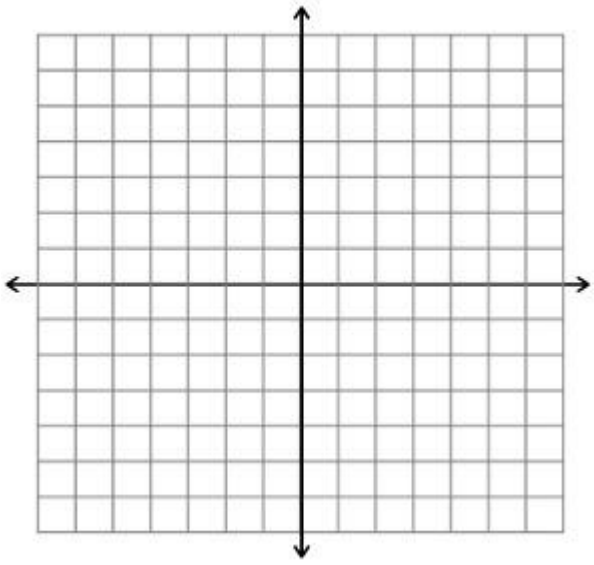
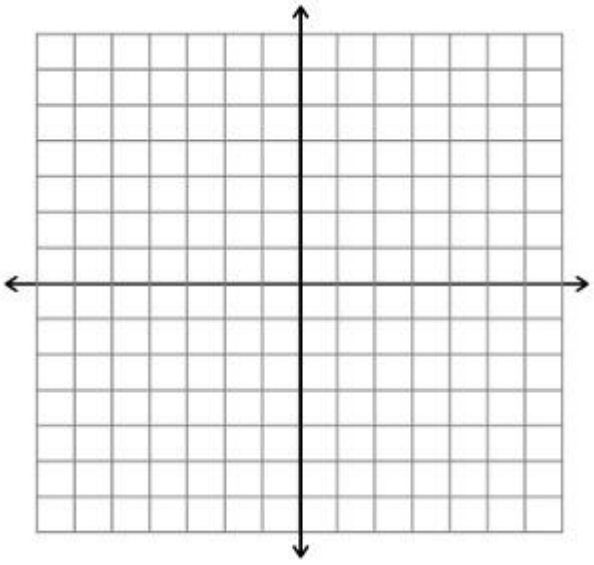
**Example** Show how to graph the following piecewise functions.

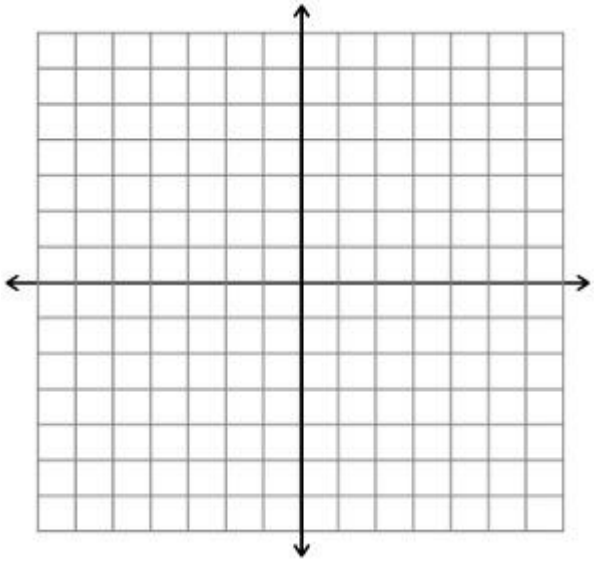
a. $f(x) = \begin{cases} 4, & x > 5 \\ -1, & x \leq 5 \end{cases}$



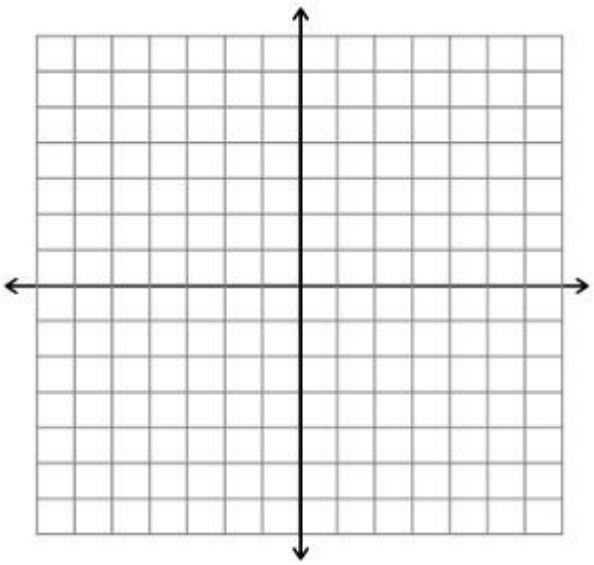
b. $f(x) = \begin{cases} 3x + 1, & x > -3 \\ -2x, & x \leq -3 \end{cases}$



c. $f(x) = \begin{cases} 3, & x > 2 \\ 2x, & -1 < x \leq 2 \\ -x + 3, & x \leq -1 \end{cases}$



d. $f(x) = \begin{cases} 3x + 1, & x \geq 1 \\ 6, & -1 \leq x < 1 \\ x^2, & x < -1 \end{cases}$





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**Lesson: The Greatest Integer Function**

Show how to graph the greatest integer function $f(x) = [\![x]\!]$

**Example** Graph $f(x) = \left[\!\left[\frac{1}{2}x - 2\right]\!\right]$



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**Lesson: Review of Triangles and Similar Triangles**

The sum of angle measures of any triangle is:

Draw the different types of triangles:

**Examples** Draw the triangles in the example, then classify the triangles as acute, right, or obtuse. Also classify as equilateral, isosceles, or scalene.



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What makes 2 triangles similar?

What makes 2 triangles congruent?

What is the key relationship between similar and congruent triangles?

**What are the 2 conditions for similar triangles?**

In other words, triangle ABC is similar to triangle DEF if:



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**Example** Draw the similar triangles in the video, and then find any missing angle measures.

**Example** Draw the similar triangles in the video, and then find any missing side lengths.

**Example** Draw the similar triangles in the video, and then find any missing measures.



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**Lesson: The Pythagorean Theorem, The Distance Formula, and Intro to Trig Functions**

State the Pythagorean theorem:

State the distance formula:

How does the Pythagorean theorem relate to the distance formula?

Now we will introduce **trigonometric functions!** Explain what makes trig functions unique:



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How do trig functions relate to the distance formula? Draw the picture associated with this.

Explain the 6 trig functions and how they relate to what we discussed above.

Sometimes you will see expressions such as $\sin^2 90°$. What does this mean?



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**Example** The terminal side of an angle $\theta$ in standard position passed through the point (-12, -5). Find the values of the 6 trig functions of angle $\theta$.

**Example** Find the 6 trig function values of an angle $\theta$ in standard position if the terminal side of $\theta$ is defined by

$3x + 5y = 0, x \geq 0$



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**Example** Suppose the point $(x, y)$ is in the indicated quadrant. Determine whether given ratio is positive or negative.

$$III, \frac{y}{r}$$



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**Lesson: Right-Triangle Based Definitions of Trigonometric Functions and Cofunctions**

Instead of using standard position to describe trig functions, we can alternatively use right triangles.

Draw a right triangle to orient yourself.

What does A represent?

Show how to define the 6 trig functions with this right triangle.

1.

2.

3.

4.

5.

6.



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What mnemonic device can help you remember the trig functions?

**Example** Draw the triangle in the video, and then show how to find the values of sine, cosine, and tangent.

**Example** Using the same example above, show how to find the values of cosecant, secant, and cotangent.



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Draw a new right triangle, and show an alternative angle measure we can find from this picture:

These relationships are called the **cofunction identities,** summarize these below:



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**Examples** Write each function in terms of its cofunction

a. $\sin 10°$

b. $\cot 65°$

c. $\csc 34°$

**Example** Find one solution for each equation. Assume all angles involved are acute angles.

a. $\cos\theta = \sin(2\theta - 30°)$

b. $\sec(\theta + 15°) = \csc(2\theta + 25°)$



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**Lesson: Fundamental Identities of Trig**

Review the test for symmetry (and the results):

We know that the sine function is odd, show what this means:

We know that the cosine function is even, show what this means:

These are called **identities.**



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Using these identities, what other identity can we determine about the tangent function:

These are called the **even-odd identities.**

We have already seen a handful of identities, and we call these the fundamental identities. Summarize them below.

**Reciprocal Identities**

**Quotient Identities**

**Pythagorean Identities**

**Even-Odd Identities**



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**Example** If $\cos\theta = \frac{3}{4}$ and $\theta$ is in quadrant I, find $\sin\theta$.

**Example** If $\tan\theta = -\frac{1}{4}$ and $\theta$ is in quadrant IV, find the values of all the other 5 trig functions.



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We use the identities to work on problems called transformations.

**Example** Write $\tan x$ in terms of $\sec x$.

**Example** Write the following in terms of sine and cosine, and then simplify until no quotients appear and all functions are of $\theta$ only.

a. $\csc\theta \cos\theta \tan\theta$

b. $\cos\ \theta(\cos\theta - \sec\theta)$



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**Examples: Using Fundamental Trig Identities to Find Values of Other Trig Functions**

Find the five remaining trig functions given the following information.

a. $\csc\theta = -\frac{5}{2}$, $\theta$ in quadrant III.

b. $\sec\theta = \frac{7}{2}$, $\tan\theta < 0$



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**Examples: Writing Trig Functions in Terms of Other Trig Functions (Fundamental Identities)**

a. Write $\cot x$ in terms of $\sin x$

b. Write $\csc x$ in terms of $\cos x$



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**Review: The Difference Quotient**

State the difference quotient.

**Examples** Find the difference quotient.

a. $f(x) = 2x - 5$

b. $g(x) = 4x^2$

c. $f(x) = 3x^2 + 1$



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**Lesson: Translations and Reflections of Functions**

Using the graph $f(x) = |x|$ as the "base function," sketch how vertical translations work (summarize what was discussed in the video).

How do you describe vertical translations in function notation?

Using the graph $f(x) = x^2$ as the "base function," sketch how horizontal translations work (summarize what was discussed in the video).

How do you describe horizontal translations in function notation?



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Use the graph $f(x) = \sqrt{x}$ to show how reflections over the x- and y-axis work (summarize what was discussed in the video).

How do you describe reflections over the x-axis and y-axis using function notation.

Use the graph $f(x) = |x|$ to show how stretching and shrinking works (summarize what was discussed in the video).



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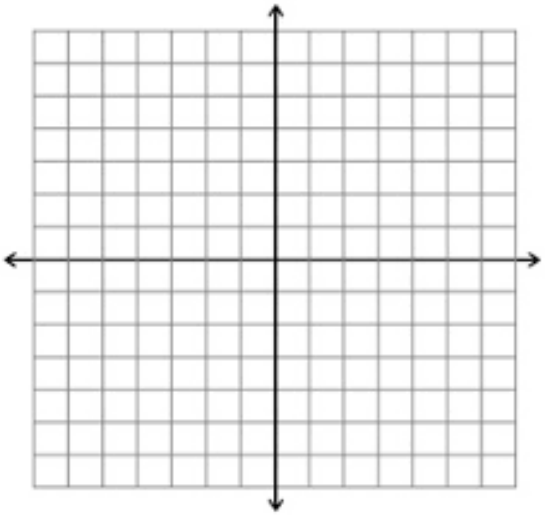
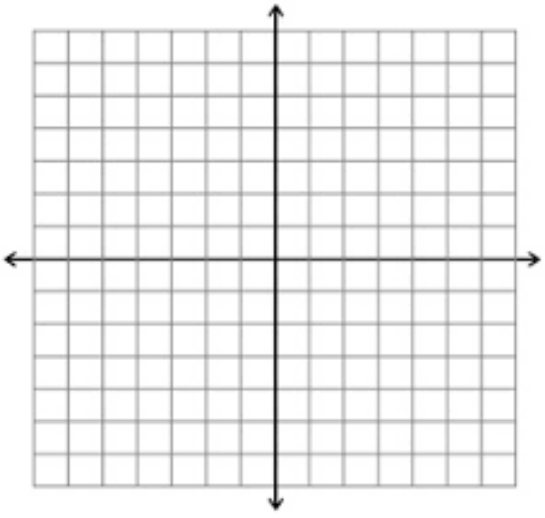
Explain how stretching and shrinking works with function notation.

**Example** Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $f(x) = -x^2 + 3$



b. $g(x) = (x + 2)^3 - 3$





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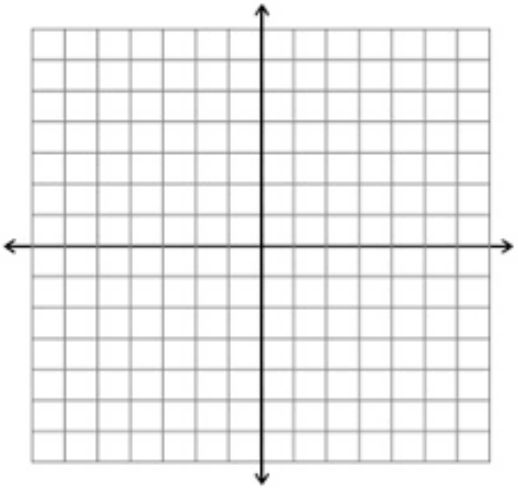
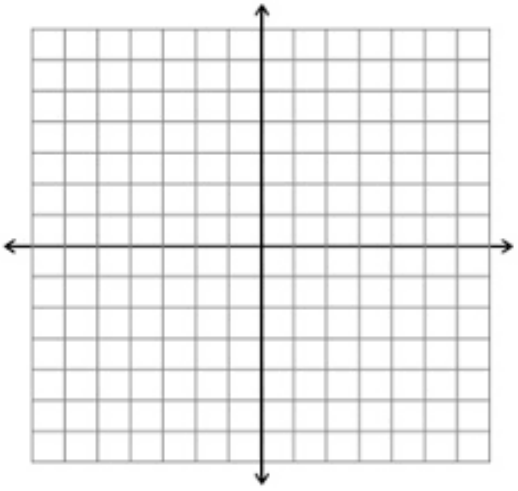
**Example** Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $g(x) = -2(x+1)^2 - 5$



b. $h(x) = 2\sqrt{-x} + 1$



**Examples** Write the equation of the given functions with the following characteristics.

a. The function $f(x) = x^2$ moved 3 units left and 5 units down.

b. The function $g(x) = \sqrt[3]{x}$ reflected over the y-axis, translated 2.5 units right and 1.75 units up.



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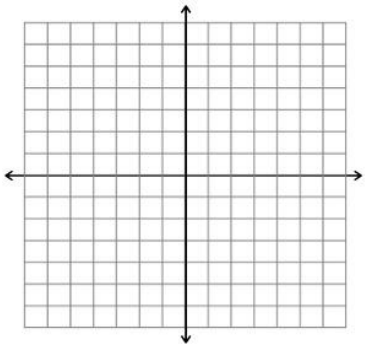
**Transformations of Functions**

These are looking at graphs of things like:

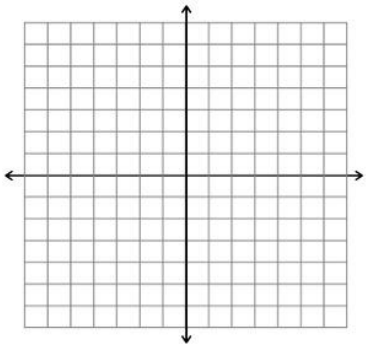
$$f(x) = x^2 - 5 \quad \text{or} \quad f(x) = |x + 3| - 2 \quad \text{or} \quad f(x) = \sqrt{x - 3}$$

First we need to discuss the shape of a few basic functions. Write these down and note the important points.

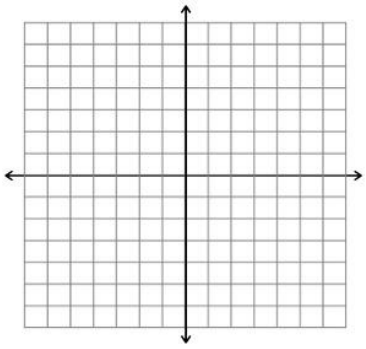
a. $f(x) = x^2$



b. $f(x) = |x|$



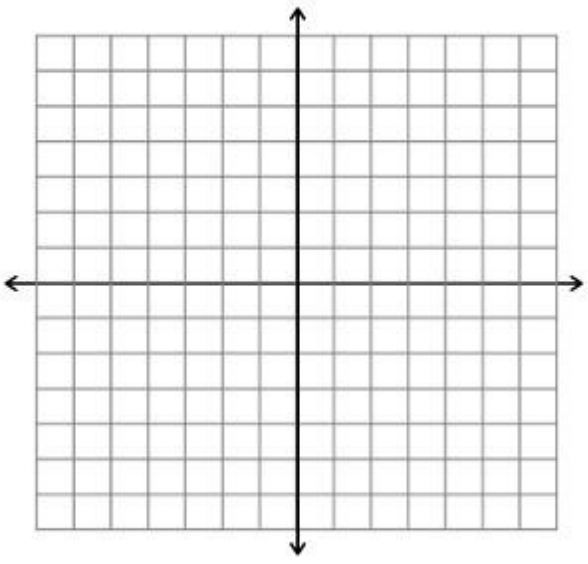
c. $f(x) = \sqrt{x}$



Using a few different colors, sketch out what the graphs of

a. $f(x) = x^2$ versus b. $g(x) = x^2 + 3$ versus c. $h(x) = x^2 - 1$

look like all on the same graph below:





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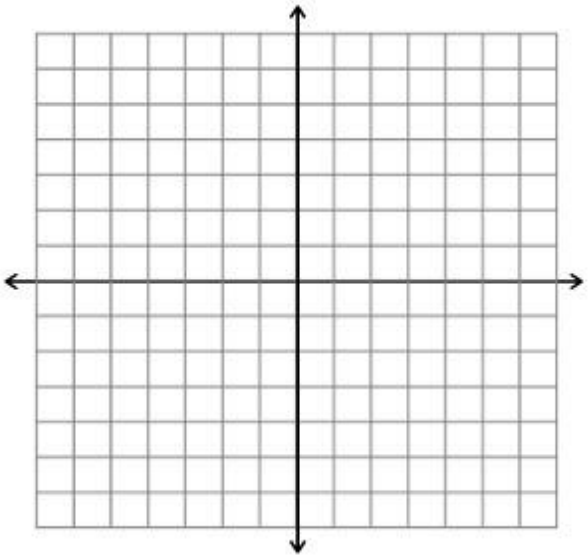
This is an example of a vertical shift. Explain what this means in mathematical terms:

Using a few different colors, sketch out what the graphs of

a. $f(x) = |x|$ versus b. $g(x) = |x - 1|$ versus c. $h(x) = |x + 2|$

look like all on the same graph below:



This is an example of a horizontal shift. Explain what this means in mathematical terms:



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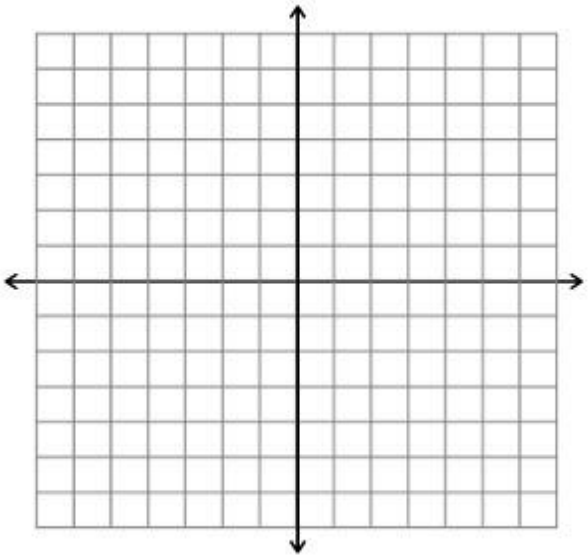
Using a few different colors, sketch out what the graphs of

$$\text{a. } f(x) = |x| \text{ versus b. } g(x) = -|x|$$

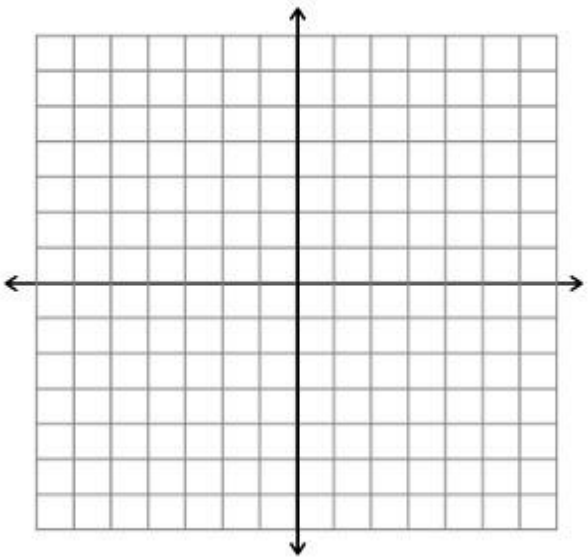
look like all on the same graph below:



This is an example of a reflection over the x-axis. Explain what this means in mathematical terms:

**Example** Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

$$f(x) = |x| + 3$$





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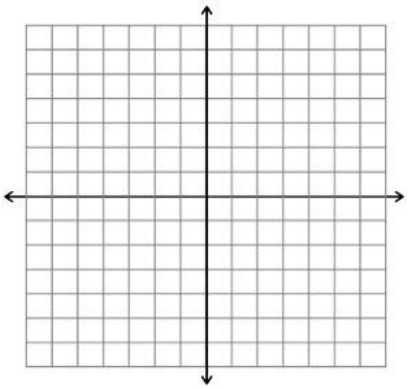
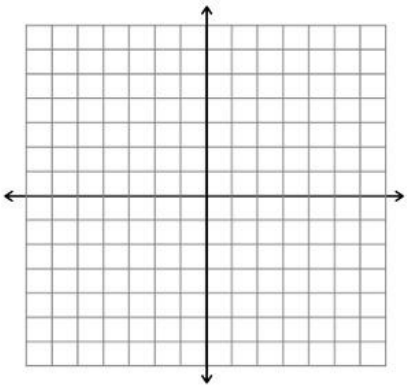
**Examples** Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

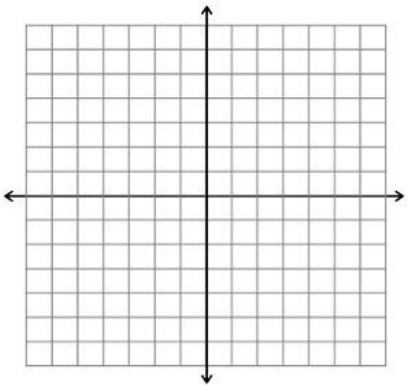
a. $f(x) = (x-1)^2$



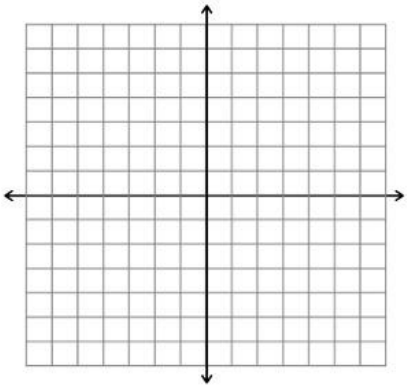
b. $g(x) = \sqrt{x} - 2$



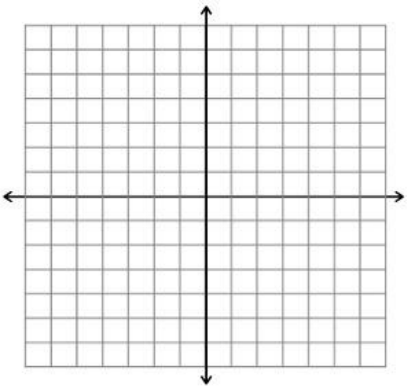
c. $h(x) = |x+1| - 2$



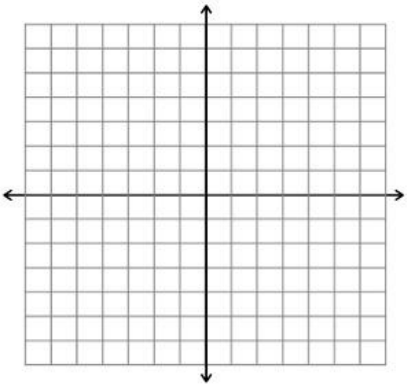
d. $k(x) = (x-2)^2 + 4$



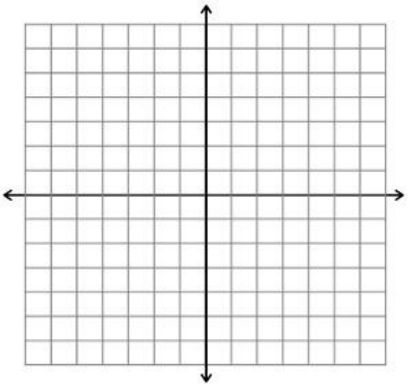
e. $f(x) = -(x+3)^2 - 2$



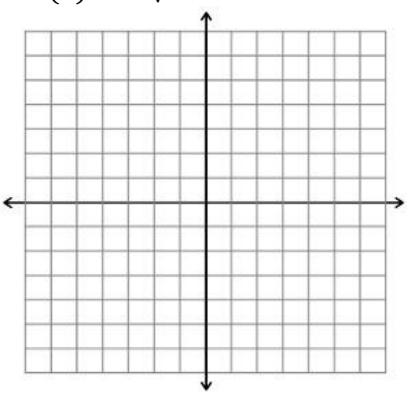
f. $g(x) = -|x+3| - 1$



g. $h(x) = -\sqrt{x-2}$



h. $k(x) = -\sqrt{x+1} - 3$





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**Examples** Write the equations that represent the listed transformations.

a. A function $f(x) = \sqrt{x}$ is shifted 7 units down.

b. A function $g(x) = x^3$ is shifted 3 units left and 2 units up.

c. A function $h(x) = |x|$ is shifted 2.5 units right and reflected over the x-axis.

d. A function $f(x) = e^x$ is shifted 4 units up and reflected over the x-axis.

e. A function $g(x) = \sqrt[3]{x}$ is shifted 3.2 units right, 5.7 units up, and reflected over the x-axis



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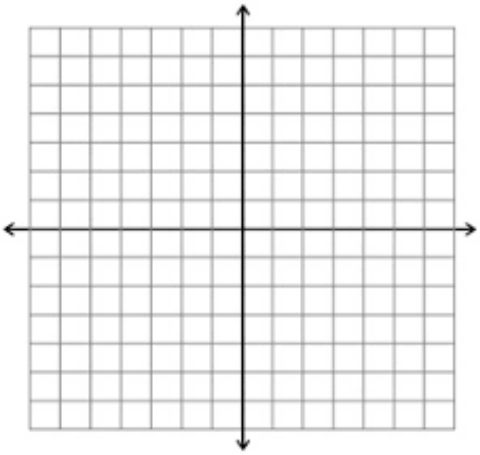
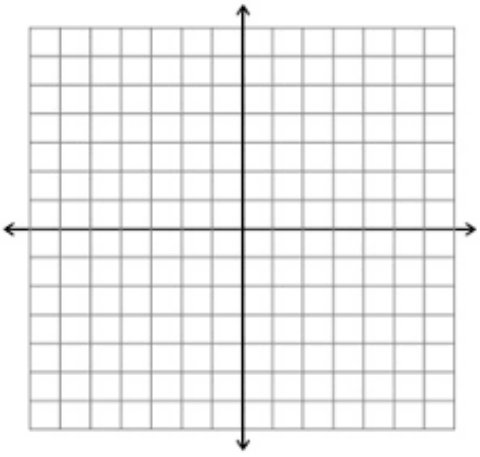
**Examples: Translations and Reflections of Functions**

Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

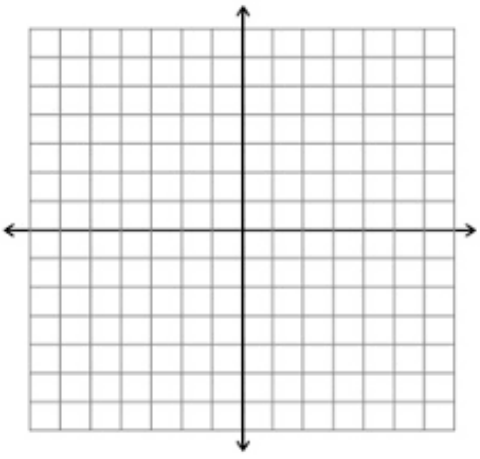
a. $f(x) = -(x+2)^2 - 3$



b. $g(x) = \frac{1}{2}(x-1)^3 + 2$



c. $h(x) = -\frac{1}{3}\sqrt{x+1} + 1$





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**Lesson: Symmetry and Even or Odd Functions**

Explain when a graph is symmetric with respect to the y-axis:

Explain when a graph is symmetric with respect to the x-axis:



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Explain when a graph is symmetric with respect to the origin:

What makes a function odd?

What makes a function even?



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So what is the test for even or odd functions?

**Examples** Determine what type of symmetry the following functions have:

a. $y = x^2 + 4$

b. $x^2 + y^2 = 25$



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**Examples** Determine what type of symmetry the following functions have:

a. $y = x^3 - 5$

b. $y = 7x^2 + 2x - 4$

c. $y = 2x^{\frac{1}{3}} - 3x$



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**Lesson: Review of Function Operations**

Given that $f$ and $g$ are functions, how do we define the function operations?

$(f+g)(x) =$

$(f-g)(x) =$

$(fg)(x) =$

$\left(\frac{f}{g}\right)(x) =$

**Example** Let $f(x) = 3x+1, g(x) = x^2-4, h(x) = 3x^2+7x+2$, and $k(x) = x^2+6x+9$. Find each of the following:

a. $(f+g)(x) =$

b. $(f+g)(3) =$

c. $(g-k)(1) =$



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**Example** Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = 3x^2 + 7x + 2$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $\left(\frac{h}{f}\right)(x) =$

b. $(gh)(x) =$

c. $\left(\frac{k}{h}\right)(1) =$



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**Lesson: Review of Composition of Functions**

For two functions $f$ and $g$, what is the composition $(f \circ g)(x)$?

**Example** Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(f \circ g)(x) =$

b. $(g \circ f)(x) =$



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**Example** Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(k \circ f)(1) =$

b. $(g \circ h)(4) =$

c. $(h \circ k)(x) =$



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**Lesson: The Difference Quotient**

Explain what the difference quotient is and draw some sketches to explain how it works. In other words, watch the video explanation and then explain it in your own words on this page.



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State the formula for the difference quotient here:

**Example** Calculate the different quotient for the following.

a. $f(x) = 4x + 5$

b. $f(x) = x^2 - 1$



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**Lesson: The Difference Quotient**

Find the difference quotient.

a. $f(x) = \frac{1}{x+3}$

b. $f(x) = x^2 - 4x + 1$

c. $f(x) = -x^2 + 3$



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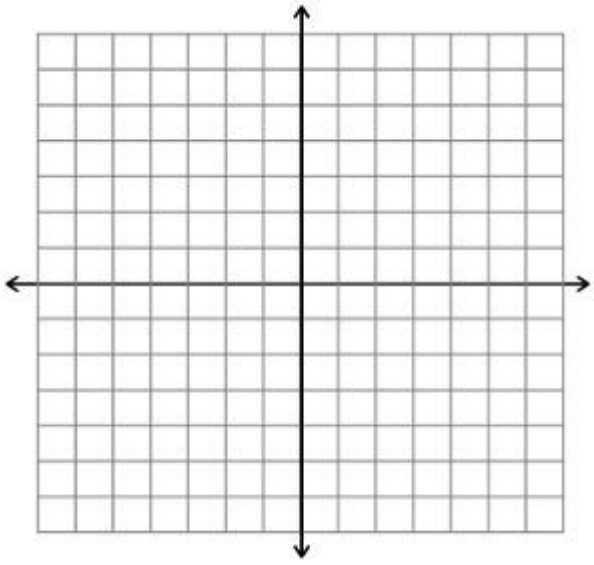
**Lesson: Exponential Functions**

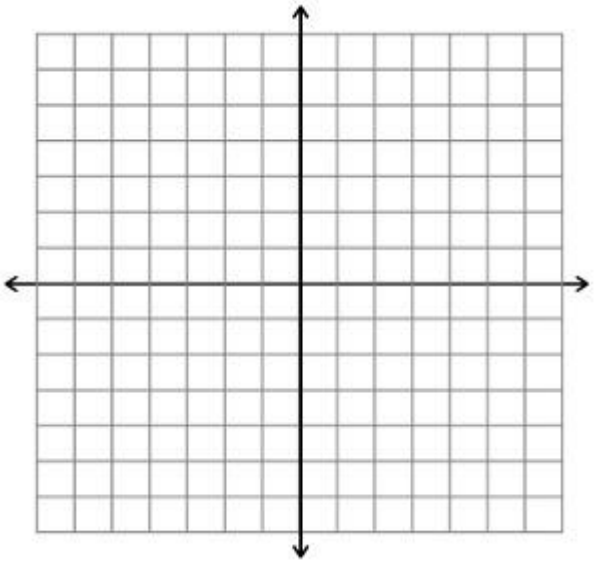
An exponential function is a function of the form _____ where $a > 0, a \neq 1$.

**Examples** Graph the following functions, then determine their domain and range.

a. $f(x) = 2^x$



b. $g(x) = \left(\frac{1}{2}\right)^x$



Using the graphs above, summarize some of the characteristics of these graphs:

The number e is an irrational number that is frequently used in math. What is an approximation of e?



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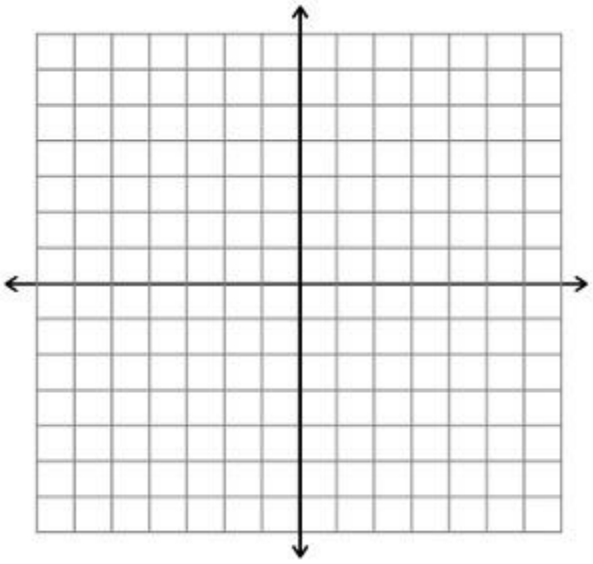
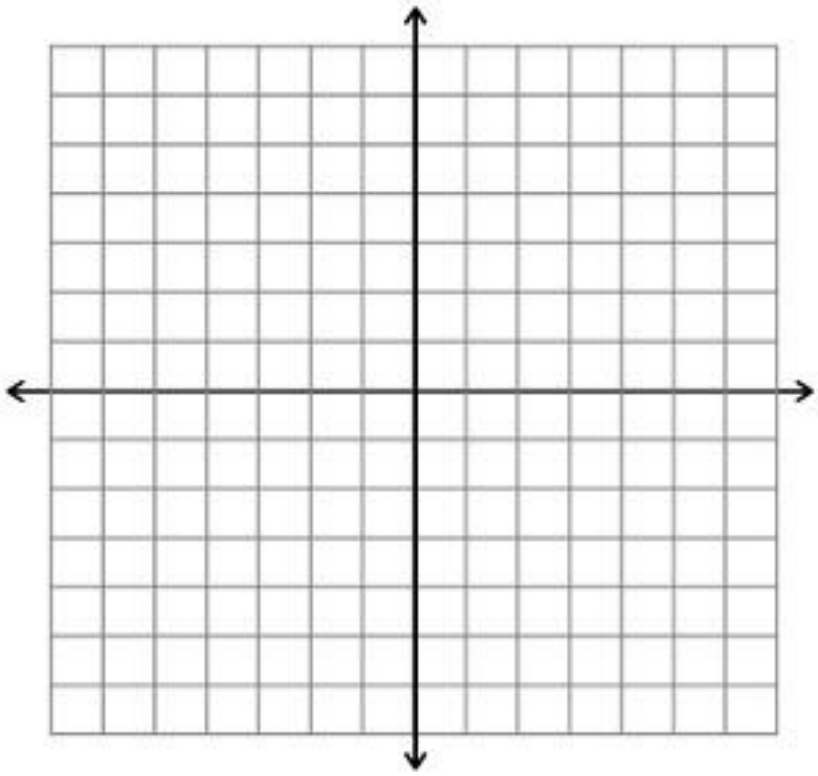
**Example** Graph $f(x) = 2^x, g(x) = e^x, h(x) = 3^x$ all on the same graph below for comparison:



Translations work similarly to other graphs we have discussed. Using the graph of $f(x) = 2^x$ as a "base graph", draw the following graphs together on the same graph: $f(x) = 2^x, g(x) = 2^x + 3, h(x) = 2^{x-1}$





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**Lesson: Solving Exponential Equations**

In this lesson, we will solve equations that look like this:

$$3^x = 27 \qquad \text{or} \qquad 4^{x+3} = 8^{x-2}$$

To do this, we will write directions around the following example: $2^{3x} = 8^{x+1}$

**Examples** Solve:

a. $49^{x-3} = 7^{2x}$

b. $9^{4x} = 27^{x+4}$



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**Examples** Solve:

a. $5^{x-2} = \frac{1}{25}$

b. $\left(\frac{3}{5}\right)^{2x} = \left(\frac{27}{125}\right)^{x+2}$

c. $27^{3x+1} = 9^{4x+2}$



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**Review: Exponents for Calc/Pre-calc Students**

The 3 Main Rules:

**Examples**

a. $(3x^4y^7)^2 * (x^7y^3)$

b. $\frac{(-2x^3y^6)^3}{(3xy^2)^4}$



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**Examples**

a. $(4x^3y^7z^{10})^{-3}$

b. $\left(\frac{3x^{-4}y^{-7}}{x^3y^{-10}}\right)^{-2}$



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**Examples**

a. $\sqrt[3]{x^2} * \sqrt{x}$

b. $\frac{x^{\frac{1}{2}}x^{\frac{1}{3}}}{x^{\frac{1}{4}}}$



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**Lesson: Determining if Functions are One-to-One**

Before we begin, you should already be familiar with the concept of functions and the vertical line test. First let's do some review.

**Review** Determine whether the following relations are functions, then determine the domain and range.

a. $\{(2,3),(5,7),(2,4)\}$

b.

Now we will discuss a specific type of function: a one-to-one function. Define this below:

How can we visualize this?



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**Example** Determine if any of the following are i) functions and ii) one-to-one functions

a. $\{(2,4),(-1,7),(3,7)\}$

b. $\{(3,6),(-1,2),(-3,-4)\}$

Similar to how we have a test to determine if a graph represents a function, we have another test to determine if a function is one-to-one.

Describe the horizontal line test below:

**Examples** Determine whether the following graphs represent i) functions and ii) one-to-one functions (draw your own):

a. b. c.



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**Lesson: Finding the Inverse of Functions**

One-to-one functions have inverses. Describe what the inverse of a function is:

**Example** Find the inverse of the following function: $\{(1,4),(7,8),(3,-2)\}$

Describe the steps to find the inverse of a function:



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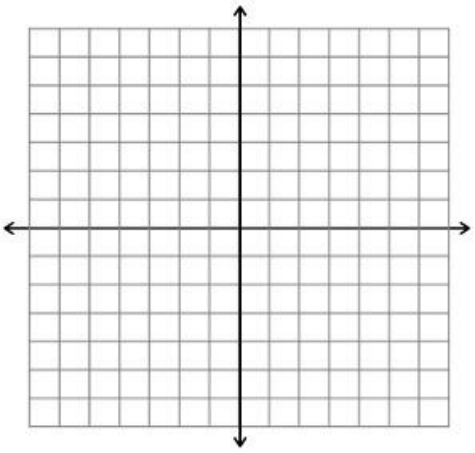
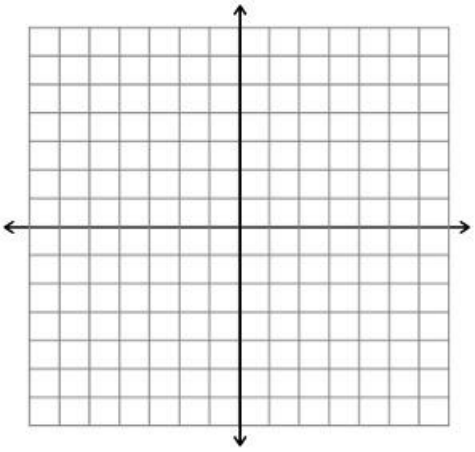
**Examples** Find the inverse of the following one-to-one functions. Then plot the original function on the graph with its inverse.

a. $f(x) = 3x + 1$



b. $g(x) = -\frac{1}{2}x + 4$



What is the relationship of a graph with its inverse?



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**Examples** Find the inverse of the following one-to-one functions.

a. $h(x) = \sqrt[3]{x}$

b. $f(x) = x^2, x \geq 0$



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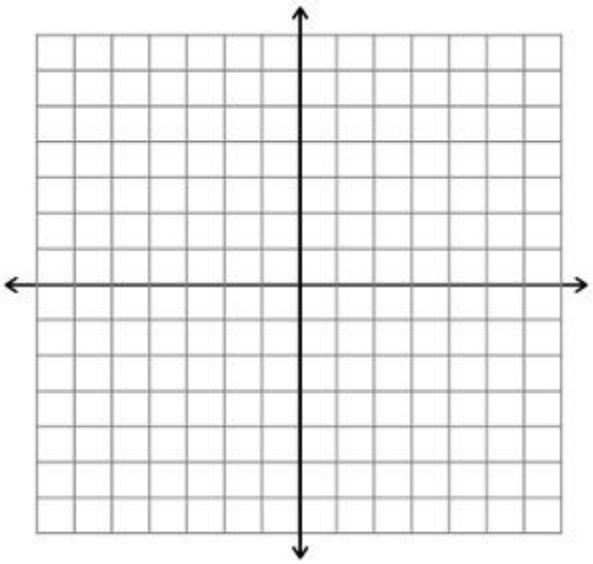
**Lesson: Intro to Logarithmic Functions**

First, we want to review how to function inverses of functions. Find the inverse of: $f(x) = 2x + 8$

Graph the exponential function $f(x) = 3^x$ below:



Recall the exponential functions are one-to-one, which means that they have inverses. Attempt to find the inverse of $f(x) = 3^x$ below:



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As we can see, there is an issue with finding the inverse – we have no way to solve for y! As such, we need new mathematics to help solve this.

Write down the definition of a logarithm:

**Examples** Write the following in exponential form:

a. $\log_3 81 = 4$

b. $\log_2 \frac{1}{8} = -3$

c. $\log_5 125 = 3$

d. $\log_9 9 = 1$

**Examples** Write the following in logarithmic form:

a. $2^4 = 16$

b. $3^0 = 1$

c. $6^{-2} = \frac{1}{36}$

d. $8^{\frac{1}{3}} = 2$



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Now we can finally determine what is the inverse of $f(x) = 3^x$:

There are a few common logarithms:

- $\log x$ means:

- $\ln x$ means:

There are also 2 key properties of logarithms:

1. $\log_a a = 1$: explain why this makes sense

2. $\log_a 1 = 0$: explain why this makes sense:



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**Lesson: Basics of Solving Logarithmic Equations and Evaluating Logarithms**

We want to solve equations that look like:

$$\log_3 x = 4 \qquad \text{or} \qquad \log_x 36 = 2$$

Before you begin, you should already be able to do the following:

a. Put $3^5$ in logarithmic form:

b. Put $\log_5 5 = 1$ in exponential form:

Now we can explain how to solve logarithmic equations using the example $\log_3(2x + 1) = 3$

**Example** Solve: $\log_x 16 = 4$



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**Example** Solve:

a. $\log_9 x = 2$

b. $\log_6(2x + 4) = 2$

Now we can also discuss how to evaluate logarithms. Explain how to evaluate: $\log_3 27$:

**Examples** Evaluate:

a. $\log_5 \frac{1}{25}$

b. $\log 100$



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**Examples** Evaluate:

a. $\ln e$

b. $\log_3 27$

c. $\log_{25} 5$

d. $\log \frac{1}{10}$



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**Lesson: How to Graph a Logarithm**

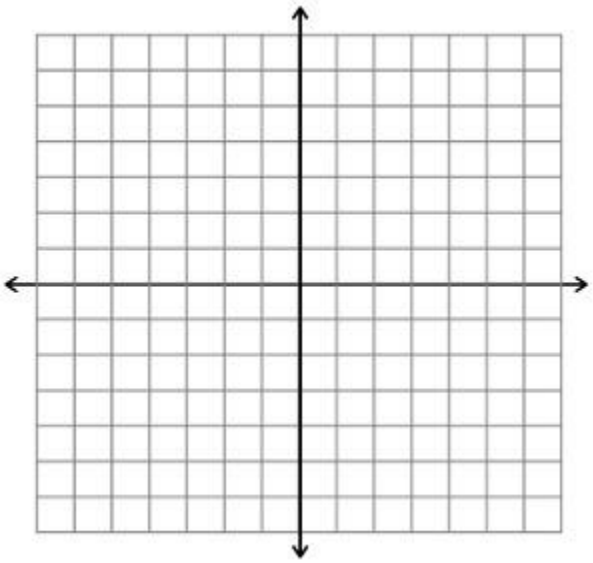
Before we begin, you should be able to do the following:

a. Convert to logarithmic form: $25 = 5^2$

b. Convert to exponential form: $\log_2 8 = 3$

c. Find the inverse of $y = 2^{\mathrm{x}}$

Now we can explain how to graph $y = \log_3 x$



For comparison, also graph $f(x) = 3^x$ on the same graph above.



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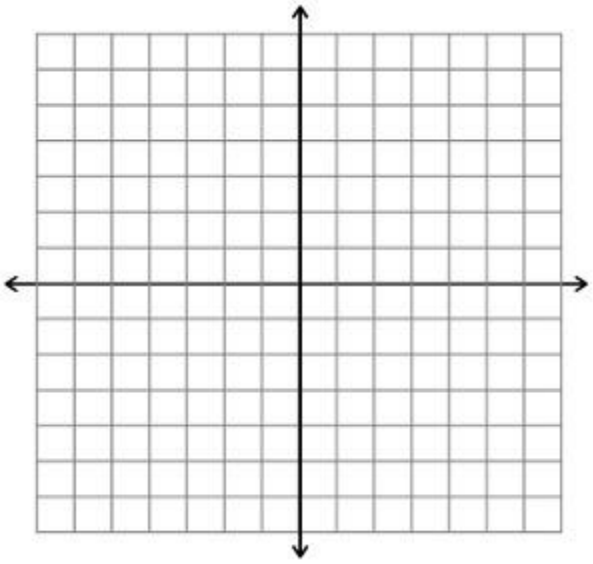
**Example** Graph $y = \log_{\frac{1}{2}} x$ below:





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**Review: Logarithms and Properties of Logarithms for Precalc/Calc Students**

Write down the definition of the logarithm:

**Example** Write as a logarithm:

a. $2^3 = 8$

b. $25^{\frac{1}{2}} = 5$

**Example** Write in exponential form:

a. $\log_7 49 = 2$

b. $\log_3 1 = 0$

Write down the common logarithms and what they mean:

Write down all the properties of logarithms:



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**Example** Expand as much as possible:

a. $\log 3xy^7$

b. $\log \frac{10x^3}{y^{10}z}$

c. $\ln \sqrt[3]{2xy^4}$

**Example** Write as a single logarithm:

a. $\ln 3 - 2\ln x$

b. $4\ln 2 + 5\ln y - 3\ln x$

c. $\frac{1}{2}\log 4 + \frac{1}{2}\log y^3 - 7\log y$



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**Review: How to Solve Logarithmic Equations**

What are the 5 main properties of logarithms?

When solving logarithmic equations, what else should you remember about logarithms?

**Examples**

a. $\log x + \log x^2 = 3$

b. $\log_5(3x + 5) + \log_5(x + 1) = 1$



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c. $\log(5x-6)-\log(x+1)=\log\ 3$

d. $\log_2(5+4x)+\log_2(x+3)=\log_2 3$



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**Examples: Finding Average Rates of Change**

Calculate the average rate of change of the functions over the given intervals.

a. $f(x) = x^2 + 3x; [0, 3]$

b. $f(x) = 2 + \sin x; \left[\frac{\pi}{2}, \ \pi\right]$

c. $f(x) = \sqrt{4x + 1}; [0, 2]$



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**Examples: Determining the Slope at a Point and the Tangent Line**

Find the slope of the curve at the given point and find the equation of the tangent line at that point. Do this only using the formula for the slope of the secant line.

a. $f(x) = x^2 - 1, (2, 3)$

b. $f(x) = x^2 - 4x; (2, -4)$

c. $y = \frac{1}{x}; (2, \frac{1}{2})$



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**Lesson: An Intro to Limits**

Graph $f(x) = \frac{(x^2-4)}{x-2}$ below. What does it look like this graph should equal when x=2?

Explain this previous question now in mathematical terms

**Example** Find $\lim_{x\to 1} f(x)$ when $f(x) = \begin{cases} x^2+1 & x \neq 1 \\ -2 & x = 1 \end{cases}$. Draw a picture of this situation.

What is the value of $f(1)$?



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**Example** What is $\lim_{x \to 2} f(x)$ when $f(x) = x$? Draw a graph to accompany your answer.

What do we call the function $f(x) = x$?

What can we say about the limit of these types of functions?

**Example** What is $\lim_{x \to -3} f(x)$ when $f(x) = 4$? Draw a graph to accompany your answer.

What do we call the function $f(x) = c$, where c is a constant?

What can we say about the limit of these types of functions?



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Write out all the limit laws on this space:

Write out the other 2 handy theorems:

Theorem 1: Limits of Polynomials

Theorem 2: Limits of Rational Functions



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**Example** Using the limit laws, calculate the following limits:

a. $\lim_{x\to 3}(2x+1)$

b. $\lim_{x\to 1}\frac{5x+2}{x-4}$

c. $\lim_{x\to 2}\sqrt{5x+4}$

d. $\lim_{x\to -3}5\sqrt{2-3x}$



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**Examples: Intro to Calculating Limits**

Calculate the following limits:

a. $\lim_{x\to 3}(x^2+2x+1)$

b. $\lim_{x\to 0}(8-2x)^{\frac{2}{3}}$

c. $\lim_{x\to \frac{2}{5}}(5x+2)(3x-1)$

d. $\lim_{x\to \frac{1}{3}}\sqrt[3]{3x+7}$



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**Lesson: One-Sided Limits**

Use the following function to describe one-sided limits: $f(x) = \begin{cases} 3, & x > 2 \\ 1, & -1 \leq x \leq 2 \\ -2, & x < -1 \end{cases}$

Evaluate the following:

a. $\lim_{x \to 2^+} f(x)$

b. $\lim_{x \to 2^-} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to -1^+} f(x)$

e. $\lim_{x \to -1^-} f(x)$

f. $\lim_{x \to -1} f(x)$



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Draw the graph presented in the video then evaluate the following:

a. $\lim_{x\to 3^+} f(x)$

b. $\lim_{x\to 3^-} f(x)$

c. $f(3)$

d. $\lim_{x\to -2^+} f(x)$

e. $\lim_{x\to 4^-} f(x)$

f. $\lim_{x\to 4^+} f(x)$

g. $\lim_{x\to 4} f(x)$

h. $f(4)$



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Write down the theorem that explains when $\lim_{x\to c} f(x)$ exists:

**Examples** Evaluate the following:

a. $\lim_{x\to 2^+} \sqrt{\frac{x-2}{x+1}}$

b. $\lim_{x\to 2^-} \left(\frac{1}{x+1}\right)\left(\frac{x-3}{x}\right)\left(\frac{4}{4-x}\right)$

c. $\lim_{x\to -3^-} \frac{(x+1)|x+3|}{x+3}$



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**Examples: Basic One-Sided Limits**

a. $\lim_{x\to 2^+} \sqrt{x^2-4}$

b. $\lim_{x\to 0^+} \frac{\sqrt{2x^2+3x+1}-1}{x}$

c. $\lim_{x\to 3^-} \begin{cases} 4, & x > 3 \\ -2, & x \leq 3 \end{cases}$

d. $\lim_{x\to 5^-} \frac{x-5}{|x-5|}$



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**Examples: One-Sided Limits with Absolute Value**

a. $\lim_{x\to 2^+} \frac{2-x}{|x-2|}$

b. $\lim_{x\to 5^-} \frac{(x-4)|x-5|}{x-5}$

c. $\lim_{x\to 2^+} \frac{\sqrt{3x}(x-2)}{|x-2|}$

d. $\lim_{x\to 0^-} \frac{\sin x}{|\sin x|}$



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**Lesson: The $\boldsymbol{lim_{x\to 0}\frac{sin\,x}{x}=1}$**

This lesson is about the tricky but very important limit:

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Or equivalently:

$$\lim_{x\to 0}\frac{x}{\sin x}=1$$

The proof is interesting and worth noting, you can write it down if you wish but you should minimally attempt to understand the reasoning behind it. It's difficult, but difficult things will expand your mind!



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**Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\, x}{x} = 1}$**

a. $\lim_{x\to 0}\frac{\sin 3x}{5x}$

b. $\lim_{x\to 0} 12x^2(\cot 2x)\,(\csc 3x)$

c. $\lim_{x\to 0}\frac{x^3-x+\sin 5x}{3x}$

d. $\lim_{x\to 0}\frac{\sin x}{\sin 3x}$



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**Step-by-Step Examples: Limits using $\boldsymbol{lim_{x\to 0}\frac{sinx}{x}=1}$**

a. $\lim\limits_{x\to 0}\frac{\sin 7x}{x}$

b. $\lim\limits_{x\to 0} x\cot 3x$



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**More Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\, x}{x} = 1}$**

a. $\lim_{x\to 0}\frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)$

b. $\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}$

c. $\lim_{x\to 0}\frac{1-\cos 2x}{x}$

d. $\lim_{x\to 0}\frac{\sin 3x}{\mathrm{x}\cos 5x}$



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Examples: Troubleshooting with $\lim_{x\to 0}\frac{\sin x}{x}$ problems – examples of what not to do

a. $\lim_{x\to 0}\frac{\tan 5x}{x}$

b. $\lim_{x\to 0}\frac{\sin 3x}{\sin 7x}$

**Lesson: Troubleshooting Limits With Algebraic Tricks (When You Get 0/0)**

Trick #1 (name this trick here):

Note what happens if we try to evaluate $\lim_{x\to 2} \frac{x^2-4}{x-2}$

Draw what this situation looks like graphically.

How can we solve this problem?

**Example** Evaluate $\lim_{x\to 3} \frac{(x^3-27)}{x-3}$



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Trick #2 (name it):

**Examples** Evaluate:

a. $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$

b. $\lim_{x\to 1} \frac{2-\sqrt{x^2+3}}{x-1}$



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Trick #3 (name it):

**Examples** Evaluate

a. $\lim_{x\to 3} \frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{x-3}$

b. $\lim_{x\to 1} \frac{\left(1-x^{-1}\right)}{1-x}$



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**Examples: Algebraic Tricks for Limits**

Evaluate the following:

a. $\lim_{x\to -4} \frac{\left(\frac{1}{\sqrt{12+x}}-\frac{1}{3}\right)}{x+4}$

b. $\lim_{x\to 0} \frac{\left(\frac{1}{3+x}-\frac{1}{3}\right)}{x}$

c. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$



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**Lesson: Limits that Do Not Exist**

List the types of limits that do not exist, as well as an example and graph



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**Lesson: The Sandwich Theorem and Basic Trig Limits**

Explain the idea of the Sandwich Theorem:

State the Sandwich Theorem

**Example** Suppose there is a function $f(x)$ where $3 - x^2 \leq f(x) \leq 3\cos x$. Find $\lim_{x \to 0} f(x)$.



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Show how to use the Sandwich Theorem to evaluate the following limits:

a. $\lim_{x \to 0} \sin x$

b. $\lim_{x \to 0} \cos x$



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**Lesson: The Precise Definition of a Limit**

What is the issue with the idea of limits?

Explain the example that went with this (it's better to watch this then write this out after):

State the Precise Definition of a Limit



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**Examples: The Precise Definition of a Limit with a Specific Epsilon**

Given the function $f(x)$, a point $c$, and a specific $\epsilon$, find $L$ and the specific $\delta$ that fulfills the definition of a limit.

a. $f(x) = 3x + 1, c = 2, \epsilon = 0.01$

b. $g(x) = x^2 + 1, c = 1, \epsilon = 0.1$

c. $f(x) = \sqrt{1 - 4x}, c = -2, \epsilon = 0.02$



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**Examples: The Precise Definition of a Limit with a Specific Epsilon**

Given the function $f(x)$, a point $c$, and a specific $\epsilon$, find $L$ and the specific $\delta$ that fulfills the definition of a limit.

$$f(x) = \frac{1}{x}, c = 1, \epsilon = 0.01$$



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**Examples: The Precise Definition of a Limit with a No Specific Epsilon**

Find the specific $\delta$ for each limit that would correspond to the precise definition of a limit.

a. $\lim_{x\to 1} \sqrt{5-x} = 2$

b. $\lim_{x\to 1} f(x) = 3$ when $f(x) = \begin{cases} 2x+1, & x \neq 1 \\ 6, & x = 1 \end{cases}$

c. $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$



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**Lesson: Continuity**

The idea of continuity is probably intuitive at the point. Draw the example in the video to demonstrate continuity.

For a point c, define when a function is:

1. Continuous

2. Right – continuous

3. Left – continuous



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Draw the graph in the video to demonstrate these ideas.

**Example** Draw the graph in the video, then describe the continuity at the following points:

x = 1 x = 3 x = -4 x = -2



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A function is continuous at a point c is it meets these 3 conditions:

1. f(c) exists

2. the limit $\lim_{x \to c} f(x)$ exists

3. $\lim_{x \to c} f(x) = f(c)$

Describe the different kinds of discontinuities below:

What does it mean for a function to be continuous?

If a function is not continuous, then it is _____

If $f(x)$ and $g(x)$ are continuous functions at the point x=c, then the following are continuous at c:

1. sum/differences:

2. constant multiples (for any number k):

3. products:

4. quotients:

5. powers:

6. roots:

7. compositions:



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**Example** Describe where the following functions are continuous:

a. $f(x) = \frac{x+3}{x-2}$

b. $f(x) = \sqrt{2x}$

What can we say about the limits of a composition of continuous functions $f(x)$ and $g(x)$?

**Example** Find the limit $\lim_{x \to 0} (x + 2)e^{\sin x}$



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**Examples: Continuity**

Determine where the following functions are continuous:

a. $f(x) = \frac{1}{(x+3)^2} - 2$

b. $g(x) = \begin{cases} 1 + x, & x \geq 3 \\ 4, & -2 \leq 0 < 3 \\ 2 - x, & x < -2 \end{cases}$

c. $h(x) = \begin{cases} |x + 1|, & x \geq -2 \\ -5, & x < -2 \end{cases}$



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**Examples: Using Continuity with Limits**

Find the limits.

a. $\lim_{x\to 0} \cos\left(x + \cos(\frac{\pi}{2} + x)\right)$

b. $\lim_{x\to \frac{\pi}{2}} \sin(\cos(\tan x))$

c. $\lim_{x\to 0} \sin^{-1}\left(e^{\cos\left(\frac{\pi}{2}+x\right)}\right)$



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**Lesson: The Intermediate Value Property and Continuous Functions**

What is the Intermediate Value Property?

There are 2 consequences of this:

1. Connectedness:

2. Root Finding:



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**Example** Determine the value of a that would make the function continuous at every x:

$$f(x) = \begin{cases} x^2 + 1, & x > 2 \\ ax, & x \leq 2 \end{cases}$$

**Example** Show that $x^3 - 14x + 2 = 0$ has 3 solutions in the interval $[-4, 4]$



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**Examples: Continuous Extensions**

Determine the value of a that would make the following functions continuous:

a. $f(x) = \begin{cases} 2x, & x > 1 \\ ax^2, & x \leq 1 \end{cases}$

b. $g(x) = \begin{cases} \frac{2x+a}{a+2}, & x > 0 \\ x^2 + a, & x \leq 0 \end{cases}$



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**Step-by-Step Examples: Continuous Extensions**

Use the space below to find a (or a and b) to make the following functions continuous:

a. $f(x) = \begin{cases} x + a, & x \geq 1 \\ 3, & x < 1 \end{cases}$

b. $f(x) = \begin{cases} 3, & x \geq 1 \\ ax + b, & -2 < x < 1 \\ -3, & x \leq -2 \end{cases}$



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**Lesson: Finite Limits as $x \to \infty$ and Limits That Go To Infinity, the Conceptual Idea**

Recreate the graph below, then find the following limits:

a. $\lim_{x\to\infty} f(x)$

b. $\lim_{x\to-\infty} f(x)$

c. $\lim_{x\to3^+} f(x)$

d. $\lim_{x\to2^-} f(x)$

e. $f(2)$

f. $\lim_{x\to2^+} f(x)$

g. $\lim_{x\to3} f(x)$

h. $\lim_{x\to-4} f(x)$



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**Lesson: Finite Limits as $x \to \infty$ and Rational Functions**

Do the limit laws apply to limits as $x \to \infty$? Yes No

How do you evaluate limits at infinity for rational functions?

In the following examples, we will see all the different ways this trick might be carried out:

**Examples** Evaluate the following:

a. $\lim_{x\to\infty} \frac{4x+3}{x^2+4x+3}$

b. $\lim_{x\to-\infty} \frac{5x^2+3x+1}{7x^2+5x-4}$

c. $\lim_{x\to\infty} \frac{6x^2+3x+1}{x+2}$



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This trick can be extended.

**Examples**

a. $\lim_{x\to\infty} \sqrt{\frac{x^2+5}{3x^2-6}}$

b. $\lim_{x\to-\infty} \frac{x^{\frac{2}{3}}-x^{\frac{1}{3}}+2}{x^2+3}$

c. $\lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+3}}$



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**Examples: Limits as $x \to \infty$**

Find the horizontal asymptotes / evaluate the following:

a. $\lim_{x\to\infty} \frac{\sin x}{x^2}$

b. $\lim_{x\to\infty} \frac{7x^4-2x^3+8}{9x^4+2x^2+x-1}$

c. $\lim_{x\to-\infty} \frac{\sqrt{25x^2+x}}{x}$



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d. $\lim_{x\to\infty}\left(x-\sqrt{x^4+2x^2}\right)$

e. $\lim_{x\to-\infty}(5x^7+2x^3+8)$

f. $\lim_{x\to-\infty}\frac{x^3+2x+5}{x^2+3x+1}$



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**Example: Limits as x approaches infinity**

a. $\lim_{x\to-\infty} \frac{2x^6+3x^2-1}{6x^3-2x+1}$

b. $\lim_{x\to\infty} \frac{5x^{\frac{8}{3}}-3x^{\frac{5}{3}}+2}{x^{\frac{13}{5}}+2x-x^{\frac{1}{2}}}$



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**Example: Comparing the Conjugate Trick as $x \to \infty$**

a. $\lim_{x\to\infty}\left(x^2-\sqrt{x^4+3}\right)$

b. $\lim_{x\to\infty}(x^2-\sqrt{x^6+3x^4})$

c. $\lim_{x\to-\infty}(5x+\sqrt{25x^2+5x-9})$



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**Lesson: Horizontal Asymptotes and Limits as $x \to \infty$**

Define a Horizontal Asymptote in terms of limits:

**Examples** Find the horizontal asymptotes of the following:

a. $f(x) = \frac{|x|^3+2}{x^3-1}$

b. $f(x) = 3^x$

c. $f(x) = \frac{3x+1}{5x+2}$



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Sometimes we can have horizontal asymptotes in surprising places.

**Example** Find the horizontal asymptotes of $x - \sqrt{x^2 + 4}$

**Example** Use the Sandwich Theorem to find any horizontal asymptotes of $y = 3 + \frac{\sin x}{x}$



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**Lesson: Infinite Limits**

To begin this conversation, we investigate the graph of $f(x) = \frac{1}{x}$. Graph this below then evaluate the following:

a. $\lim_{x \to 0^+} f(x) =$

b. $\lim_{x \to 0^-} f(x) =$

c. $\lim_{x \to 0} f(x) =$

Show how to use a table to prove this result:



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Now compare this to $g(x) = \frac{1}{x^2}$. Graph this below:

a. $\lim_{x \to 0^+} g(x) =$ b. $\lim_{x \to 0^-} g(x) =$ c. $\lim_{x \to 0} g(x) =$

Show how to use a table to prove this result:



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Define vertical asymptotes using limits:

**Example** Evaluate the following and state any vertical asymptotes.

a. $\lim_{x\to 2^+} \frac{1}{x-2}$

b. $\lim_{x\to 4} \frac{3}{(x-4)^2}$



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**Examples: Infinite Limits**

Evaluate the following:

a. $\lim_{x\to 0^+} \frac{1}{x^{\frac{1}{5}}}$

b. $\lim_{x\to 0^-} \csc x$

c. $\lim_{x\to 5} \frac{1}{x^2(x-5)}$

d. $\lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}}$



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**Lesson: Using Substitution with Limits as $x \to \infty$**

In this lesson, we are going to look at a few limits that end up using clever substitutions.

**Examples** Evaluate the following:

a. $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$

b. $\lim_{x\to 0^-} e^{\frac{1}{x}}$



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**Examples: Horizontal and Vertical Asymptotes**

Find any horizontal, vertical, or oblique asymptotes

a. $f(x) = \frac{x^2-5x+6}{x-2}$

b. $f(x) = \frac{\sqrt{25x^2+100}+x^2}{5x^2-25}$

c. $f(x) = \frac{x^2}{x-1}$



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**Example: Horizontal, Oblique, and Vertical Asymptotes**

Find any horizontal, oblique, or vertical asymptotes.

a. $f(x) = \frac{\sqrt{x^2+1}}{x}$

b. $g(x) = \frac{x^2-4}{x-3}$

c. $y = \frac{3x}{x^2-1}$



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**Lesson: The Formal Definition of the Derivative**

One of the things we learn in pre-calculus is about secant lines. You can write a review of this below:

When the distance between two points for a secant line goes to zero, what do we get?

What is the formal definition of the derivative?

What is the alternate formula for the derivative?

Which formula should you use?



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How do you find the derivative at a specific point?

What 4 things does the derivative at a point represent?

1.

2.

3.

4.

**Example** Calculate the derivative of $f(x) = x^2 + 2$

**Example** What is the value of the derivative of $f(x) = x^2 + 2$ at $(1, 3)$?



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**Example** Find the equation of the tangent line for $f(x) = x^2 + 2$ at $(1, 3)$

List all the different ways to represent a derivative below:

Write out the theorem that discusses differentiability and continuity

What types of functions will not have a derivative at a point?



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**Examples: Calculating Derivatives Using the Formal Definition Only and Finding Tangent Lines**

Use the formal definition of a derivative to calculate the follow derivatives of each function. Then determine the tangent line at the given point.

a. $f(x) = 4 - x^2; (-1, 3)$

b. $y = \frac{1-x}{2x}, (1, 0)$

c. $g(x) = x^{\frac{3}{2}}, (4, 8)$



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**Examples: Using the Alternative Formula for Derivatives**

Use the alternative formula for derivatives to calculate the derivative of the following function. Compare this calculation to the original formula.

a. $y = x^3 - x^2$

b. $y = \frac{1}{x+3}$

c. $f(x) = 2 + \sqrt{x}$



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**Lesson: The Derivative as a Rate of Change**

We've talked about that the derivative at a point $x_0$ represents:

- the slope of the curve at $x_0$
- the derivative at $x_0$
- the slope of the tangent line at $x_0$
- the instantaneous rate of change at $x_0$

We will focus on the derivative as a rate of change in this lesson.

Often times, we omit the word "instantaneous" when we talk about this.

What represents position?

What is displacement?

What is average velocity?

How do we find velocity (instantaneous velocity)?

How is velocity different from speed?

How do you find speed?



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How do you find acceleration?

What comes after acceleration?

If I move at a constant speed, what would the function and graphs that model this situation from a position, velocity, and acceleration standpoint?



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If I ride down a hill, what would the function and graphs that model this situation from a position, velocity, and acceleration standpoint?

Now let's model the situation where I drop a tennis ball from a height of 6ft (1.8m). The equation that models this situation is $s(t) = -4.9t^2 + 1.8$

a. When do the ball hit the ground?

b. When the ball hits the ground, what was its velocity? What about its speed?

c. What is the acceleration of the ball at any time?



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**Examples: Position, Velocity, and Acceleration**

Let $s = -t^3 + 9t^2 - 24t$ represent the position of a moving object, in feet, at time t, in seconds.

a. Find the acceleration of the object when the velocity is 0.

b. What is the speed of the object when the acceleration is 0?

c. What is the total distance traveled from t=0 to t=7?



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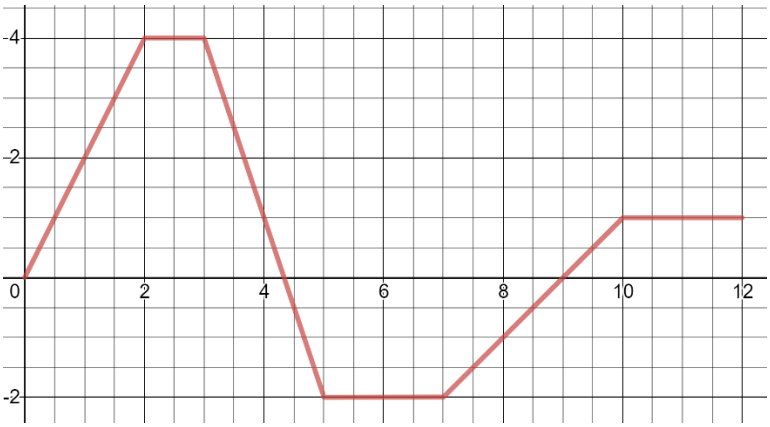
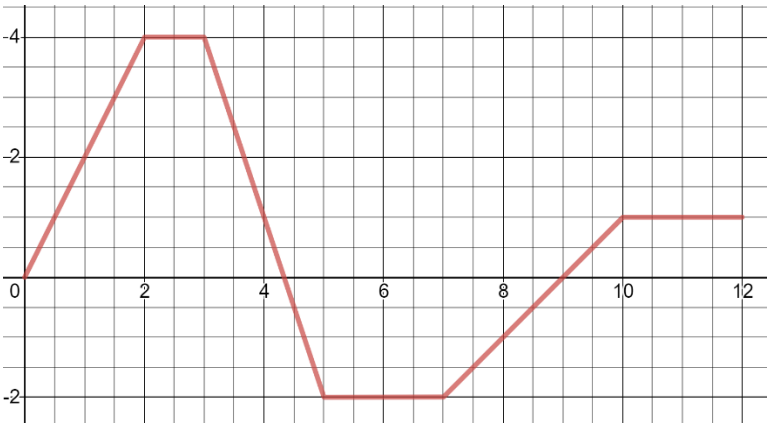
**Examples: Position, Velocity, and Acceleration from a Graph**

This is the graph of velocity for an object. The y-axis is m/s, the x-axis is seconds; the graph is from desmos.com. I have reprinted the graph for each question is case you want to mark it up.




When does the object reverse its direction?




When is the object going a constant speed?



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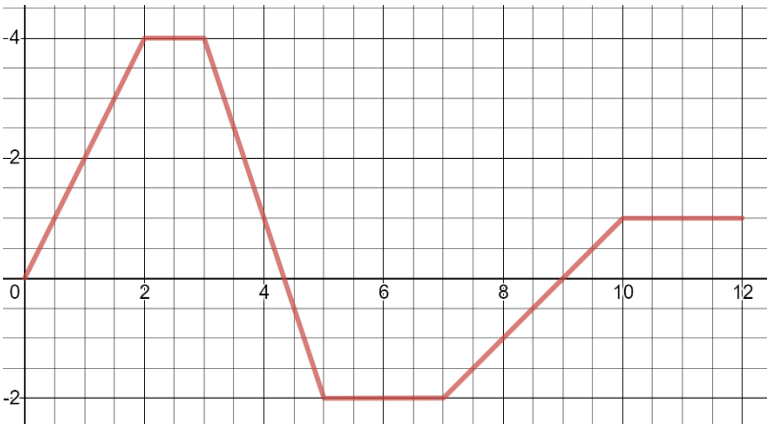
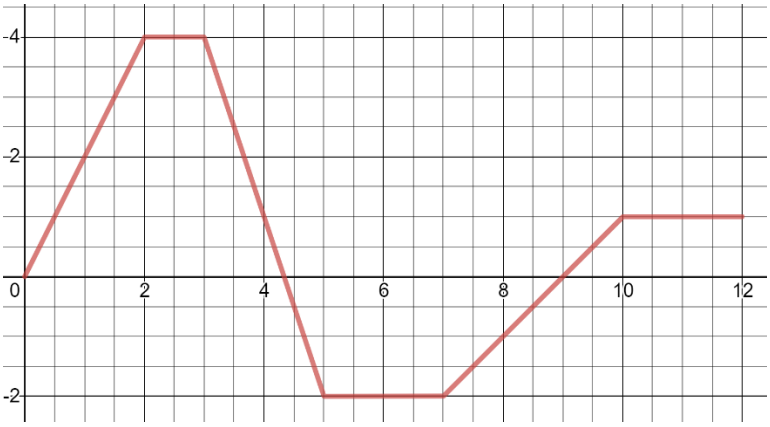



What would the graph of the object's speed look like?




What would the graph of the object's acceleration look like?



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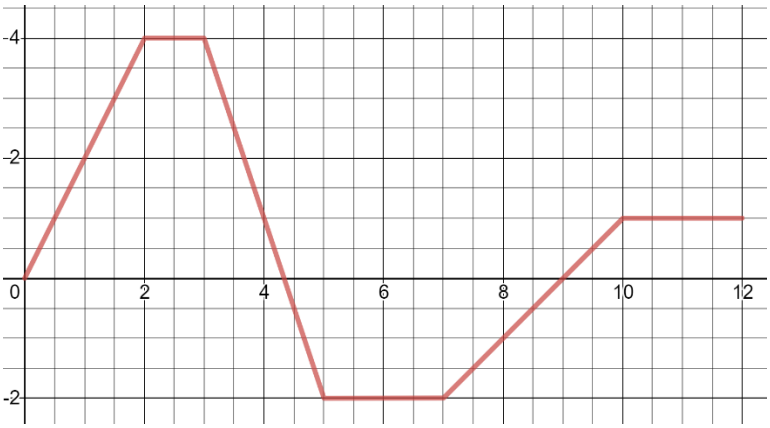
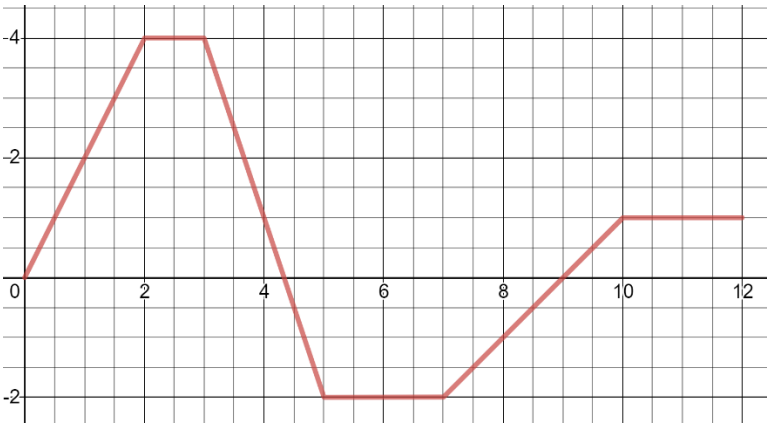



When is the object's acceleration …

Positive?

Negative?

Constant?




When does the object attain its greatest speed?



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**Step-by-Step Examples: Comparing the Definition of the Derivative to the Alternative Formula**

a. $f(x) = x + \frac{3}{x}$

b. $g(x) = x^3 - 5x^2 + 1$



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**Example: Finding the Derivative of a Square Root with the Definition of the Derivative**

Find the tangent line at (6,5) for $y = \frac{5}{\sqrt{7-x}}$ using only the definition of the derivative.



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**Lesson: Other Notes about the Derivative**

Define right-hand derivatives

Define left-hand derivatives

Explain how $y = |x|$ is differentiable on each side



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Interpreting Graphs of Derivatives

The derivative of $f(x) = x^2$ is $f'(x) = 2x$. Compare their graphs below:



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**Lesson: Basic Differentiation Rules**

Now we will introduce the basic differentiation rules, which will make life much easier. Let $f(x)$ and $g(x)$ be differentiable functions. Then:

1. Derivative of a Constant Function

**Example** Find the derivative of $f(x) = 6$

2. Power Rule

**Example** Find the derivative of $f(x) = x^{\frac{3}{2}}$

3. Constant Multiple Rule

**Example** Find the derivative of $f(x) = 4x^7$

4. Sum Rule

**Example** Find the derivative of $f(x) + 5x^2 + 2$



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5. The Natural Exponential Function

**Example** Find the derivative of $f(x) = 5e^x - 7\sqrt{x}$

6. Product Rule

**Example** Find the derivative of $f(x) = (3x^2 - 5)e^x$

7. Quotient Rule

**Example** Find the derivative of $f(x) = \frac{5x}{e^x}$



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**Examples: Finding Tangent Lines**

a. Find the line tangent to $y = 4x^2 + 3$ that has a slope of $16$. State the point that this occurs at on the graph, and also state the equation of the tangent line.

b. Find the equation of any tangent lines perpendicular to the x-axis for $f(x) = x^3 - 4x^2 + 4$

c. Find the equation of the line that is tangent to the curve $f(x) = e^x$ and goes through the origin.



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**Examples Finding a Point Tangent to One Graph, Perpendicular to Another Line**

Find all the points on $f(x) = \frac{x}{x-2}$ where the tangent lines are perpendicular to $f(x) = 2x + 7$



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**Lesson: Higher Derivatives**

How do you find the second derivative?

Notation for the second derivative:

How do you find and notate higher derivatives?

**Examples** Find the 4th derivatives of the following:

a. $y = x^7 - 5x^3 + 2x$

b. $f(x) = e^x - 7x^2 + x^3$



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**Example: The Most Basic Derivative Examples: Constant, Constant Multiple, Power Rule, and Sum Rule**

Take the derivative of the following only using the constant, constant multiple, power, or sum rules:

a. $f(x) = 5x^7 - 3x^5 + \pi x^2$

b. $g(x) = 3x^{-5} - 2\sqrt{x} + \frac{1}{2}x^{\frac{9}{5}} + x^{\pi}$

c. $y = \frac{x^5 - 3x^2 - 5}{x^2}$

d. $y = (2x + 1)(5x - 4)$



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**Example: The Product and Quotient Rules**

Take the derivative of the following only the product and quotient rules. Do not simplify these expressions, the point is to ues the rules.

a. $f(x) = (3x^2 - 6x)(2x + 1)$

b. $g(x) = \frac{3x^7}{2x^5}$

c. $y = \left(5\sqrt{x} + 3x^{\frac{4}{5}}\right)(7e^x)$

d. $y = \frac{(\sqrt{x}-1)(x+1)}{(\sqrt{x}+1)(x-3)}$



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**Step-by-Step Examples: The Quotient Rule**

Find the derivatives:

a. $y = \frac{1+x+6\sqrt{x}}{x}$

b. $y = \frac{\sqrt{x}+1}{\sqrt{x}-1}$



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**Examples: The Quotient Rule for Derivatives Onlyl**

Take the derivatives:

a. $f(x) = \frac{\sin x}{\sqrt{x}}$

b. $y = \frac{9\cos x - 2\tan x}{e^x + \sqrt[3]{x} - 8\sin x}$

c. $f(x) = \frac{(5x^7 - 4x^3 + 3x)}{\sin x - x^{\frac{3}{4}}}$



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**Lesson: Derivatives of Trigonometric Functions**

| Name of Function | Function Form | Derivative |
|---|---|---|
| Sine | $f(x) = \sin x$ | |
| Cosine | $f(x) = \cos x$ | |
| Tangent | $f(x) = \tan x$ | |
| Cotangent | $f(x) = \cot x$ | |
| Secant | $f(x) = \sec x$ | |
| Cosecant | $f(x) = \csc x$ | |

**Examples** Find the following derivatives:

a. $f(x) = 3x^7 - \cos x$

b. $g(x) = e^x \sin x$

c. $y = \frac{5}{\cos x} - \frac{3}{\tan x}$



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**Examples: Derivatives of Trig Functions**

*Note: these examples assume you do not know the chain rule, you will strictly use the rules for trig derivatives here.*

Find the derivatives of the following:

a. $f(x) = \sqrt{x}\csc x - \frac{1}{x^2}$

b. $g(x) = \sin^2 x$

c. $y = \frac{\sin x}{\cos^2 x}$

d. $y = x^2 \sin x$



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**Lesson: The Chain Rule**

The Chain Rule focuses on what types of functions?

State the Chain Rule:

**Examples** Take the derivatives of the following functions:

a. $y = (x^2 - 8)^{12}$

b. $f(x) = e^{\sin(5x)}$



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The key to the chain rule is understanding how functions are being composed with one another. So before we start taking any derivatives, we will first focus on understanding compositions of functions.

The basic rules that we have for derivatives are (you should have this memorized, quiz yourself!):

| | |
|---|---|
| Constant Rule $(k)' =$ | Sine Derivative $(\sin x)' =$ |
| Power Rule $(x^n)' =$ | Cosine Derivative $(\cos x)' =$ |
| Constant Multiple Rule $\big(kf(x)\big)' =$ | Tangent $(\tan x)' =$ |
| Sum Rule $\big(f(x) + g(x)\big)' =$ | Cotangent $(\cot x)' =$ |
| Natural Exponential Rule $(e^x)' =$ | Secant $(\sec x)' =$ |
| Product Rule $\big(f(x) * g(x)\big)' =$ | Cosecant $(\csc x)' =$ |
| Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' =$ | |



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The first thing we must do is identify when we actually need the chain rule.

**Example** Determine if the chain rule is needed for each function (is it a composition of the derivatives in the previous list?), and if so, then identify how many compositions of functions there are.

| Function | Chain Rule Needed? | How Many/Which Functions are Being Composed? |
|---|---|---|
| a. $f(x) = \sin(3x)$ | Yes No | |
| b. $y = e^{\sin 4x}$ | Yes No | |
| c. $g(x) = (x^4 - 5)^7$ | Yes No | |
| d. $y = \sin x \tan x$ | Yes No | |
| e. $f(x) = \sin^2(\cos(5x))$ | Yes No | |
| f. $g(x) = (x^2 + 7)^5(3x + 1)^{-2}$ | Yes No | |
| g. $y = \frac{1}{3}\cos x\, e^x$ | Yes No | |



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**Examples** Now take the derivative of each function:

a. $f(x) = \sin(3x)$

b. $y = e^{\sin 4x}$

c. $g(x) = (x^4 - 5)^7$

e. $f(x) = \sin^2(\cos(5x))$

f. $g(x) = (x^2 + 7)^5(3x + 1)^{-2}$



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**Examples: The Chain Rule**

Find the derivatives of the following functions:

a. $y = \left(\frac{1}{3}x^3 - 7x^2 + 1\right)^8$

b. $f(x) = (4x + 1)e^{3x+1}$

c. $y = \cos\left(\tan\left(e^{\sin 4x}\right)\right)$

d. $y = \sin\left(\frac{x}{\sqrt{x+2}}\right)$



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e. $f(x) = x^3 e^{-2x} \sin 5x$

f. $y = 5x(e^{-x} - 4x + 2)^5$

g. $y = \left(\frac{\sqrt{x}-7}{e^{\sin x}}\right)^{-3}$

h. $f(x) = \frac{e^{\sin(4x)}}{x^2 \cos(3x-1)}$



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**Lesson: Help! The Chain Rule is messing with my head!**

The key to the chain rule is understanding how functions are being composed with one another. So before we start taking any derivatives, we will first focus on understanding compositions of functions.

The basic rules that we have for derivatives are (you should have this memorized, quiz yourself!):

| | |
|---|---|
| Chain Rule $(f(g(x))' =$ | Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' =$ |
| Constant Rule $(k)'$ = | Sine Derivative $(\sin x)' =$ |
| Power Rule $(x^n)' =$ | Cosine Derivative $(\cos x)' =$ |
| Constant Multiple Rule $(kf(x))' =$ | Tangent $(\tan x)' =$ |
| Sum Rule $(f(x) + g(x))' =$ | Cotangent $(\cot x)' =$ |
| Natural Exponential Rule $(e^x)' =$ | Secant $(\sec x)' =$ |
| Product Rule $(f(x) * g(x))' =$ | Cosecant $(\csc x)' =$ |



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**Example** Take the derivatives of the following:

a. $f(x) = \sin(7x)$

b. $g(x) = e^{x^2}$

c. $h(x) = (3x + 7)^{21}$

**Example** Take the derivatives of the following:

a. $f(x) = x\sin(7x)$

b. $g(x) = e^{x^2}\tan x + 77$



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**Example** Take the derivatives of the following:

c. $h(x) = \frac{(3x+7)^{21}}{9x}$

d. $y = e^{x^2}\sin(7x)$

e. $f(x) = \frac{e^{x^2}}{(3x+7)^{21}}$

f. $f(x) = \sin\left(7x + e^{x^2}\right)$

g. $f(x) = \tan(3x)\, e^{4x-\cos(3x)}$



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**Examples: The Chain Rule Mixed with Quotient Rule or Product Rule (messy functions)**

Find the following derivatives.

a. $y = \left(\frac{\sin 4x - e^{\tan x}}{\sec(3x-1) - x^7 + 9}\right)^8$

b. $f(x) = \sin(5x+1)\, e^{4x - \csc(x^3 - 2x)}$

c. $y = \left(\frac{\sin(4x)e^{5x} + x^3}{(x-7)^4 \tan^2 x}\right)^9$



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**Step-by-Step Example: The Chain Rule and the Quotient Rule**

Take the derivative: $y = \frac{3x}{(x^3+5)^6}$



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**Examples: Trig Functions in the Chain Rule**

a. $y = \sin\left(\frac{1}{\sqrt{x+7}}\right)$

b. $y = e^{\sin^2(7x+2)}$

c. $y = \frac{(x+3)^7}{\ln(\sin x)}$



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**Examples: The Chain Rule and Second Derivatives**

Find the first and second derivatives of the following functions:

a. $y = \sin 6x^2$

b. $f(x) = xe^{-x}$

c. $y = x(3x + 5)^3$



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**Lesson: Implicit Differentiation**

Not all equations are modeled with y on one side and x's on the other. When you have an equation like:

$$x = y^2,$$

How do you take the derivative?

But there's another way we can take the derivative: implicit differentiation. Show how this works below.



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Let's look at a few other examples of how to do this:

a. $2x + 3y - y^2 = 5$

b. $x^3y^5 = 5$

c. $\sqrt{x + 3y} = 8$



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**Examples: Implicit Differentiation**

Use implicit differentiation to find $\frac{dy}{dx}$:

a. $x^2 = \frac{3-y}{4x+5y}$

b. $x^6 - \cos(xy) = x^2y^5$

c. $y\cos(3y) = 1 - 4xy$



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**Step-by-Step Examples: Implicit Differentiation**

Find $\frac{dy}{dx}$: $5x - 7y = 3x^2y^5$



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**Lesson: Implicit Differentiation, Tangent Lines, and Normal Lines**

Explain where we might encounter a normal line:

Show how to find the tangent and normal line to the curve at the following point:

$x^2 + y^2 = 100$ at $(8, -6)$



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**Examples: Implicit Differentiation, Tangent Lines, and Normal Lines**

Find the tangent and normal line to the curve at the following point:

a. $x^2y^2 = 25, (25, -1)$

b. $x^2 - x + y - y^2 = 4, (-2, 2)$



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**Examples: Implicit Differentiation and Finding Tangent/Normal Lines With Different Characteristics**

a. Find 2 points where $x^2 - xy + y^2 = 5$ crosses the x-axis. Are the tangent lines at these points parallel?

b. Find any normals to the curve $3xy + 4x - y = 0$ that are parallel to $y = -4x + 2$.



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**Lesson: Second Derivatives with Implicit Differentiation**

Find the first and second derivatives of the following:

a. $x^2 + y^2 = 9$

b. $y^2 = x + 3$



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**Examples: Second Derivatives with Implicit Differentiation**

Find the first and second derivatives of the following:

a. $y^2 = e^{5x} - 5x$

b. $3 - \cos y = 4x + x^2$



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**Examples: Implicit Differentiation and Simplifying with a Fraction within a Fraction**

Find $y''$: $3x^2 - 2y^2 = 5$



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**Lesson: Derivatives of Inverse Functions and Logarithms**

Find the inverse of each function:

a. $y = x^2, x > 0$

b. $y = x^3 + 2$

Select a random point for each function. How does that point relate to each function's inverse? Write down your choice of point here and how it would appear in the inverse below:

a. Choice:

b. Choice:

Now take the derivative of each function and the derivative of its inverse below:

a. Derivatives:

b. Derivatives:

What is the relationship between the points you found when you plug them into the derivatives?



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**Write down the derivative rule for inverses below:**

Explain how to leverage this rule to find the derivative of $f(x) = \ln x$



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**Lesson: How to Find the Derivatives of $a^x$ and $log_a x$**

Explain how to find the derivative of $a^x$:

Explain how to find the derivative of $\log_a x$:



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**Example** Find the derivatives of the following:

a. $f(x) = 3^{x^2}$

b. $y = \log_5(3x + 2)$



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**Examples: Derivatives with $e^x, ln\ x, log_a x, a^x$**

Find the derivatives of the following functions:

a. $f(x) = \ln(x^2)$

b. $g(x) = e^{\sin x + \ln x}$

c. $y = 5^{\ln x}$



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d. $f(x) = \log_3(x) + \log_3(x^4)$

e. $g(x) = \ln(\ln(\ln(\ln(4x))))$

f. $y = 7^{\log_2 x}$



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**Examples: Using Properties of Logarithms to Simplify Derivatives**

Take the derivatives of the following:

a. $y = \log_2 \left(\frac{3x+1}{x-1}\right)^{\ln 2}$

b. $y = \log_7 \sqrt[3]{\left(\frac{x+1}{4x-6}\right)^{\ln 7}}$

c. $y = x \log_4 e^{(\tan 2x)(\ln 4)}$



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**Lesson: Logarithmic Differentiation**

Logarithmic differentiation is another way to take derivatives.

Show how to use logarithmic differentiation to take the following derivatives:

a. $y = \frac{x\sqrt{5x+1}}{(x+9)^7}$

b. $f(x) = (x+1)^x$



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**Examples: Logarithmic Differentiation**

Use logarithmic differentiation to find the following derivatives:

a. $y = x(x-4)(x+5)(x+2)^7$

b. $f(x) = \sqrt[3]{\frac{\sin x}{(x^2+5)^3}}$



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c. $x^y = y^x$

d. $y = x^{\sin x}$



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**Lesson: Understanding the Derivatives of Inverse Trig Functions**

Show how to find the derivative of $y = \sin^{-1} x$

Show how to find the derivative of $y = \tan^{-1} x$

Show how to find the derivative of $y = \sec^{-1} x$



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State the Inverse Function-Inverse Cofunction Identities

Explain how these relate to finding the derivatives of $y = \cos^{-1} x,\ y = \cot^{-1} x,\ y = \csc^{-1} x$



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**Examples: The Derivatives of Inverse Trig Functions**

Find the derivatives of the following:

a. $y = \cos^{-1}(x^5)$

b. $f(x) = \cot^{-1}\sqrt{3x+1}$

c. $y = \ln\left(\sec^{-1}(2x-1)\right)$

d. $f(x) = \sin^{-1}(e^{-x})$



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**Examples: Separable Differential Equations, Initial Value Problems**

**Examples** Solve the initial value problems

a. $\frac{dy}{dx} = \sqrt{y}; y(0) = 4$

b. $\frac{dy}{dx} = \frac{x}{y}; y(2) = 6$

c. $\frac{dy}{dx} = 3x^2y^2 - y^2; y(0) = \frac{1}{2}$



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**Examples: Population Growth**

We can study how diseases die when properly treated by using the following logic let $\frac{dy}{dt}$ represent the rate at which the number of infected people changes is proportional to the number y. The number of people who are cured is proportional to y, the number who are infected with the disease. As of April 23, 2020, the number of coronavirus cases was 2,659,557. For simplicity, let's round that number to 2,660,000. Let's suppose that we can reduce the number of cases of the disease by 50% in a year. If there are 2,660,00 cases, then how many years will it take to the reduce the number to 2,000 cases?



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**Examples: Population Growth**

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**Lesson: Understanding Related Rates with an Example**

A 12-ft ladder is leaning against a wall when the base starts to slide. By the time the base is 10 feet from the house, the base is moving at the rate of 4 ft/sec.

a. How fast is the top of the ladder sliding down the wall?

b. What is the rate that the area is changing for the triangle formed between the wall, ladder, and floor?



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**Example: Related Rates – Volume**

A conical tank is being filled with water at a rate of $45 m^3$/min. The tank is 40 m tall and 10 m wide. How fast is the water level rising when the water is 15 ft deep?



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**Example: Related Rates – Two Moving Entities**

A hot air balloon is riding straight up from a point at a constant rate of 2 ft / sec. When the balloon is 50 ft about the ground, a motorcycle passes underneath is traveling at a constant speed of 30 ft / sec. How fast is the distance between the balloon and the motorcycle increasing 3 seconds later?



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**Example: Related Rates – A Cone and a Cylinder**

Este is making tea by pouring hot water into a conical filter with tea leaves and allowing them to drain into cylindrical pot. The water is draining into the pot at a rate of $11\frac{in^3}{min}$.

a. How fast is the level in the pot rising when the water in the cone is 5 inches deep?

b. How fast is the level in the cone falling?



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**Example: Related Rates – Problems with Angles**

a. A particle is moving along the curve $y = 3x^2$ in the first quadrant of a graph. The x-coordinate is increasing by a rate of 15 m/sec. How fast is the angle of inclination $\theta$ of the line joining the origin to the particle changing when x=1?

b. A sun is shining over a building creating a shadow. The building is 88 feet tall and the shadow behind the building is 66 feet long. The angle $\theta$ that the sun makes with the ground is changing at a rate of 0.00436 radians per min. What is the rate that the shadow is decreasing?



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**Lesson: Linearization**

Linearization is a way to estimate certain difficult calculations without a calculator. Explain the basic idea behind this below:

How do we define the linearization of a function f at a point a?

**Example** Find the linearization of an integer near the point a at which the given function and its derivative are easy to calculate. Then compare the actual value of the function at a to the approximated value.

$$f(x) = x^{-1}, a = -0.9$$



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**Example** Find the linearization of an integer near the point a at which the given function and its derivative are easy to calculate. Then compare the actual value of the function at a to the approximated value.

$$f(x) = \sqrt[3]{x}, a = 28.5$$



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**Lesson: Differentials**

Leibniz notation $\frac{dy}{dx}$ represents the derivative and does not represent a ratio, however we will now introduce some very similar notation $dx$ and $dy$, and when their ratio exists then it is equal to the derivative.

Define the differential below:

**Example** Find

a. $dy$ if $y = x^3 - 5x^2$

b. the value of $dy$ if $x = 2$ and $dx = 0.1$



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Differentials are related to linearization and can also be used to do estimation.

**Example** Use differentials to estimate

a. $8.1^{\frac{1}{3}}$

b. $\sqrt{6}$



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**Lesson: Intro to Hyperbolic Functions**

The hyperbolic sin and hyperbolic cosine functions are defined by the equations:

How do we say “sinh x” ?

What about “cosh x” ?

Using these definitions, we get some interesting identities. Determine the value of $2\sinh x\ \cosh x$:

Now determine the value of $\cosh^2 x - \sinh^2 x$



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List all of the relevant identities for hyperbolic functions:



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**Lesson: Derivatives of Hyperbolic Functions – A Quick Overview**

Find the derivative of $\sinh x$

Find the derivative of $\operatorname{csch} x$

Find the derivative of $\tanh x$



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Write the derivative rules for all of the hyperbolic functions below:

Write the integral formulas for all of the hyperbolic functions:



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**Extreme Values of Functions on Closed Intervals**

The general idea for closed intervals:

To begin, we need to understand the following definitions

Let $f(x)$ be a function. We say that:

- $f(x)$ has an absolute **maximum value** at a point c if $f(x) \leq f(c)$ for all x in the domain
- $f(x)$ has an absolute **minimum value** at a point c if $f(x) \geq f(c)$ for all x in the domain

What are some other words for maximum and minimum values?

**The Extreme Value Theorem**



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Here are some ways that extreme values can occur on closed intervals:

**Example** Sketch each graph and determine whether the function has any absolute extrema. Does this contradict the extreme value theorem?

$$f(x) = |x + 1|, 0 < x < 2$$



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**Example** Sketch each graph and determine whether the function has any absolute extrema. Does this contradict the extreme value theorem?

a. $f(x) = \begin{cases} \frac{1}{x^2}, & -1 \le x < 0 \\ x + 3, & 0 \le x \le 2 \end{cases}$

b. $y = 2\cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

If we focus on maximum or minimum values that occur within the interior of the interval, what do we notice about the tangent lines at these points? How can we think of these points?

A function $f(x)$ has a **local maximum value** at a point c with its domain if:

A function $f(x)$ has a **local minimum value** at a point c with its domain if:



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What is another word for local maxima/minima?

Critical Points

How to Find Absolute Extrema of a Continuous Function $f(x)$ on a Finite Closed Interval

1.

2.

3.

**Example** Find the critical points for the function, and then determine the absolute maximum and minimum on the given interval.

$$f(x) = \frac{2}{5}x - 2, \qquad -5 \leq x < 0$$



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**Example** Find the critical points for the function, and then determine the absolute maximum and minimum on the given interval.

a. $f(x) = -\frac{1}{x}, 1 \leq x \leq 3$

b. $f(x) = \sec x\,, -\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$

c. $f(x) = x^{\frac{4}{3}}, -1 \leq x \leq 27$

d. $f(x) = 3x^{\frac{2}{3}},\ 1 \leq x \leq 8$



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**Examples: Absolute Extrema on Closed Intervals**

Find the absolute extrema for the following functions on the given closed intervals.

a. $f(x) = 4 - x^2, -1 \leq x \leq 2$

b. $f(x) = \sin x\,, 0 \leq x \leq \frac{\pi}{2}$

c. $f(x) = \frac{1}{x} + \ln x\,, 2 \leq x \leq 4$



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**Review: Factoring with Rational Exponents (Trick for Critical Points)**

Find the critical points and domain:

a. $y = x^{\frac{4}{5}}(x + 3)$

b. $y = x^{\frac{3}{4}}(x - 1)$



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**Review: Factoring Out Fractions and Rational Exponents from Expressions**

Factor the following out of each expression.

a. $\frac{1}{4}x^{-\frac{1}{3}}$ from $\frac{7}{4}x^{\frac{5}{3}}+\frac{3}{4}x^{-\frac{1}{3}}$

b. $\frac{2}{3}x^{-\frac{2}{5}}$ from $\frac{4}{3}x^{\frac{3}{5}}-\frac{2}{3}x^{-\frac{2}{5}}$

c. $\frac{1}{2}x^{\frac{4}{7}}$ from $6x^{\frac{11}{7}}+5x^{\frac{4}{7}}$



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**Rolle's Theorem**

**State Rolle's Theorem**

Justify why this makes sense:



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**Example** Show that $x^3 + 4x + 2 = 0$ has exactly 1 real solution.

**Example** Show $f(x) = x^4 + 4x + 1$ has exactly one zero in the interval $[-2, -1]$.



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**The Mean Value Theorem**

**State the Mean Value Theorem**

Explain what this means:



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**Examples** Find the value or values of c that satisfy the equation:

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

a. $f(x)=\sqrt{x+1}, [0,3]$

b. $f(x)=3x^2+6x-5, [-2,1]$

**Examples** Determine whether the functions satisfy the hypotheses of the mean value theorem on the given interval, with reasoning.

a. $f(x)=x^{\frac{4}{5}}, [0,1]$

b. $f(x)=3x^{\frac{2}{3}}, [-1,1]$



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**Consequences of the Mean Value Theorem**

Write the 2 corollaries associated with the mean value theorem

**Examples** Find a function with the following derivative that goes through the given point.

a. $f'(x) = 4x; f(0) = 5$

b. $f(x) = \frac{1}{x^2} + 2x; (-1, 1)$



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**Examples** Given the velocity function and initial position of an object, determine the object's position at time t.
a. $v = 9.8t + 5; s(0) = 10$

b. $v = \sin t,\ s(0) = 1$

**Examples** Find all possible functions with the given derivatives:
a. $y' = \cos x$

b. $y' = \frac{1}{\sqrt{x}}$

c. $y' = \sin\frac{x}{2}$



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**Increasing and Decreasing Functions and the First Derivative Test**
**Part 1: Theory and An Example**

Before we begin, let's first make sure we have an intuitive understanding of what increasing and decreasing functions look like:

There is a third corollary to the Mean Value Theorem regarding whether functions are increasing or decreasing:

What does monotonic mean?



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Review: What is a critical point? How do you find critical points?

Now we can state the **First Derivative Test for Local Extrema:**
Let $f(x)$ be a continuous differentiable function at every point in some interval, except possibly at a critical point c itself. If we examine this interval from left to right:

1.

2.

3.

We can visualize this:



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**Examples** Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this first set of examples, you will be given the derivative:

$$f'(x) = (x-1)(x+2)(x-3)$$



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**Increasing and Decreasing Functions and the First Derivative Test**
**Examples 1**

**Examples** Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this first set of examples, you will be given the derivative:
a. $f'(x) = \frac{(x+1)(x-4)}{(x+3)(x+6)}$

b. $f'(x) = x^{-\frac{1}{3}}(x+3)$



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**Increasing and Decreasing Functions and the First Derivative Test**
**Examples 2**

**Examples** Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this set of examples, you will have to find the derivative:
a. $f(x) = 4x^2 + 8x + 13$

b. $f(x) = x^2 e^{-\frac{x}{3}}$



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**Increasing and Decreasing Functions and the First Derivative Test**
**Examples 3**

**Examples** Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this set of examples, you will have to find the derivative:
a. $f(x) = x^2 - 4x - 8\ln x$

b. $f(x) = \sin x - \cos x, \quad 0 \le x \le 2\pi$



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**Increasing and Decreasing Functions and the First Derivative Test**
**Examples 2**

**Examples** Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

$$f(x) = -6x^{\frac{6}{5}} + 3x^{\frac{2}{5}}$$



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**Lesson: Concavity and the Second Derivative Test for Concavity**

Explain concavity in non-technical terms:

For a differentiable function $f(x)$, the function is:
1. **concave up** on an open interval if I if:

Another word for concave up is: _____

2. **concave down** on an open interval I if:

**The Second Derivative Test for Concavity:**
Let $f(x)$ be a twice-differentiable function on some interval. Then:

1. If $f''(x) > 0$ on that interval, the graph is concave up on that interval
2. If $f''(x) < 0$ on that interval, the graph is concave down on that interval

When concavity changes from up to down (or vice versa), what do we have? Draw a picture of this idea:



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How do you find points of inflection?

**Examples** Estimate the points of inflection from the graphs (draw your own graphs, sorry!)

a.

b.

**Examples** Determine i) where the graph is concave up or concave down and ii) any points of inflection

$$f(x) = 2x^3 - 3x^2 - 12x + 3$$



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**Examples: Concavity and the Second Derivative Test for Concavity**

**Examples** Determine i) where the graph is concave up or concave down and ii) any potential points of inflection

a. $f(x) = 1 + x^{\frac{1}{3}}$

b. $f(x) = 5 + \cos 2x,\ [0, \pi]$



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**Second Derivative Test for Local Extrema**

How can concavity indicate maximums or minimums? Draw some graphs that illustrate this idea.

**The Second Derivative Test for Local Extrema**

Suppose $f''(x)$ is continuous on an open interval that contains $x = c$.

1.

2.

3.



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**Examples** Use the second derivative test for local extrema to find local maxima and minima. Then compare this to the first derivative test.

a. $f(x) = x^2 - 4x + 3$

b. $f(x) = x^4 - 8x^2$



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**Examples: Second Derivative Test for Local Extrema**

**Examples** Use the second derivative test for local extrema to find local maxima and minima.

a. $f(x) = x^2(x-1)^2$

b. $f(x) = \cos^2 x, -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

c. $f(x) = (x^2 - x)e^{-x}$



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**Lesson: L'Hopitals Rule**

Does $\frac{0}{0} = 1$?

List all of the indeterminate forms:

Write down L'Hopital's Rule

**Examples** Determine the following limits.

a. $\lim_{x\to 0} \frac{5x+\sin x}{4x}$

b. $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

c. $\lim_{x\to 0} \frac{x-\sin x}{2x^3}$



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What must be true to use L'Hopitals Rule?

**Example** Evaluate

$\lim_{x\to 0} \frac{(x+\cos x)}{x+x^3}$

You can also combine L'Hopital's Rule with other limit knowledge

$\lim_{x\to 0} \frac{3\sin x}{x^2}$



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**Lesson: L'Hopital's Rule for $\frac{\infty}{\infty}, \infty \times \mathbf{0}, \infty - \infty$ Forms**

L'Hopitals Rule can also be applied when we find these types of indeterminate forms: $\frac{\infty}{\infty}, \infty \times 0, \infty - \infty$

a. $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x}$

b. $\lim_{x \to \infty} \frac{e^x}{x^3}$

c. $\lim_{x \to \infty} \left( x \sin \frac{1}{x} \right)$



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a. $\lim_{x\to 0^+} \sqrt{x}\ln x$

b. $\lim_{x\to\infty} (\ln 2x - \ln(x+2))$

c. $\lim_{x\to 0^+} \left(\frac{2x+1}{x} - \frac{1}{\sin x}\right)$



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**Lesson: L'Hopital's Rule for Indeterminate Powers**

L'Hopitals Rule can also be applied to indeterminate powers of these forms:

In this case, L'Hopital's Rule works a little differently. We use logarithms to help evaluate. Write down how this works below

**Examples** Evaluate.

a. $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$

b. $\lim_{x\to\infty} x^{\frac{1}{x}}$



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**Examples**

a. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$

b. $\lim_{x\to\infty}\left(\frac{x^2+2}{x+1}\right)^{\frac{1}{x}}$

c. $\lim_{x\to e^+}(\ln x)^{\frac{1}{x-e}}$



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**Lesson: What Happens when L'Hopitals Rule Doesn't Work?**

In some cases, L'Hopitals Rule may not work and you will have to think of other ways to evaluate the limit. Let's look at a few cases.

a. $\lim_{x\to\infty} \frac{\sqrt{4x+1}}{\sqrt{x+3}}$

b. $\lim_{x\to\infty} \frac{2^x-5^x}{4^x+3^x}$

c. $\lim_{x\to 0^+} \frac{x}{e^{-\frac{1}{x}}}$



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**Examples: L'Hopitals Rule**

Evaluate the following limits using L'Hopitals Rule, if possible.

a. $\lim_{x\to 0} \frac{\sin x^2}{x}$

b. $\lim_{x\to \frac{\pi}{3}} \frac{3x+\pi}{\sin\left(x+\frac{\pi}{3}\right)}$

c. $\lim_{x\to 0} \frac{\left(\left(\frac{1}{3}\right)^x - 1\right)}{x}$

d. $\lim_{x\to \infty} x^2 e^{-x}$



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e. $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$

f. $\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$

g. $\lim_{x\to 0^+} x^x$

h. $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{\sin x}}$



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**Summary: L'Hopital's Rule**

State L'Hopital's Rule

What are all the indeterminate forms that apply to L'Hopital's Rule?

For which type of indeterminate forms can you just take the derivative of the top and bottom?

**Examples** Find the following limits.

a. $\lim_{x\to 0} \frac{4x^2}{2\cos x - 2}$

b. $\lim_{x\to \infty} \frac{\ln x}{3\sqrt[3]{x}}$



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For the forms $0 * \infty$ or $\infty - \infty$, what must you try?

**Examples** Find the following limits.

a. $\lim_{x \to \infty} 3x\, e^{-x}$

b. $\lim_{x \to 0^+} (\ln 5x - \ln(\sin x))$

What must you do with indeterminate powers?

**Example** Find the limit $\lim_{x \to 0^+} x^{\frac{-1}{\ln x}}$



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What happens if L'hopitals rule continues to cycle?

**Example** Find the limit $\lim_{x\to 0^+} \frac{\cot x}{\csc x}$



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**Curve Sketching**

Demonstrate all the steps for sketching a curve while sketching $f(x) = \frac{x^2+2}{2x}$



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**Curve Sketching Examples 1**

**Example** Sketch $f(x) = 6 + 8x^2 - x^4$, showing all the relevant steps



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**Curve Sketching Examples 2**

**Example** Sketch $f(x) = \cos x + \sqrt{2}\sin x,\ 0 \leq x \leq 2\pi$, showing all the relevant steps



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**Curve Sketching Examples 3**

**Example** Sketch $x^{\frac{2}{3}}(5-2x)$, showing all the relevant steps



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**Curve Sketching Examples 4**

**Example** Sketch $x^{\frac{2}{3}}(5-2x)$, showing all the relevant steps



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**Curve Sketching when you only know y'**

In another video, we discussed how to sketch curves. Given the following derivatives, what can you confidently sketch about the curves?

a. $f'(x) = x^2 + 5x + 6$



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b. $f'(x) = (x-2)^2(2x+1)$



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**Optimization Problems**

**Example 1:** A piece of sheet metal is rectangular, 3 ft wide and 6 ft long. Congruent squares are to be cut from its 4 corners. The resulting piece of metal is to be folded and welded to form an open-top box. How should this be done to get the maximum volume?



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**Optimization Problems**

**Example 2:** What is the maximum possible area of a rectangle with a base that lies on the x-axis with two upper vertices that lie on the graph of the equation $y = 4 - x^2$?



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**Optimization Problems**

**Example 3:** Find the dimensions of the largest right circular cylinder of greatest volume that can be inscribed in a right circular cone of radius R and height H.



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**Summary: Derivative Theorems About Min/Max/Inflection Points**

In this summary, we will go through summarizing all of the theory and theorems about finding maximums and minimums with functions. We will then relate this to graphing and optimization problems.

First, we will review all of the vocabulary and theorems that usually come up in these concepts.

Absolute minimums/maximums versus local minimum/maximums

Critical point

Monotonic functions

Definition of an increasing function

Definition of a decreasing function



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Definition of concave up

Definition of concave down

Inflection point

Extreme Value Theorem

**Example** Does the extreme value theorem guarantee that $f(x) = \frac{1}{x}$ will have an absolute minimum or maximum point on the interval $[-3, 3]$?

**Example** Does the extreme value theorem guarantee that $f(x) = \frac{1}{x}$ will have an absolute minimum or maximum point on the interval $[1, 5]$?



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The First Derivative Test for Local Extrema

The Second Derivative Test for Local Extrema

**Examples** For each of the following functions, use a) the first derivative test for extrema to find all local/absolute extrema, and then compare that result with b) the second derivative test for extrema. Then c) find the intervals of increasing/decreasing.

a. $f(x) = x^3 - 3x + 3$

b. $f(x) = x^{\frac{2}{3}}(x - 2)$



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What are the steps for graphing a function?

**Examples** For each of the following functions, use a) the second derivative test for concavity to determine the intervals of concavity. Then determine b) if there are any inflection points. Then c) complete the graph of the function.

$$f(x) = x^3 - 3x + 3$$



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**Examples** For each of the following functions, use a) the second derivative test for concavity to determine the intervals of concavity. Then determine b) if there are any inflection points. Then c) complete the graph of the function using the steps above.

a. $f(x) = x^{\frac{2}{3}}(x-2)$

b. $f(x) = x + \cos x, 0 \leq x \leq 2\pi$



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One other application with min/max theorems are optimization problems.

**Example** A rectangular garden next to a barn needs to be fenced in. Since the garden is next to the barn, the side on the barn does not need fencing. If there is 600 feet of fencing, what is the largest area that can be enclosed?

**Example** Determine the rectangle of largest area that can be inscribed in a semicircle of radius 4.



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**Newton's Method**

What is the idea and derivation behind Newton's Method?

Write the iterative formula for Newton's Method here:



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**Example** Use Newton's method to approximate a root for the following function. Stop when the first 5 decimal places do not change over 2 iterations.

$$f(x) = x^3 + x^2 + 1; x_0 = -1.5$$



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**Example** Use Newton's Method to find a solution to $4x - \sin x = 4$ on $[1, 2]$.



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**Antiderivatives versus Indefinite Integrals versus Differential Equations**

Define an antiderivative and any special notes about it:

Define an indefinite integral and any special notes about it:

Define a differential equation and any special notes about it:



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**Example** Find the antiderivative to the function: $2x$

**Example** Find the indefinite integral: $\int 2x\, dx$

**Example** Find the general solution to: $\frac{dy}{dx} = 2x$

**Example** Solve the initial value problem: $\frac{dy}{dx} = 2x; y(0) = 7$



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List all of the relevant antiderivatives you know here:



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**Examples of Basic Antiderivatives / Indefinite Integrals**

Find the antiderivatives of the following functions:

a. $\frac{1}{x^2}$

b. $3x^2$

c. $x^{-\frac{1}{2}}$

d. $\frac{2}{3}x^{-\frac{1}{3}}$

Find the indefinite integrals:

a. $\int 2\sin x \ dx$

b. $\int \sec^2 x \ dx$

c. $\int \frac{\pi}{2}\cos\frac{\pi}{2}x \ dx$

d. $\int -\csc 3x \cot 3x \ dx$

e. $\int e^{-2x}dx$

f. $\int \frac{2}{\sqrt{1-x^2}}dx$

g. $\int 3^x dx$



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**Examples of Basic Antiderivatives / Indefinite Integrals**

Integrate the following:

a. $\int \frac{5}{x} dx$

b. $\int 5x^{\sqrt{5}} dx$

c. $\int \frac{1+\cos 3x}{2} dx$

d. $\int \cot^2 x \, dx$

e. $\int \cos x \, (\tan x + \sec x) dx$



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**Examples of Basic Differential Equations**

Solve the initial value problems:

a. $\frac{dy}{dx} = 5 - 3x^2; y(0) = 4$

b. $\frac{dy}{dx} = 3x^{-\frac{2}{3}}; y(8) = 1$

c. $\frac{dy}{dx} = -\pi \sin \pi x \, ; y(0) = 1$



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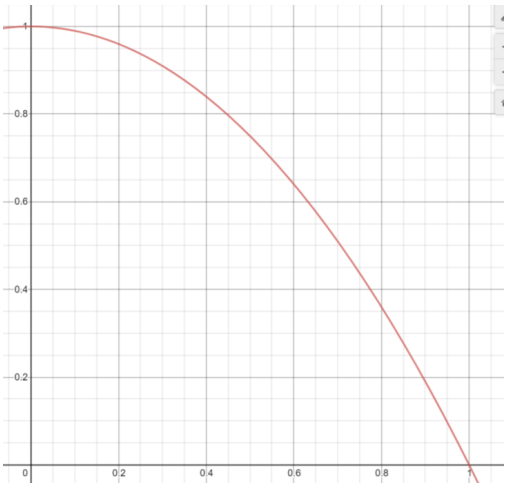
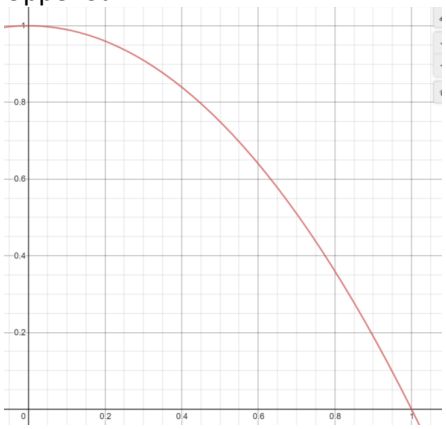
**Area and Estimating with Finite Sums**

First we will try to understand the idea behind this. Draw out one way to find the area of the curve $y = 1 - x^2$ below.

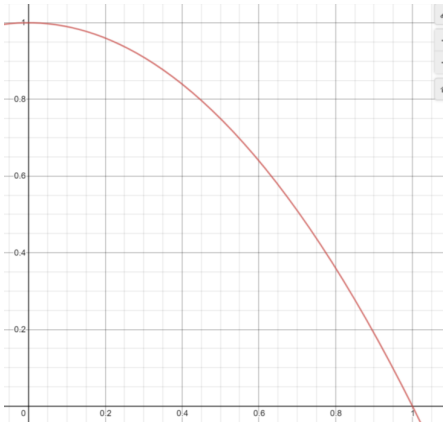


Now we need more formality around this idea. Show the idea of how to create sums using a more algorithmic approach. Detail what each sum means:

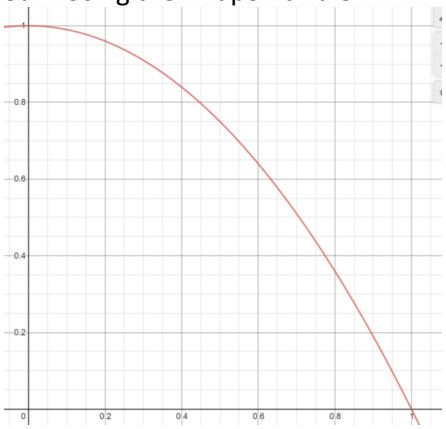
Upper Sum



Lower Sum



Sum Using the Midpoint Rule





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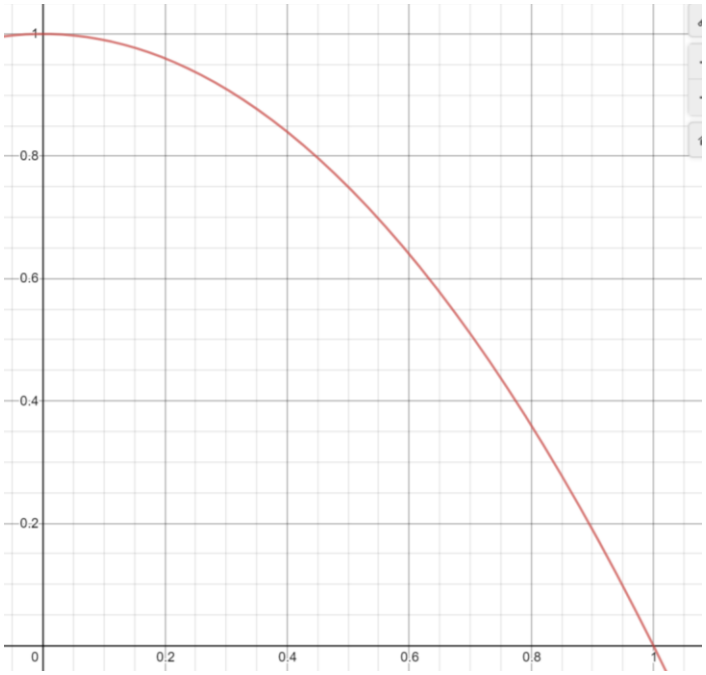
Now we need some vocabulary. Separate the graph in 4 parts and then label it with the correct terminology:




Define a Riemann Sum:

What is a left Riemann sum?

What is a right Riemann sum?

What is a midpoint Riemann sum?



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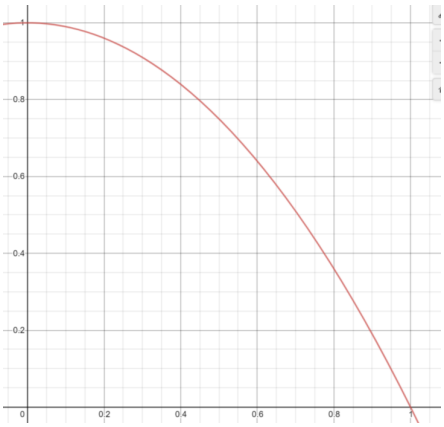
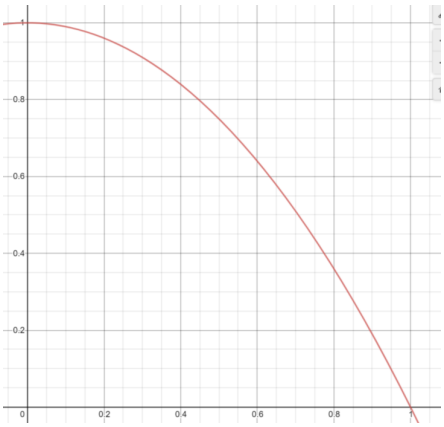
Show how you can make a more precise sum:

8 Subintervals

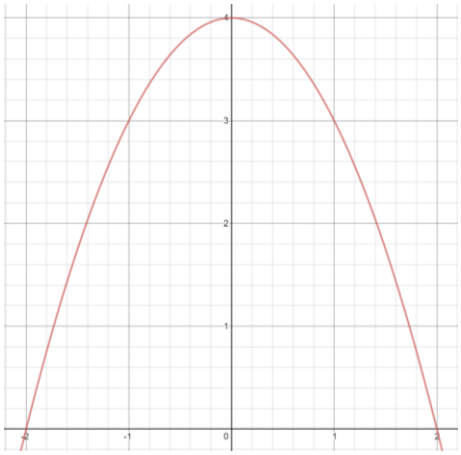


16 Subintervals

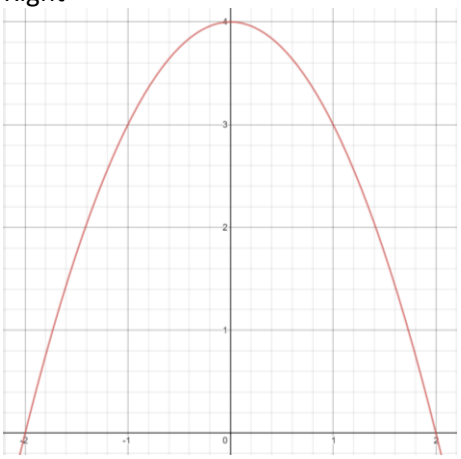


**Example** Compute the left, right, and midpoint Riemann sums for $f(x) = 4 - x^2$ between $[-2, 2]$ using 4 subintervals

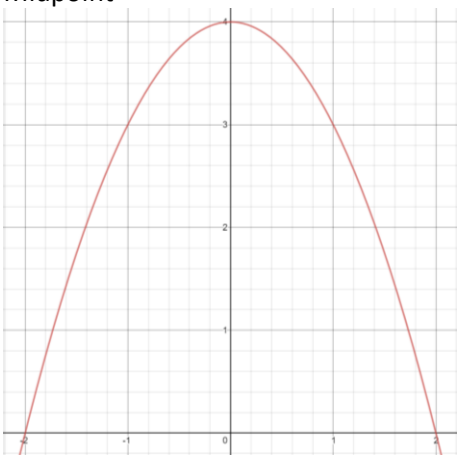
Left



Right



Midpoint



*Graphs were created using Desmos.com. They have the greatest graphing software for free, check them out!*



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**Examples of Riemann Sums**

Compute the left, right, and midpoint Riemann sum for $f(x) = 3 + x$ from $[1,3]$ using 4 subintervals.



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**Examples of Riemann Sums – When It Goes Negative?**

Compute the left, right, and midpoint Riemann sum for $f(x) = x^3$ from $[-2,2]$ using 4 subintervals.



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**Examples: Different Ways to Compute Area Estimations**

Using $f(x) = -x^2 + 2x + 3$, compute an area estimate using 4 subintervals from x=-1 to x=3.

Compute this estimate using an upper, lower, left-hand, and right-hand sum.



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**Lesson: The Basics of Sigma Notation**

Explain what sigma notation looks like and what it means:

**Examples** Compute the following sums:

a. $\Sigma_{k=1}^{4} k$

b. $\Sigma_{i=3}^{7} (2i - 1)$

c. $\Sigma_{k=1}^{4} (-1)^{k+1} k$



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We have a few handy rules for sigma notation:

1. Sum Rule:

2. Difference Rule:

3. Constant Multiple Rule:

4. Constant Value Rule:

**Examples** Use the rules above to help calculate the following sums:

a. $\Sigma_{k=1}^{3}(k+5)$

b. $\Sigma_{k=1}^{4}(-3k-2)$



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**Lesson: Using Sigma Notation and Sum Formulas to Calculate Larger Sums**

We've already discussed these basic algebraic rules for sums:

**Sum Rule**

$$\Sigma_{k=1}^{n}(a_k + b_k) = \Sigma_{k=1}^{n}a_k + \Sigma_{k=1}^{n}b_k$$

**Difference Rule**

$$\Sigma_{k=1}^{n}(a_k - b_k) = \Sigma_{k=1}^{n}a_k - \Sigma_{k=1}^{n}b_k$$

**Constant Multiple Rule**

$$\Sigma_{k=1}^{n}\, ca_k = c * \Sigma_{k=1}^{n}a_k$$

**Constant Value Rule**

$$\Sigma_{k=1}^{n}\, c = n * c$$

Now we will add on a few other handy formulas for sums:

**The First n Terms**

**The First n Squares**

**The First n Cubes**



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These formulas are particularly handy for larger sums.

**Examples** Sum the following using the sum formulas:

a. $\Sigma_{k=1}^{15} 3k^2$

b. $\Sigma_{i=1}^{52} (i^3 + 4i - 2)$

c. $\Sigma_{k=1}^{35} k(2k + 5)$



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**Lesson: Sigma Notation with Riemann Sums**

Our goal in this lesson is to figure out how to use Riemann Sums with Sigma Notation. To get you warmed up for this idea, we will practice figuring out how to write certain Riemann sums:

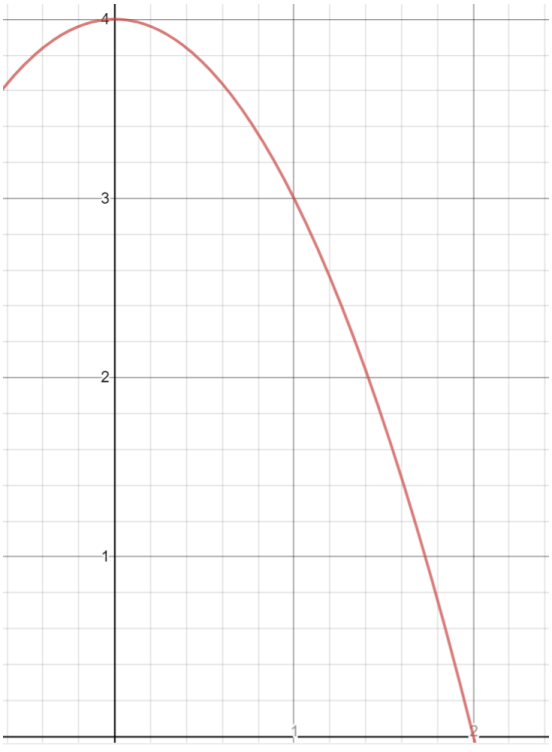
**Examples** Write the following sums using sigma notation.

a. $1 + 2 + 3 + 4 + 5$

b. $2 + 4 + 6 + 8 + 10 + 12$

c. $1 - 3 + 5 - 7$

Previously we had discussed how to computer Riemann sums. Using $f(x) = 4 - x^2$ on $[0, 2]$, create 4 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?



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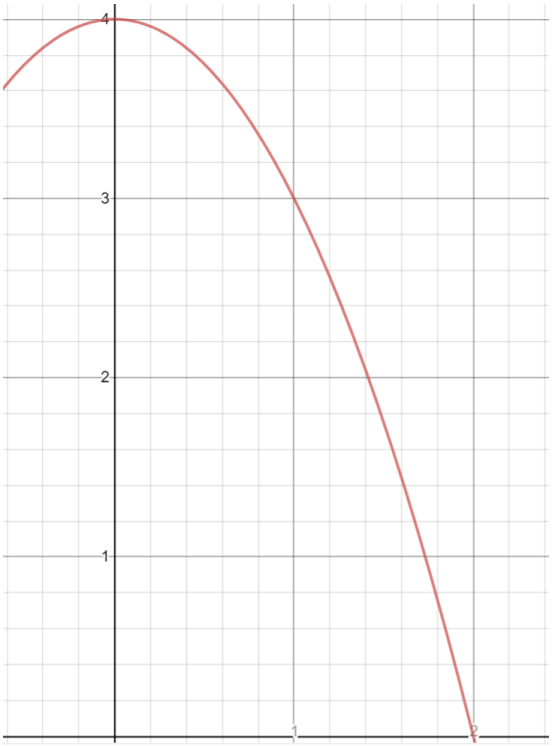
We will continue the pattern. Continuing to use $f(x) = 4 - x^2$ on $[0, 2]$, create 8 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?




Can you determine how you might generalize this for n subintervals?

Write out the Riemann sum formula for using right-hand endpoints:



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How can we manipulate this formula so that we use left-hand endpoints instead? Write this down:

**Example** Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^2 + 1$ on $[0,2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.



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**Example: Evaluating Riemann Sums as $n \to \infty$**

**Example** Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = 4 - x^2$ on $[0, 2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.



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**Example: Evaluating Riemann Sums as $n \to \infty$**

**Example** Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^3$ on $[0,1]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.



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**Lesson: The Basics of Definite Integrals**

Before we begin, we need to define some terminology. Draw and label the following:

Let $f(x)$ be a function defined on $[a, b]$. We say that a number J is the **definite integral of f over** $[\boldsymbol{a}, \boldsymbol{b}]$ and that J is the limit of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\Delta x_k$ if the following is true:

> Given any $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for every partition P = $\{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of $c_k$ in $[x_{k-1}, x_k]$ we have:
>
> $$|\Sigma_{k=1}^{n} f(c_k)\Delta x_k - J| < \epsilon$$

This looks very similar to the definition of a limit in some ways, which I will state here:

> We have $\lim_{x \to c} f(x) = L$ if, for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that:
>
> $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

However, if the limit exists and using equal width partitions, we can instead state this (label what you write down):

Write down and label the definite integral notation:



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There are a few rules of definite integrals that we will want to note. When $f(x)$ and $g(x)$ are integrable functions over $[a, b]$, then we can use the following rules:

1. Order of Integration

2. Zero Width Interval

3. Constant Multiple

4. Sum and Difference

5. Additivity

6. Max-Min Inequality

7. Domination

**Example** if $\int_3^5 f(x)\,dx = 5$, $\int_0^3 f(x)\,dx = 2$, and $\int_0^5 g(x)\,dx = 9$, $\int_3^5 g(x)\,dx = 4$ find:

a. $\int_0^5 f(x)\,dx$

b. $\int_0^3 g(x)\,dx$

c. $\int_5^3 f(x)\,dx$

d. $\int_0^5 \big(3f(x) + g(x)\big)dx$



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Finally, we have one final theorem to state that helps us determine when a function is integrable.

**Theorem: Integrability of Continuous Functions**

A question that usually arises from this is, “What is a function that isn’t integrable then?” This is the classic example:



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**Lesson: Finding Definite Integrals Using Area Formulas**

If $f(x)$ is nonnegative and integrable over $[a, b]$, how do we define the area under the curve $y = f(x)$?

This means that we can calculate definite integrals using known area formulas when possible, as opposed to using Riemann sums.

**Example** Sketch the following curves and use known area formulas to evaluate the definite integrals.

a. $\int_{-1}^{1}(3 - |x|)\ dx$

b. $\int_{0}^{3}(2x)\ dx$



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c. $\int_0^2 (2x + 1)\ dx$



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**Lesson: The Average Value of a Function**

If a function $f(x)$ is integrable on $[a, b]$, then its average value or mean is calculated by:

In this lesson, we will calculate integrals only using known area formulas.

**Examples** Find the average value of the following functions:

a. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$

b. $f(x) = |x| - 1$ on $[-1, 1]$



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**Lesson: The Fundamental Theorem of Calculus, part 1 proof**

Define the initial function we are working with and the corresponding graph:

Compute F’(x), including any relevant graphs. Interpret what this means:



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State the first part of the Fundamental Theorem of Calculus

**Example** Find $\frac{dy}{dx}$ given $y = \int_3^x (t^5 + 1)\, dt$



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**Examples: The Fundamental Theorem of Calculus, part 1**

Differentiate the following in 2 ways:

i. evaluate the integral then differentiate

ii. differentiate the integral directly

a. $\frac{d}{dx}\int_3^x \sin t \ dt$

b. $\frac{d}{dx}\int_1^{\cos x} 2t \ dt$

c. $\frac{d}{dx}\int_4^{\sqrt{x}} (5t^3 - \cos t) \ dt$



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**Examples: The Fundamental Theorem of Calculus, part 1**

Compute $\frac{dy}{dx}$ for each of the following:

a. $y = \int_4^x \sqrt{3t+2}\, dt$

b. $y = \int_x^5 \sin t \ \ dt$

c. $y = \int_{\sin x}^4 3t^2 \ \ dt$

d. $y = \int_{-3}^{e^{4x}} \frac{1}{\sqrt{3t+1}}\ dt$



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**Step by Step Example: Fundamental Theorem of Calculus, part 1**

Find $\frac{dy}{dx}$ for $y = x\int_5^{x^2} \sin t^3 \ dt$



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**Lesson: The Fundamental Theorem of Calculus, part 2 proof**

State the Mean Value Theorem for Definite Integrals

**Example** if $\int_a^b f(x)\,dx = 0$, must there be a point c in $[a, b]$ where $f(c) = 0$?

From the Fundamental Theorem of Calculus part 1, we know that the antiderivative of $f$ exists in the form:

$$F(x) = \int_a^x f(t)\ dt$$

Now, we know that the most general form of an antiderivative for $f$ would be $F(x) + c$. Now we want to make this more precise.

What is $F(b) - F(a)$?



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State the second part of the Fundamental Theorem of Calculus

**Examples** Compute the following:

a. $\int_1^3 2x\,dx$

b. $\int_0^{\frac{\pi}{2}} \cos x\,dx$



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**Examples: The Fundamental Theorem of Calculus, part 2 – basic examples**

Evaluate the following integrals:

a. $\int_1^4 (3x^2 - x)\ dx$

b. $\int_0^{\frac{\pi}{3}} 5\sec^2 x\ dx$

c. $\int_0^4 \left(4x^2 + \sqrt{x}\right) dx$



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**Examples: The Fundamental Theorem of Calculus, part 2 – tricky trig examples**

Evaluate the following integrals:

a. $\int_0^{\frac{\pi}{4}} \frac{1+\cos x}{2}\, dx$

b. $\int_0^{\frac{\pi}{3}} \frac{1+\cos 2x}{2}\, dx$

c. $\int_{\frac{\pi}{2}}^{\pi} \sin^2 x\ dx$

d. $\int_0^{\frac{\pi}{2}} (\sec x + \cos x)^2\ dx$



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**Examples: The Fundamental Theorem of Calculus, part 2 – tricky non-trig examples**

Evaluate the following integrals:

a. $\int_3^5 (x+1)^2 \ dx$

b. $\int_1^3 \left(\frac{x^6}{3} - \frac{1}{x}\right) dx$

c. $\int_0^4 \frac{x+\sqrt{x}}{x} \ dx$

d. $\int_{-2}^2 |x| \ dx$



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**Examples: Which Formula Should I Use for Integrals?**

In this video, we will compute the integral $\int_1^4 (2x+1)\,dx$ 3 ways:

Way 1: Area Formulas

Way 2: Riemann Sum

Way 3: Using the fundamental theorem of calculus / evaluation formula

Use the space below to write out each way.



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**Summary: Everything about Indefinite and Definite Integrals**

Compare $\int 2x\,dx$, $\int_0^3 2x\ dx$, and $\int_3^0 2x\ dx$



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**Lesson: Indefinite Integrals and the Substitution Method**

Before we begin, let's just review how indefinite integrals work:

$\int (3x^4 - \sin 5x + 3e^{3x})\, dx$

Now we can explain substitution using the following example: $\int (4x^3 + 2x)(x^4 + x^2)^{18}\, dx$

**Example** Integrate:

a. $\int (4x^3 - 6x^2)(x^4 - 2x^3)^8\, dx$

b. $\int 3\sin(3x)\ dx$



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Sometimes we must manipulate the problem to get substitution to work.

Use the example $\int (6x + 9)^{12}\ dx$ to demonstrate this idea:

**Examples** Integrate:

a. $\int 4\sec^2(3x + 1)\ dx$

b. $\int \frac{1}{\sqrt{3x-5}}\ dx$

c. $\int \tan x\ dx$



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Sometimes we must go nuclear with our manipulations. Below are a few examples of creative ways to manipulate the problems:

a. $\int \sin^5 x \cos^3 x \ dx$

b. $\int (x-4)(x+3)^{15} \ dx$

c. $\int x^3 \sqrt[3]{x^2+4} \, dx$

d. $\int \frac{dx}{1-e^x} \ dx$



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**Lesson: Indefinite Integrals and the Substitution Method to Solve Certain Trig Integrals**

In this lesson, it is assumed that you already understand the basics of the substitution method. Here we will focus on how to use substitution to integrate the following integrals:

a. $\int \tan x \ dx$

b. $\int \cot x \ dx$

c. $\int \sec x \ dx$

d. $\int \csc x \ dx$



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**Basic Examples of Indefinite Integrals and the Substitution Method**

a. $\int 2x\sqrt{x^2+5}\ dx$

b. $\int \frac{1}{2\sqrt{x}\left(3+\sqrt{x}\right)^2}\ dx$

c. $\int 3\sin^3 3x\cos 3x\ dx$

d. $\int -\frac{1}{x^2}\sqrt{3+\frac{1}{x}}\ dx$



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**Examples of Indefinite Integrals and the Substitution Method Where Some Manipulation is Required**

a. $\int 5x\sqrt{3x^2+2}\,dx$

b. $\int 5x^{\frac{1}{2}}\cos\left(3x^{\frac{3}{2}}+1\right)\,dx$

c. $\int \frac{\cos(5x+2)}{\sin^2(5x+2)}\,dx$

d. $\int \sqrt{\frac{x^8}{x^5+4}}\,dx$



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**Difficult Examples of Indefinite Integrals and the Substitution Method Where Creativity is Needed**

a. $\int \sqrt{\frac{x^2-x}{x^6}}\ dx$

b. $\int 5x^5\sqrt{4x^3+1}\ dx$



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c. $\int \frac{7}{4+25x^2}\ dx$

d. $\int \frac{3}{x\sqrt{1-x^4}}\ dx$



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**Lesson: Substitution with Definite Integrals**

This lesson assumes you are already familiar with substitution for indefinite integrals. To get warmed up, evaluate the following:

a. $\int \frac{6x^2}{\sqrt{1-x^3}}\ dx$

b. $\int \frac{x}{\sqrt{1+x}}\ dx$

Using these examples, we can demonstrate 2 different ways to evaluate definite integrals with substitution.

$\int_1^3 \frac{6x^2}{\sqrt{1-x^3}}\ dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution



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Now you try using $\int_0^3 \frac{x}{\sqrt{1+x}}\, dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution



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**Examples: Substitution with Definite Integrals**

Evaluate the following:

a. $\int_1^4 \frac{1}{2x(1+\sqrt{x})^2}\ dx$

b. $\int_0^{\frac{\pi}{3}} \cos^{-2} x \sin x\ \ dx$



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c. $\int_1^4 x\sqrt{1+3x}\,dx$

d. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+e^{\tan x})\sec^2 x\;\;dx$



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**Example: Definite Integrals and Substitution**

$\int_2^4 \frac{dx}{x(\ln x)^6}$



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**Step by Step Example: Substitution and Definite Integrals – Trig Problem!**

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{x}\sin^2\left(x^{\frac{3}{2}}\right)\,dx$$



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**Lesson: Definite Integrals of Symmetric Functions (Proof and Examples)**

In this lesson, we will look at how substitution can be used to prove the following theorem:

**Theorem** Suppose that $f(x)$ is a continuous function on $[-a, a]$. Then:

a. if $f(x)$ is even, then $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{2} f(x)\ dx$

b. if $f(x)$ is odd, then $\int_{-a}^{a} f(x)\ dx = 0$

Prove this below:



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**Examples** Evaluate the following integrals:

a. $\int_{-4}^{4}(x^3 + x)\, dx$

b. $\int_{-5}^{5}(|x| - x^4 + x^2)\, dx$



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**Lesson: Calculating the Area Between 2 Curves With Integration**

An integral is calculating what area?

Draw out an explanation of how we can understand using integration to find the area between 2 curves. Watch the explanation, and then rewrite it in terms that make sense to you.



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**Example** Find the area enclosed between $y = x^2 - 1$ and $y = 3$.



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**Example** Find the area between $x = 2y^2$, $x = 0$, and $y = 3$



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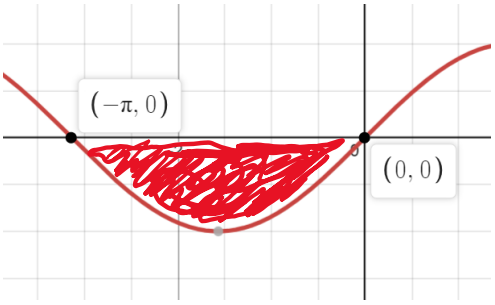
**Examples: Total Area vs Area**

What's the difference between area and total area? Demonstrate this concept with a graph.

**Example** Find the total area for the curve $y = \sin x$ in the shaded region (graph from Desmos.com)



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**Examples: The Area Between 2 Curves With Respect to x**

**Example** Find the area between $y = x^2 - 2x$ and $y = -x^2 + 4x$



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**Examples: The Area Between 2 Curves With Respect to y**

**Example** Find the area between $x = 2y$ and $x = y^2 - 3$



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**Example** Find the area enclosed by $x + 4y^2 = 4$ and $x + y^4 = 1$ for $x \geq 0$



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**Examples: Finding Area Between Hard to Visualize Curves**

a. Find the area between the curves enclosed $y = x^4 - 9x^2 + 1$ and $y = x^2 - 8$



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b. Find the area enclosed by the curves $9x^2 + y = 9$ and $x^8 - y = 1$



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c. Find the area of the region enclosed by the curve $x = 3\sin y\ \sqrt{\cos y}$ and the line $x = 0, 0 \leq y \leq \frac{\pi}{2}$



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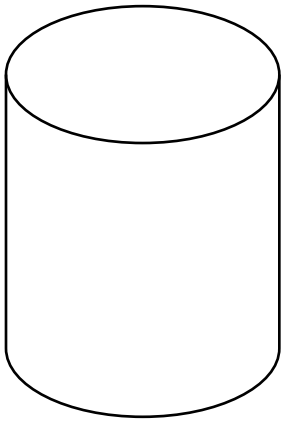
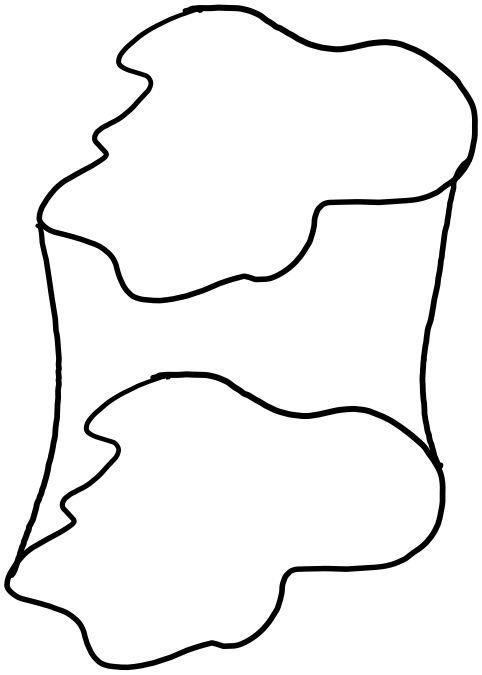
**Lesson: Understanding Volume by Slicing with Parallel Planes**

We're going to discuss how integration relates to volume. One way is using cross-sections.

What is a cross-section?

Below we have a few different kinds of cylinders. What would the cross-sections look like?







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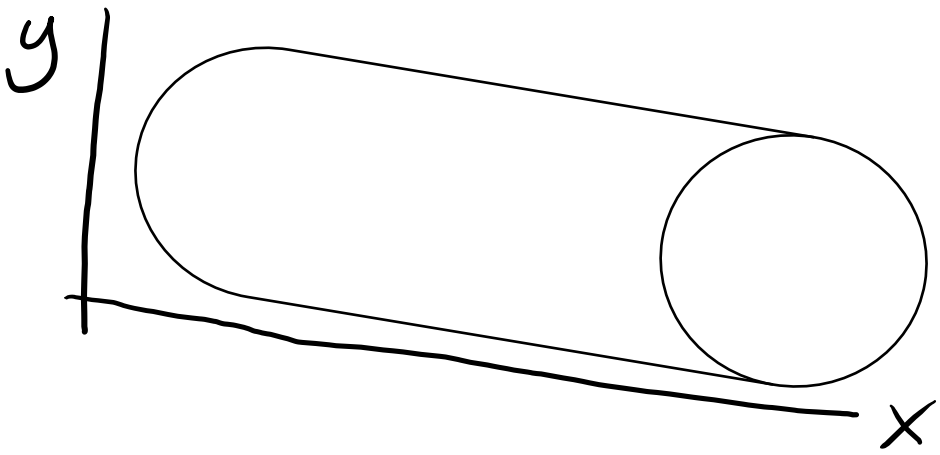
Which way do you want these slices to be oriented?

Below is how we would orient the circular cylinder.




Draw a sketch showing how we would orient the second (odd looking) cylinder on the x- and y-axis.



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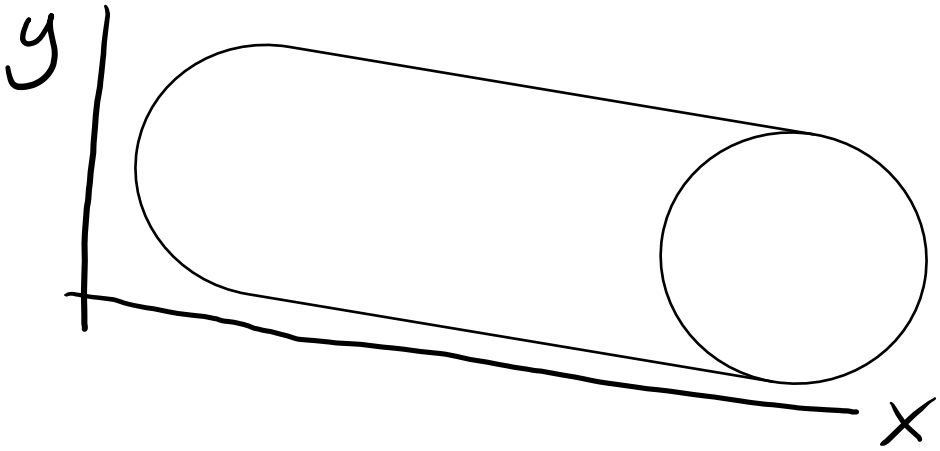
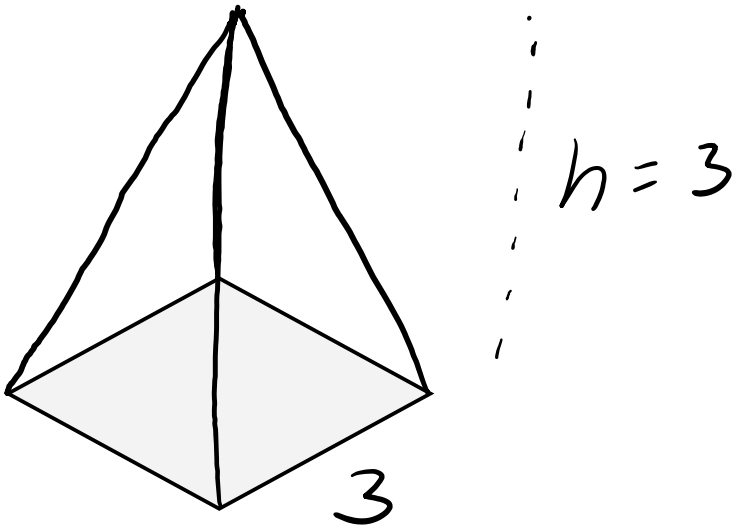
Explain how we derive volume using cross-sections for the cylinder.




Using the following pyramid with a square base, draw how we might opt to orient this object to make it easier to analyze the cross-sections.



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Summarize the explanation of how we would opt to take the volume of cross-sections in the case of the pyramid. In your explanation, explain how Riemann sums fit into this.

Write down both formulas for finding volume with cross-sections.



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What are the steps to calculating the volume of a solid with cross-sections?

**Example** Find the volume of a solid where the base is the region between the curve $y = 4\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis, where the cross-sections are equilateral triangles with the bases running from the x-axis to the curve y.



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**Examples: Volumes Using Cross-Sections**

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 4 - x^2$. Find the volume.



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The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x$. Find the volume when the cross-sections perpendicular to the x-axis are:

a. isosceles triangles of height 4

b. semicircles with diameters running across the base of the solid



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**Examples: Volume by Slicing, Harder Examples**

A square of side length 2x lies in a plane perpendicular to a line L. One vertex of the square lies on L. As the square moves a distance h along L, the square turns one revolution about L to generate a corkscrew shaped column with square cross sections. What is the volume?



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Find the volume of the tetrahedron. Please refer to the video for the reference picture.



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The base of the solid is the disk $x^2 + y^2 = 4$. The cross-sections by planes perpendicular to the y-axis between $y = -2$ and $y = 2$ are isosceles right triangles with one leg in the disk. Find the volume.



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**Lesson: Solids of Revolution via the Disk Method**

Before you do this lesson, you need to make sure you understand the lesson about volume by cross-sections!

The disk-method is basically an application of volume by cross-sections.

What is a solid of a revolution?

What are the cross-sections in this case?

How can we represent the area of these disk?

**Formula for Finding Volume by Disks for Rotation About the x-axis**



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**Formula for Finding Volume by Disks for Rotation About the y-axis**

**Example** Find the volume of the solid generated by revolving the bounded region about the x-axis:

$$y = x^3, y = 0, x = 1$$



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**Example** Find the volume of the solid generated by revolving the bounded region about the y-axis:

$$x = \sqrt{4 \sin y}\,, 0 \leq y \leq \pi, x = 0$$



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**Example** Find the volume of the solid generated by revolving the bounded region about the line $y = 1$.

$$y = \sqrt{2x}, y = 1, x = 2$$



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**Examples: The Disk-Method, Revolved around the x-axis or y-axis**

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = \sec x\,, y = 0, x = -\frac{\pi}{4}, x = \frac{\pi}{4}$ around the x-axis



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B. $x = \frac{2}{\sqrt{y+1}}, x = 0, y = 0, y = 3$



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**Examples: The Disk-Method, Revolved Around a Line (not an axis)**

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 4x, y = 0, x = 1$ about the line $x = 1$

B. $y = \sqrt{x}, y = 2, x = 0$ about the line $y = 2$



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**Lesson: Solids of Revolution via the Washer Method**

You should understand the idea of volume by cross-sections and volume by the disk method before you try this lesson.

The washer method is used with solids of revolution, but what characteristic do these solids have?

What do we need to know about the washers?

**Formula for Volume by Washers for Rotation about the x-axis**



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**Example** Find the volumes of the solid generated by revolving the enclosed region about the x-axis.

$$y = x, y = 2, x = 0$$

**Example** Find the volumes of the solid generated by revolving the enclosed region about the y-axis.

$$x = \tan y\,, y = 0, x = 1, y \leq 0 \leq \frac{\pi}{4}$$



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**Examples: The Washer Method**

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 2\sqrt{x}, y = 2, x = 0$ about the x-axis



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B. $y = \sec x,\ y = \sqrt{2}, -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$



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**Examples: Volume by Disks and Washers Around Different Axes**

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, y = 0, x = 2$ about:

a. The x-axis

b. The line $y = -2$

c. The line $y = 6$



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d. The y-axis

e. The line $x = 3$

f. The line $x = -3$



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**Examples: When to Use Subtraction with Finding the Radius**

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, x = 0, y = 4$ about:

a. The x-axis

b. The y-axis

c. The line $y = 4$



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**Examples: Help! I Feel Like I'm Bad At Sketching**

Sketch the region bounded by:

$y = 3x, x = 2, y = 0$

a. Show the solid of a revolution obtained by revolving this region around the x-axis

b. Show the solid of a revolution obtained by revolving this region around the y-axis

c. Show the solid of a revolution obtained by revolving this region around the line $x = 2$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

Sketch the region bounded by:

$y = \sqrt{x}, y = 2x - 1, x = 0$

a. Show the solid of a revolution obtained by revolving this region around the y-axis

b. Show the solid of a revolution obtained by revolving this region around the line $x = 4$

c. Show the solid of a revolution obtained by revolving this region around the line $y = 3$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

**Lesson: Volume by Cylindrical Shells**

Sometimes cross-sections can be awkward.

Demonstrate why cross-sections might be awkward using the example $y = 2x - x^2$ revolved around $x = -1$. Focus on the area enclosed by the x-axis.



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Another way to calculate volume is by using a cylindrical shell. Draw how a cylindrical shell is constructed. How do we compute the volume of a shell?

Now draw how a cylindrical shell might fit into your sketch. Your sketch does not have to be perfect – you just need to use it as a way to wrap your head around the idea. Draw multiple sketches if needed. Label the pieces of your sketch.



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One of the keys to using cylindrical shells is what?

Explain how the volume formula will relate to integration:

**Shell Formula for A Revolution Around Any Vertical Line x=L**



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What assumptions do we make about calculating volume in this manner?

**Example** Find the volume generated by revolving the bounded region about the y-axis:

$$y = x, y = -\frac{1}{2}x, x = 4$$



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**Example** Find the volume generated by revolving the bounded region about the line $y = -1$

$$x = \frac{1}{2}y, y = 0, x = 4$$



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**Examples: Volume by Using Cylindrical Shells with a Piecewise Function**

Let $g(x) = \begin{cases} \frac{\sin x}{x}, & 0 < x \leq \pi \\ 1, & x = 0 \end{cases}$

a. Show the $x * g(x) = \sin x,\ 0 \leq x \leq \pi$

b. Find the volume of the solid generated by revolving the region enclosed by the curve and the x-axis about the y-axis.



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**Examples: The Shell Method Where The Integral is Split Up**

Using the region enclosed by $y = x^2$ and $y = x + 6$, use the shell method to find the volume of solids generated by revolving the regions about the given lines:

a. $x = 3$

b. $x = -4$

c. the x-axis

d. $y = -2$



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**Examples: Volume by Using Cylindrical Shells About Different Lines**

For the region bounded by the curves:

$$y = 2x, y = 0, x = 3$$

Find the volume generated by revolving the region about:

a. The line $x = -1$

b. The line $y = -2$



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**Lesson: Volume by the Washer Method vs Volume by Cylindrical Shells**

Compute the volume of the solid bounded by the curves:

$$y = \frac{1}{2}x + 1, y = x, x = 0$$

In the following ways:

a. Revolved about the x-axis using the washer method

b. Revolved about the y-axis using the shell method

c. Revolved about the line x=3 using the shell method

d. Revolved about the line y=4 using the washer method



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**Summary: How to Set Up Volume by Integration Problems**

There are a few keys to setting up volume by integration problems:

1. Always draw the region **and** draw a sketch of the solid – you should always have 2 drawings
2. Decide what formula is appropriate for the situation and with respect to which variable
3. Label your solid to go with the formula
4. Fill in the formula and integrate

Let's review a few concepts.

Write out the formulas for the disk method.

Write out the formulas for the washer method.



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When do you use the washer method?

How can you determine if your revolution is a solid or has a hole in it?

With the disk/washer method, when do you integrate with respect to x?

With the disk/washer method, when do you integrate with respect to y?

In what circumstances do you have to “adjust” the radius? How does this adjustment look? What does this mean about rotating about the x-axis or y-axis?



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Write out the formulas for the shell method:

With the shell method, when do you integrate with respect to x?

With the shell method, when do you integrate with respect to y?

If you rotate around the follow axis, what is the radius in that situation?

a. x-axis

b. y-axis



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When do you need to subtraction with finding the shell radius?

If you need to use subtraction to find the shell radius, how do you determine the radius?

In the disk, washer, or shell methods: if you are integrating with respect to x, how can you determine the limits of integration?

In the disk, washer, or shell methods: if you are integrating with respect to y, how can you determine the limits of integration?



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If you struggle to sketch your solids, I have an entire video just focusing on this. *You absolutely need to be able to quickly sketch a solid, and it is worth your time to do this video if you have anxiety about doing this.*

**Example** Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

First, sketch the region:

a. Now we will find the volume generated by revolving this region about the x-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.



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**Example** Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

b. Now we will find the volume generated by revolving this region about the y-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

c. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.



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d. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

e. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.



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f. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

g. Now we will find the volume generated by revolving this region about the x-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.



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h. Now we will find the volume generated by revolving this region about the y-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

i. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.



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j. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

k. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.



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l. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.



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**Lesson: A Guide to Approaching Work Problems**

Constant force work formula:

Variable force work formula:

A note on calculating force:

The units are very important because they will likely guide you on how to do the problem.

Let's note units too for the following:

| Measure | SI Units | US Customary |
|---|---|---|
| Force | | |
| Work | | |
| Mass | | |
| Length | | |
| Acceleration due to gravity | | |



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What are some important notes regarding how we think of force in different applications?

With problems involving objects:

With pumps/volume problems:

If you have completely forgotten how to do dimensional analysis, *please watch my review video on this topic.* This will really help you!

Going back to our integral, rewrite the integral to explain how we should be watching out for units.

A guideline for how to get to these units:

With objects

With pumps/volume



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**Example** You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

Why is it obnoxious to say “weighs” here? What’s a better way to think about this?

What elements go into how this formula is set up?

When we set up our integral, what ultimately needs to happen?

Show how to finish the problem and show how the units guide us that we are right.



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**Example** You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

What if I wanted to use (25-x) instead of x to model the length left of the rope?

What does this mean?

Note the following fact about integration:



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**Example** You want to haul a 15-m length of hanging rope. How much work will it take if the rope weighs 0.08 kg/m.

**Example** A bag of sand weighing 160lbs was lifted at a constant rate. As it rose, sand leaked out at a constant rate. One-quarter of the sand was gone when the bag had been lifted to 20ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment).



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**Example** A vertical right-circular cylindrical tank measures 20 ft high and 10 ft in diameter. It is full of milk weighting 64 $\frac{lb}{ft^3}$. How much work does it take to pump the milk to the level of the top of the tank?

**Example** Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 5 ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to the rim of the tank? (See picture)



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**Example** Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 8ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to 5ft above the top of the tank?



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**Lesson: Hooke's Law for Springs**

What is Hooke's law?

What are the relevant units to pay attention to?

**Example** It took 1500 J of work to stretch a spring from its natural length of 2m to a length of 4m. Find the spring's force constant.



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**Example** If a force of 80 N stretch a spring 2 m beyond its natural length, how much work does it take to stretch the spring 6 m beyond its natural length?

**Example** A spring has a natural length of 8 in. A 500-lb force stretches the spring to 13 in.

a. Find the force constant

b. How much work is done in stretching the spring from 8 in to 13 in?

c. How far beyond its natural length will a 800-lb force stretch the spring?



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**Lesson: Fluid Forces**

Fluid Force on a Constant-Depth Surface

Fluid Force Against a Vertical Flat Plate (Integration Formula)

**Example** Refer to the video for the sketch. Calculate the fluid force on one of the triangular plate using the given coordinates.



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**Example** In a pool fill with water to a depth of 12 ft, calculate the fluid force on one side of a 4ft x 6ft rectangular plate if the plate rests vertically at the bottom of the pool on its 4-ft edge.



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**Examples: Fluid Forces**

Calculate the fluid force on one side of a semicircular plate of radius 4 ft that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 7 feet.



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A metal cube that is 8ft on each side has a parabolic gate on one side at the bottom. See the picture in the video. It is designed to withstand a fluid force 500 lb without rupturing. The liquid you plan to store has a weight-density of 62.4 lb/ft^3.

a. What is the fluid force on the gate when the liquid is 6 feet deep?

b. What is the maximum height to which the container can be filled without exceeding the gate's design limitation?



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**Lesson: Integration by Parts**

State the formula for **integration by parts**:

We will look at a few basic examples in this video.

**Basic Example** $\int x \cos \pi x \ dx$



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**Basic Example** $\int \ln x \ dx$

**Example of Using Integration by Parts Multiple Times** $\int x^3 e^x \ dx$



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**Example Where the Integral Cycles** $\int e^x \sin x \ dx$



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**Lesson: Integration by Parts with Definite Integrals**

Let's look at an example of how to use integration by parts with definite integrals.

$\int_0^4 x\, e^{-x}\ dx$



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**Examples: Integration by Parts, Examples Where You Have to Use Integrate Multiple Times**

a. $\int x^2 \cos x \ dx$

b. $\int (x^2 - 2x + 1)e^x \ dx$



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**Examples: Integration by Parts, Examples Where The Integral Cycles Back to Itself**

a. $\int e^{4x} \cos 3x \ dx$



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b. $\int e^{-x} \cos x \, dx$



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**Examples: Using Substitution AND Integration by Parts Together**

a. $\int e^{\sqrt{4x+1}}dx$



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b. $\int x\,(\ln x)^2\,dx$



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c. $\int \cos\sqrt{x}\, dx$



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**Lesson: Trigonometric Integrals with Different Powers of Sine and Cosine**

***I. Integrals of the Form*** $\int sin^m x\, cos^n x\ dx$

A. When m is even, what should you do?

**Example** $\int \sin^3 4x \cos^2 4x\ dx$



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B. When n is odd, what should you do?

**Example** $\int \cos^3 2x \ dx$



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C. When both m and n are odd, what should you do?

**Example** $\int \sin^4 x \cos^2 x \ dx$



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**Examples: Integrals with Powers of Sines and Cosines**

a. $\int \sin^4 3x \cos 3x \ dx$

b. $\int \cos^3 4x \sin^5 4x \ dx$



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c. $\int 4\,sin^4\,2\pi x\;\;dx$



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**Lesson: Trigonometric Integrals with Square Roots or Different Products of Sine and Cosine**

***II. Integrals with Square Roots***

When an integral has a square root with sine or cosine, what identity should we try to leverage?

**Example** $\int_0^{\frac{\pi}{2}} \sqrt{1-\cos 2x}\ dx$



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**Example** $\int_0^{\frac{\pi}{2}}(1-\cos^2 x)\,dx$

**Example** $\int_0^{\frac{\pi}{6}}\sqrt{1+\sin x}\,dx$



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***III. Integrals with Different Products of Sine and Cosine***

For integrals of the form:

- $\int \sin mx \sin nx \, dx$
- $\int \cos mx \cos nx \, dx$
- $\int \sin mx \cos nx \, dx$

What trig identities can be used to help integrate these types of integrals?

**Example** $\int \sin 7x \cos 3x \ \ dx$



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**Examples: Integrals and Square Roots with Trig Functions**

a. $\int_0^{\pi} \sqrt{1-\cos^2 x}\ dx$

b. $\int_0^{\frac{\pi}{2}} x\sqrt{1-\cos 2x}\, dx$



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**Examples: Integrals and Products of Sines and Cosines**

a. $\int \sin 3x \cos 2x \, dx$

b. $\int \cos^2 2x \sin x \, dx$



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**Lesson: Trigonometric Integrals with Tangent, Cotangent, Secant, and Cosecant**

For integrals with $\tan x$ and $\sec x$, what identities should be used to help with integration?

For integrals with $\cot x$ and $\csc x$, what identities should be used to help with integration?

**Examples** Integrate the following.

a. $\int \tan^3 x \ dx$



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b. $\int \sec^4 x \ dx$



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c. $\int \cot^2 x \csc^4 x \ dx$



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**Examples: Integrals and Powers of Tangent and Secant**

a. $\int \sec^2 x \tan^2 x \ dx$

b. $\int \sec^3 x \tan x \ dx$



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c. $\int \csc^4 x \ dx$



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**Lesson: Trigonometric Substitution**

Trig substitution is a special way to use substitution for integrals of a special form.

***Reference Guide***

| If the integral involves… | Then substitute: | And use: |
|---|---|---|
| | | |
| | | |
| | | |

**Example** Write out instructions for how to approach these types of problems: $\int \frac{dx}{\sqrt{9+x^2}}$



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What is another way to integrate $\int \frac{dx}{\sqrt{9+x^2}}$?

$\sinh^{-1}\frac{x}{a}$ is equivalent to:

**Example** Evaluate $\int \frac{x^2}{\sqrt{25-x^2}}\,dx$



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**Example** $\int_0^2 \frac{dx}{50+2x^2}$



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**Example** Use substitution before using trig substitution: $\int_{\frac{1}{12}}^{\frac{1}{4}} \frac{3}{\sqrt{x}+4x\sqrt{x}}\ dx$



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**Examples: Basic Trig Substitution Examples**

a. $\int \frac{dx}{\sqrt{25+x^2}}$

b. $\int \sqrt{16-x^2}dx$



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**Examples: More Complicated Trig Substitution Examples**

a. $\int \sqrt{\frac{x+1}{1-x}}\, dx$

b. $\int \frac{dx}{(x^2-4)^{\frac{3}{2}}}$



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c. $\int \frac{6}{(4x^2+1)^2}\, dx$



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**Examples: Integrals That Use Both Substitution and Trig Sub**

$\int \frac{e^x}{1+e^{2x}}\, dx$



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**Review: How to Complete the Square (A Few Ways)**

Complete the square on the polynomial:

a. $x^2 - 4x$

b. $x^2 + 6x + 9$

c. $x^2 + 3x$

d. $5x^2 - 20x - 10$

e. $6 - 2x - x^2$



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**Review: Completing the Square for Advanced Integration Techinques**

First, let's review a few examples of completing the square in general. Complete the square on the following:

a. $5x^2 + 10x - 20$

b. $4 - 2x - x^2$

c. $x^2 + 3x - 7$



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Now that we have thoroughly reviewed this skill, let's use it for 2 different integrals.

Integrate the following.

a. $\int \frac{dx}{\sqrt{2x+x^2}}$

b. $\int \sqrt{5-4x-x^2}\ dx$



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**Lesson: Integration by Partial Fractions**

What are the keys to writing an expression $\frac{f(x)}{g(x)}$ with partial fractions?

I. Nonrepeated Linear Factors

$\int \frac{3x+11}{x^2-x-6}\, dx$



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II. Repeated Linear Factors

$\int \frac{3x-2}{x^2-2x+1}\,dx$



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III. Irreducible Quadratic Factors

$\int \frac{x}{x^4-x^2-2}\, dx$



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IV. Improper Fractions

$\int \frac{x^3+1}{x^3-x}\, dx$



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Summary of How to Solve Partial Fractions

| Factor in Denominator | Form for Partial Fractions |
|---|---|
| $ax + b$ (unique factor) | |
| $(ax + b)^k$ | |
| $ax^2 + bx + c$ | |
| $(ax^2 + bx + c)^k$ | |



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**Examples: Integration with Partial Fractions Where the Denominator Has Linear Factors Only**

a. $\int \frac{dx}{x^3+x^2-2x}$



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b. $\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx$



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c. $\int \frac{2x+1}{x^2-7x+12}\, dx$



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**Examples: Integration with Partial Fractions Where the Denominator Has Quadratic Factors**

a. $\int \frac{x^2-x+2}{x^3-1}$



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b. $\int \frac{2x^3+5x^2+8x+4}{(x^2+2x+2)^2}$



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**Examples: Integration with Partial Fractions With an Improper Fractions**

$$\int \frac{16x^3}{4x^2 - 4x + 1} \, dx$$



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**Examples: Some Hard Integrals that Involve Partial Fractions and Creativity**

a. $\int \frac{\cos x}{\sin^2 x - \sin x - 6}\, dx$

b. $\int \frac{1}{x^{\frac{3}{2}} - x^{\frac{1}{2}}}\, dx$



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c. $\int \frac{1}{\cos x + \sin 2x}\, dx$



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**Lesson: A Review of "Basic" and Creative Integration Techniques**

Write down all of the "basic" integration formulas we should know at this point:



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Now we will review some more creative “basic” integration techniques.

I. More Creative Substitution Ideas with Completing the Square

$\int \frac{dx}{\sqrt{2x-x^2}}$

II. Manipulating an Expression to Use Substitution

$\int \frac{dx}{\sqrt{e^{2x}-1}}$



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III. Multiply by the Conjugate

$\int \frac{dx}{1-\sec x}$

IV. Multiply by an Expression Equivalent to 1

$\int \frac{1}{e^x+e^{-x}}\, dx$



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V. Split Into 2 Parts

$\int \frac{1+\sin x}{\cos^2 x}\, dx$

VI. Use Long Division

$\int \frac{4x^2-7}{2x+1} dx$



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**Lesson: Completing the Square to Integrate**

$\int \frac{dx}{\sqrt{2x-x^2}}$



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**Lesson: Multiplying by a Strategic Fraction to Integrate**

a. $\int \frac{dx}{1-\sin x}$

b. $\int \frac{dx}{\sec x+\tan x}$



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**Lesson: Improper Fractions and Division**

What makes an improper fraction?

a. $\int \frac{x^2}{x^2+1} dx$



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b. $\int \frac{4x^3-x^2+16x}{x^2+4}\ dx$



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**Lesson: Separating Fractions to Integrate**

$\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\, dx$



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**Lesson: Using Substitution More Creatively**

a. $\int \frac{dx}{\cos^2 x \tan x}$

b. $\int \frac{2^{\ln x^3}}{16x} dx$



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**Lesson: The Trapezoidal Rule**

With Riemann sums, we use rectangles for making area estimates. However, there are other shapes we can use.

First we need to know the area of a trapezoid formula:

$$A = \frac{a+b}{2}h$$

You may want to draw a reference picture for that.

Now we can discuss how to use trapezoids for area estimates below:



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State the Trapezoidal Rule

**Example** Estimate the area of $f(x) = 2x + 1$ on $[0, 2]$ using four subintervals and the trapezoidal rule.



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**Lesson: Improper Integrals Type I – Integrals Containing Infinity**

In this lesson we will consider type 1 improper integrals. What are some examples of these types of integrals?

We will use limits to help us integrate these types of integrals. Explain the different ways to solve integrals of this type:



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If the integral has an answer, what do we say about that integral?

What if the integral does not have a finite answer?

Find the area under the curve $y = \frac{1}{x^2+1}$ from $x = 0$ to $x = \infty$ if possible.



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**Example** $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$



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**Lesson: Improper Integrals Type II – Integrals Involving Vertical Asymptote**

Explain what type of integrals are type II integrals:

How do you evaluate these types of integrals?



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Show how to integrate $\int_0^3 \frac{dx}{\sqrt{3-x}}$



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**Example** $\int_{-8}^{1} \frac{1}{x^{\frac{1}{3}}}\ dx$



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**Lesson: Comparison Tests for Integrals**

Why might we want to use a comparison test for an integral?

Explain the Direct Comparison Test



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**Examples** Use the direct comparison to evaluate the following integrals.

a. $\int_1^\infty e^{-x^2}\,dx$

b. $\int_1^\infty \frac{1}{\sqrt{x^2-0.01}}\,dx$

c. $\int_0^\pi \frac{dx}{\sqrt{x}+\sin x}$



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Explain the Limit Comparison Test

**Example** Use the limit comparison test to evaluate the integrals.

a. $\int_1^\infty \frac{1}{1+x^2}\,dx$

b. $\int_0^1 \frac{dx}{x-\sin x}$



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**Summary: How to Study for the Integration Exam**

Obviously you are told over and over again that you need to practice for an integration exam. If you go through each section of your notes / book / homework, what techniques are used?

You should be able to do a few problems of each of these types. In this video, we will take a look at one type of common integral form that can sprawl into many different types of problems.

Analyze the following, determine appropriate techniques for integration, and then integrate.

a. $\int \sqrt{4-x^2}\, dx$

b. $\int_0^1 \frac{5x+5}{\sqrt{x^2+2x}} dx$

c. $\int \frac{dx}{x^2+4x}$

d. $\int \frac{dx}{\sqrt{x^2-2x}}$

e. $\int \frac{x^3}{x^2+2x+1}\, dx$

f. $\int \frac{1}{\sqrt{x^2-4x+5}}\, dx$



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a. $\int \sqrt{4-x^2}\, dx$

b. $\int_0^1 \frac{5x+5}{\sqrt{x^2+2x}} dx$



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c. $\int \frac{dx}{x^2+4x}$

d. $\int \frac{dx}{\sqrt{x^2-2x}}$



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e. $\int \frac{x^3}{x^2+2x+1}\ dx$

f. $\int \frac{1}{\sqrt{x^2-4x+5}}\ dx$



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**Lesson: Arc Length and How the Formula is Derived**

Suppose we want to find the length of the curve $y = \sqrt{x-2}$ from $x = 2$ to $x = 6$. Draw a sketch of the curve.

To understand how we can use integration to find the length of a curve, we need to recall how we find the length of a line segment. Draw out a line segment from $(x_1, y_1)$ to $(x_2, y_2)$, and write out how we would find the formula for this line segment.

How can we apply this idea to a Riemann sum with a curve? Draw an illustration to accompany this explanation.



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Show how we can take the ideas from algebra and mold them to using integration to finding the length of the curve.



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State the arc length formula.

**Example** Find the length of the curve $y = 4x^{\frac{3}{2}}$ from $x = 0$ to $x = 1$



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**Lesson: Arc Length with Discontinuous Derivative Values**

There is an issue if we try to from the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$. Explain what the issue is.

How can we resolve this issue?

What is the formula for the length of the curve $x = g(y), c \leq y \leq d$



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Show how to find the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$.



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**Lesson: Finding the Area of a Surface of Revolution**

How do you create a solid of revolution?

How to find the area of surface from revolving the graph of f(x) around the x-axis:



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How to find the area of surface from revolving around the y-axis:

**Examples** Find the areas of surface generated by revolving the curve around the given access.

a. $f(x) = \sqrt{x+1}, 1 \leq x \leq 5$, x-axis



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b. $x = \left(\frac{1}{3}\right) y^{\frac{3}{2}} - y^{\frac{1}{2}}, 1 \leq y \leq 3$, y-axis



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**Examples: Finding Surface Areas of Revolution**

Find the surface area of the revolution generated by revolving the given curve about the given axis.

a. $x = \sqrt{4y - 3}, 1 \leq y \leq 3$, y-axis

b. $y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}, 1 \leq x \leq 2$, x-axis



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c. $x = \frac{1}{2}(e^{y} + e^{-y}), 0 \leq y \leq \ln 3$, y-axis



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**Examples: Separable Differential Equations, General Solutions Only**

**Examples** Find general solutions to the following:

a. $\frac{dy}{dx} = 2x\sqrt{y}$

b. $\frac{dy}{dx} = (xy)^{\frac{3}{2}}$

c. $\frac{dy}{dx} = \frac{(x^5-1)y^3}{x^2(3y^2-2)}$



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**Examples: Radioactivity and Exponential Decay**

Some atoms are unstable and spontaneously emit mass, which is called radioactive decay. An element that goes through this process is called radioactive. The decay of a radioactive element is described by the differential equation:

$$\frac{dy}{dy} = -ky, k > 0$$

If $y_0$ is the number of initial nuclei present, then the number at a later time is:

$$y = y_0 e^{-kt}, k > 0$$

An interesting consequence of this is that half-life can be calculated by the equation:

$$\frac{\ln 2}{k}$$

The half-life of the plutonium isotope is 24,360 years. If 15 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 75% of the isotope to decay?



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**Examples: Newton’s Law of Cooling**

A metal beam was brought from the outside cold into a machine shop where the temperature was held at $75°\text{F}$. After 10 min, the beam warmed to $45°\text{F}$ and after another 15 min it was $55°\text{F}$. Use Newton's Law of Cooling to estimate the beam's initial temperature.



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**Lesson: Parametrization of Plane Curves**

Write out the vocabulary for parametrization:



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**Example** Sketch the curve defined by the parametric equations.

a. $x = \cos\frac{\pi t}{4}, y = t, 0 \leq t \leq 8$

b. $x = t^2, y = t - 2, -\infty < t < \infty$



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**Example** The position $P(x, y)$ of a particle moving in the xy-plane is given by $x = 2t - 5, y = 2t, 0 \leq t \leq 1$. Identify the Cartesian equation for its path and graph the equation. Indicate the portion of the graph traced by the particle and the direction of motion.



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**Example** The position $P(x, y)$ of a particle moving in the xy-plane is given by $x = \frac{t}{t-1}, y = \frac{t-2}{t+1}, -1 < t < 1$. Identify the Cartesian equation for its path and graph the equation. Indicate the portion of the graph traced by the particle and the direction of motion.



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How can any function be parametrized? Given an example.

**Example** Find a parametrization for the line through the point $(a, b)$ with slope m.



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**Lesson: Tangents and Areas**

When is a parametrized curve differentiable?

State the parametric formula for $\frac{dy}{dx}$:

State the parametric formula for $\frac{d^2y}{dx^2}$:



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**Example** Find the tangent to the curve at the given value of t:

$x = 2\sin 2\pi t,\ y = \cos 2\pi t,\ t = -\frac{1}{6}$



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**Example** Find the value of $\frac{d^2y}{dx^2}$ at the point t:

$x = 2\sin 2\pi t,\ y = \cos 2\pi t,\ t = -\frac{1}{6}$



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**Example** Assuming that the equations define x and y implicitly as differentiable functions, find the slope of the curve at the given value of t:

$x = t^3 + t, y + 2t^3 = 2x + t^2, t = 1$



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**Example** Find the area enclosed by the y-axis and the curve:

$x = t - t^2, y = 1 + e^{-t}$



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**Summary: Derivative Rules**

For a derivatives exam, you will obviously need to know many formulas. To begin, we will summarize all of these formulas.

The formula for a derivative according to the definition:

The alternative formula for a derivative:

Compare using the formulas by taking the derivative of this function: $f(x) = x^2 - 3x$



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Next we will review all of the derivative rules.

**Basic Rules**

Notes: c stands for a constant, n is any number, g(x) and h(x) are functions

For each function form, state what the derivative would be.

Derivative of a constant function: $f(x) = c; \frac{d}{dx}(c) =$

Power rule: $f(x) = x^n; \frac{d}{dx}(x^n) =$

Derivative constant multiple rule: $f(x) = cg(x); \frac{d}{dx}\big(cg(x)\big) =$

Derivative sum rule: $f(x) = g(x) + h(x); \frac{d}{dx}\big(g(x) + h(x)\big) =$

Derivative of the natural exponential function: $f(x) = e^x, \frac{d}{dx}(e^x) =$

Derivative product rule: $f(x) = g(x) * h(x), \frac{d}{dx}\big(g(x) * h(x)\big) =$

Derivative quotient rule $f(x) = \frac{g(x)}{h(x)}\ (h(x) \neq 0); \frac{d}{dx}\left(\frac{g(x)}{h(x)}\right) =$

**Examples** Find the derivatives of the functions.

a. $f(x) = \frac{e^x}{\sqrt{x} - 4x^2}$

b. $y = 5x^{\pi}e^x$



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**Trig Rules and the Chain Rule**

Derivative of sine: $f(x) = \sin x$ ; $\frac{d}{dx}(\sin x) =$

Derivative of cosine: $f(x) = \cos x$ , $\frac{d}{dx}(\cos x) =$

Derivative of tangent: $f(x) = \tan x$ , $\frac{d}{dx}(\tan x) =$

Derivative of cotangent: $f(x) = \cot x$ , $\frac{d}{dx}(\cot x) =$

Derivative of secant: $f(x) = \sec x$ , $\frac{d}{dx}(\sec x) =$

Derivative of cosecant: $f(x) = \csc x$ , $\frac{d}{dx}(\csc x) =$

Chain rule: $f(x) = g\big(h(x)\big)$; $\frac{d}{dx}\Big(g\big(h(x)\big)\Big) =$

**Examples** Find the derivatives of the functions.

a. $f(x) = \sin x\, e^x$

b. $g(x) = \frac{e^x}{\tan x}$

c. $y = \sin(e^{3x+1})$

d. $h(x) = \sec(sin\, x + (3x^2 + 5)^{18})$



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**Derivatives of Logarithms and Exponentials**

Derivative of natural logarithm: $f(x) = \ln x; \frac{d}{dx}(\ln x) =$

Derivative of general exponential function: $f(x) = a^x, a > 0, a \neq 1; \frac{d}{dx}(a^x) =$

Derivative of general logarithm: $f(x) = \log_a x,\ a > 0, a \neq 1; \frac{d}{dx}(\log_a x) =$

**Derivatives of Inverse Trig Functions**

Derivative of inverse sine: $f(x) = \sin^{-1} x\,; \frac{d}{dx}(\sin^{-1} x) =$

Derivative of inverse cosine: $f(x) = \cos^{-1} x; \frac{d}{dx}(\cos^{-1} x) =$

Derivative of inverse tangent: $f(x) = \tan^{-1} x; \frac{d}{dx}(\tan^{-1} x) =$

Derivative of inverse cotangent: $f(x) = \cot^{-1} x; \frac{d}{dx}(\cot^{-1} x) =$

Derivative of inverse secant: $f(x) = \sec^{-1} x; \frac{d}{dx}(\sec^{-1} x) =$

Derivative of inverse cosecant: $f(x) = \csc^{-1} x; \frac{d}{dx}(\csc^{-1} x) =$



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**Examples** Find the derivatives of the functions.

a. $f(x) = \log_3 x^5$

b. $y = \ln(sin(4^x))$

c. $g(x) = \sin^{-1}(\sqrt{x})$

d. $y = \csc^{-1}(\sin x)$

Suggestion: for each rule, make 2 of your own exercises and take the derivatives. Use a derivative calculator on Google to check your work!

**Implicit Differentiation**

**Examples** Find the derivatives of the functions.

a. $x^2 - xy + y^2 = 4$

b. $x^2 + y \sin x = x \cos 4y$



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**Logarithmic Differentiation**

Why would you want to use this?

**Examples** Let's review a few examples.

a. $y = \frac{x^2\sqrt{(4x^2-2)^5}}{\sqrt[3]{x^3-1}}$

b. $y^{lnx} = x^y$

**Logarithms and Differentiation**

Sometimes we will want to use properties of logarithms to help use simplify functions so that we can take the derivatives of them. This is not logarithmic differentiation, this is just leveraging properties of logarithms

**Examples** Take the derivatives of the following functions.

a. $f(x) = \ln\frac{x\sqrt{x+1}}{x-4}$

b. $y = \log_4 x - \log_4 x^3$



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What are all of the different types of notation for the following?

- First derivatives

- Second derivatives

- Third derivatives

- Fourth derivatives and higher derivatives

**Examples** Find the fourth derivative of the following function:

$$f(x) = \sin 3x + x^3 - e^{-2x} + 7$$



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**Examples** Find the second derivative of the following function:

$$x^3 - y^4 = 5$$

What are 4 different interpretations for the derivative at a point?



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**Example** Find the equation of the line tangent to $f(x) = x^2 - 3x + 1$ at the point $(1, -1)$.

**Example** A function has the derivative $f'(x) = 3x^2 - 2x$.

a. Do you have enough information to determine the equation of the tangent line at the point $x = 2$? If so, find the equation of the line.

b. What is the slope of the tangent line at $x = -3$?

c. What is a potential form of the original function?

d. If you know that $(-1, -4)$ is a point on $f(x)$, can you determine the original function? If so, find it.



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**Example** Find the equation of the line that is tangent to $f(x) = 3x^2 - 4x$ and parallel to $y = 14x - 2$

**Example** At time t, the position of a body moving along the s-axis is $s = t^3 + 6t^2 + 9t$ ft.

a. What is the velocity of the object at a $t = 3$ seconds?

b. When will the object reach a velocity of zero?

d. Find the body's acceleration at a time t.

e. What is the speed of the object at $t = 3$ seconds?



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**Lesson: Length of Parametrically Defined Curves**

How can we find the length of a curve C with parametric equations?

**Example** Find the length of the curve defined by: $x = \frac{t^2}{2}, y = \frac{(2t+1)^{\frac{3}{2}}}{3}, 0 \leq t \leq 4$



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How does the formula apply for a curve defined by $y = f(x)$? Give an example.



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Explain the arc length differential:

**Example** Find the coordinates of the centroid of the curve:

$$x = \cos t + t \sin t \, , y = \sin t - t \cos t \, , 0 \leq t \leq \frac{\pi}{2}$$



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**Lesson: Areas of Surfaces of Revolution from Parametric Curves**

State the area of surface of revolution for parametrized curves:

**Example** Find the areas of the surfaces generated by revolving the curves:

$$x = \left(\frac{2}{3}\right) t^{\frac{3}{2}}, y = 2\sqrt{t}, 0 \leq t \leq \sqrt{3}$$



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**Lesson: Polar Coordinates**

How do you define polar coordinates?

**Example** Find all the polar coordinates of the point $\left(3, \frac{\pi}{4}\right)$

**Example** Graph the following:

a. $2 \leq r \leq 3$ and $\theta = \frac{\pi}{4}$



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**Example** Graph the following:

b. $-1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{4}$

c. $\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{6}$

d. $r \geq 1$



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How can you relate polar and Cartesian coordinates?

**Example** Rewrite as an equivalent Cartesian equation:

a. $r\cos\theta + r\sin\theta = 1$

b. $\cos^2\theta = \sin^2\theta$



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**Example** Rewrite as an equivalent polar equation:

a. $x - y = 5$

b. $xy = 2$



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**Lesson: Graphing Polar Coordinate Equations**

Tests for Symmetry for Polar Graphs in the Cartesian xy-Plane

**Example** Identify any symmetries of the curves.

a. $r = 2 + \sin\theta$

b. $r^2 = \sin\theta$



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How to find the slope of the curve $r = f(\theta)$

**Example** Find the slopes of the curve at the given points.

a. $r = -1 + \sin\theta$ at $\theta = 0, \pi$

b. $r = \cos 2\theta\,, \theta = 0, \pm\frac{\pi}{2}$



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How can you graph a polar equation in the Cartesian xy-plane?

**Example** Graph the following in the Cartesian xy-plane

a. $r = 2 + \sin\theta$

b. $r^2 = \sin\theta$



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**Lesson: Areas in Polar Coordinates**

Explain how to find areas with polar equations:



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**Example** Find the areas of the regions:

a. Bounded by the circle $r = 3\sin\theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

d. Inside the circle $r = 4$ and above the line $r = 2\csc\theta$



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**Lesson: Lengths of a Polar Curve**

How do you find the length of a polar curve?

**Example** Find the length of the curves:

a. The spiral $r = \frac{e^{\theta}}{\sqrt{2}}, 0 \leq \theta \leq \pi$



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b. The parabolic segment $r = \frac{4}{1+\cos\theta}$, $0 \leq \theta \leq \frac{\pi}{2}$



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**Lesson: Introduction to Sequences**

What is a sequence?

What is the index of a sequence?

How do we describe the length of a sequence?

Write does a few examples of how sequences can be presented:



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**Example** Write down the first 5 terms of each sequence.

a. $a_n = \frac{1}{n!}$

b. $a_n = 2 + (-1)^n$

What is a recursive sequence?

Find the first six terms of the recursive sequence if:

$a_1 = 2, a_n = \frac{na_{n-1}}{n-1}$



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**Lesson: Introduction to Convergence of Sequences**

When does a sequence converge?

What a sequence does not converge, what do we say about it?

What notation do we use to denote a sequence has converged?



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How do we define sequence divergence?

**Example** Use the definition of sequence convergence to prove that $\lim_{n\to\infty} \frac{1}{n} = 0$

**Example** Use the definition of sequence convergence to prove that $\lim_{n\to\infty} c = c$ for any constant c.



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**Rules for Limits of Sequences**

Let $a_n$ and $b_n$ be sequences. Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then the following rules hold:

**Example** Determine if the following converge or diverge. If they converge, then what is the limit?

a. $a_n = \frac{1-3n}{1+3n}$

b. $a_n = \sqrt{n}$



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**Example** Determine if the following converge or diverge. If they converge, then what is the limit?

a. $a_n = \frac{3+(-1)^n}{n}$

b. $a_n = \sqrt{\frac{2n}{n+1}}$

Explain the Sandwich Theorem for Sequences



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**Example** Determine if and where the sequence converge to: $a_n = \frac{\sin n}{n}$

State the Continuous Function Theorem for Sequences

**Example** Determine if and where the sequence converge to: $a_n = \sqrt{\frac{n+1}{n}}$



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Explain the theorem that connects functions to sequences:

What is the cool result of this theorem?

**Example** Explain why $\lim_{n\to\infty} \frac{\ln n}{n^2} = 0$



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**Example** Determine if and where the sequence converge to: $a_n = \left(\frac{4n+1}{4n-1}\right)^n$



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**Lesson: Commonly Occurring Limits for Sequences**

State the 6 commonly occurring limits for sequences:

**Example** Determine whether the following sequences converge and to what:

a. $a_n = \frac{(-5)^n}{n!}$

b. $a_n = \sqrt[n]{5^{2n+1}}$



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**Example** Determine whether the following sequences converge and to what:

a. $a_n = \frac{\ln n}{n^{\frac{1}{n}}}$

b. $a_n = \left(1 - \frac{1}{n}\right)^n$

c. $a_n = \frac{7^n}{n!}$



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**Lesson: Boundedness and Monotonicity for Sequences**

Explain the concepts of being bounded from above versus bounded from below:

What does it mean to be a bounded versus an unbounded sequence?



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**Example** Describe a sequence with a lower bound but no upper bound:

**Example** Describe a sequence with an upper bound but no lower bound:

**Example** Explain a bounded sequence that does not converge:



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Explain the difference between nonincreasing vs nondecreasing:

What does it mean for a sequence to be monotonic?

**Example** Write an example of each type of sequence:

a. A nondecreasing sequence

b. A nonincreasing sequence:

c. A sequence that is not monotonic:



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State the Monotonic Sequence Theorem:



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**Examples: Determining the Convergence of Sequences**

Determine if the following sequences converge or diverge.

a. $a_n = \frac{3^n}{n^3}$

b. $a_n = \frac{(2n+2)!}{(2n-1)!}$



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c. $a_n = (\ln n - \ln(n-1))$

d. $a_n = \tan^{-1} n$



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**Examples: Determining the Convergence of Recursively Defined Sequences**

Determine if the following sequences converge or diverge.

a. $a_1 = -1, a_{n+1} = \frac{a_n+6}{a_n+2}$

b. $a_1 = 3, a_{n+1} = 8 - \sqrt{a_n}$



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**Lesson: Intro to Infinite Series**

Write down an example of a sequence:

Using that sequence, show how we can turn a sequence into a series:

What is the difference between a finite series and an infinite series?

Show how we can denote summing up only n terms:



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In some cases, a pattern can emerge with infinite series. Let's investigate the sum of $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}+\cdots$

| Partial Sum | Value of Sum | Another way to think of this |
|---|---|---|
| $S_1 = 1$ | $1$ | |
| $S_2 = 1+\frac{1}{2}$ | $\frac{3}{2}$ | |
| $S_3 = 1+\frac{1}{2}+\frac{1}{4}$ | $\frac{7}{4}$ | |
| $S_n = 1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}$ | $\frac{2^n-1}{2^{n-1}}$ | |

How do we denote a sequence of partial sums?

How do we denote an infinite series?

What happens if the partial sums of an infinite series converge to a limit L?



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Use the sum $a_n = 1 + \frac{1}{n}$ show how we can visualize the sums of a series with rectangles:

Use the sum $a_n = \frac{1}{n^2}$ show how we can visualize the sums of a series with rectangles:



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**Lesson: Geometric Series**

What is the general form of a geometric series?

Show an example of a geometric series:

What's are two ways to write a geometric series?

What is the ratio of a geometric series?

True or false: The ratio of a geometric series can only be positive



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Explain how we can evaluate the value of a geometric series.

Explain when a geometric series converges versus diverges:



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**Example** Write out the first 4 terms of the series, then determine if the series converges or diverges.

$$\sum_{n=0}^{\infty}(\frac{(-1)^n}{5^n}$$

**Example** Determine if the geometric series converges or diverges. If it converges, find its sum.

$$\frac{9}{4}-\frac{27}{8}+\frac{81}{16}-\frac{243}{32}+\cdots$$



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If $\Sigma a_n = A$ and $\Sigma b_n = B$ are convergent series, then what other properties hold?

**Example** Write out the first 4 terms of the series, then determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \left( \frac{5}{2^n} + \frac{1}{3^n} \right)$$



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**Example** Express the number as the ratio of two integers: $0.\overline{231} = 0.231\ 231\ 231\ 231\ \ldots$



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**Lesson: Nth Term Test for a Divergent Series**

If $\Sigma_{n=1}^{\infty} a_n$ converges, what can be said about $a_n$?

What is the nth-Term Test for Divergence

**Examples** Why do the following series diverge?

a. $\Sigma_{n=1}^{\infty}\, n$

b. $\Sigma_{n=1}^{\infty} \left(\frac{2n+3}{n}\right)$

c. $\Sigma_{n=1}^{\infty}(-1)^{n+1}$



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**Lesson: Combing Series, Reindexing Terms in a Series, and Adding or Deleting Terms in a Series**

How do you find the value of the series $\Sigma_{n=5}^{\infty}\left(\frac{1}{4^n}\right)$? Note the index.

How can we reindex the series $\Sigma_{n=1}^{\infty}\left(\frac{1}{7^{n-1}}\right)$?



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**Examples: Working with Geometric Series**

Determine if the following series converge or diverge.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$

b. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$



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**Examples: Repeating Decimals and Geometric Series**

Express each number as the ratio of two integers.

a. $0.\overline{27} = 0.27\ 27\ 27\ 27\ \ldots$

b. $1.35\overline{127} = 1.35\ 127\ 127\ 127\ \ldots$



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**Examples: The nth Term Test for Divergence**

Use the nth-term test for divergence to show that the series is divergent OR that the test is inconclusive.

a. $\sum_{n=0}^{\infty} \frac{e^n}{e^n+n}$

b. $\sum_{n=0}^{\infty} \frac{n}{n^2+7}$



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**Examples: Partial Sums with Telescoping Series**

Find a formula for the nth partial sum, then use it to determine if the series converges or diverges.

a. $\sum_{n=1}^{\infty}(\tan n - \tan(n-1))$

b. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$



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**Lesson: The Integral Test for Infinite Series**

When does a series of nonnegative terms converge?

Explain how this idea applies to the harmonic series $\Sigma_{n=1}^{\infty} \frac{1}{n}$

State the Integral Test



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**Example** Does the following series converge: $\Sigma_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$

For what values of p does the p-series converge?

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$



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**Example** Determine if the following series converge by the Integral Test

a. $\Sigma_{n=1}^{\infty}\left(\frac{n}{n^2+5}\right)$

b. $\Sigma_{n=2}^{\infty}\left(\frac{1}{n(\ln n)^2}\right)$



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**Examples: Using the Integral Test for Series**

Determine which series converge or diverge using the integral test.

a. $\sum_{n=1}^{\infty} e^{-2n}$

b. $\sum_{n=1}^{\infty} \frac{n^2}{e^{\frac{1}{3}n}}$



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**Lesson: Comparison Tests for Series**

State the direct comparison test:

**Example** Show how to use the direct comparison test to determine if the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$$



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State the Limit Comparison Test:

**Example** Use the limit comparison test to determine if the series converges or diverges:

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$$



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**Example** Use the limit comparison test to determine if the series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$



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**Examples: Comparison Tests for Series**

Use any methods to determine if the following converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$

b. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$



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c. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$

d. $\sum_{n=1}^{\infty} \frac{\sec^{-1} n}{n^{2.3}}$



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**Lesson: The Alternating Series Test**

State the Alternating Series Test

**Example** Does the alternating harmonic series converge? $\Sigma_{n=1}^{\infty}\left((-1)^{n+1}\left(\frac{1}{n}\right)\right)$



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**Example** Determine if the alternating series converge or diverge. Not all series satisfy the conditions of the Alternating Series Test.

a. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}1}{n5^n}\right)$

b. $\Sigma_{n=1}^{\infty}((-1)^{n+1}\left(\frac{n}{5}\right)^n$

c. $\Sigma_{n=1}^{\infty}\left((-1)^n(\frac{n}{n^2+1})\right)$



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**Lesson: Conditional Convergence**

Explain what conditional convergence is:

Explain how conditional convergence applies to the alternating p-series: $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{n^p}\right)$



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What is the Rearrangement Theorem for Absolutely Convergent Series?

**Example** What happens if we multiply the alternating harmonic series by 2?



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**Lesson: Summary of Tests to Determine Convergence or Divergence**

State the 6 different tests we've discussed so far for infinite series.



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**Example** Using each test at most once, determine the convergence of the following series.

a. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^n n}{n+5}\right)$

b. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}(0.5)^n}{n}\right)$

c. $\Sigma_{n=1}^{\infty}(-1)^n\left(\frac{1}{\sqrt{n}}\right)$



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d. $\Sigma_{n=1}^{\infty}(-6)^{-n}$

e. $\Sigma_{n=1}^{\infty}\left(\frac{(-105)^n}{n!}\right)$

f. $\Sigma_{n=1}^{\infty}\left(\frac{(-1)^{n+1}(\sqrt{n}+1)}{n+1}\right)$



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**Examples: Alternating Series Test**

Determine if the following satisfy the conditions of the alternating series test, and then determine if the series converges.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+5}{n^2+4}$

b. $\sum_{n=1}^{\infty}(-1)^n \ln\left(1+\frac{1}{n}\right)$



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**Examples: Conditional Convergence**

Determine if the series converges absolutely, if it converges, or if it diverges.

a. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+2}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+3}$

c. $\sum_{n=1}^{\infty}(-1)^{n+1}\sqrt[n]{13}$



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d. $\sum_{n=1}^{\infty}(-1)^n \frac{(-2)^{n+1}}{n+5^n}$

e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \tan^{-1} n}{n^2+1}$



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**Lesson: Absolute Convergence for Infinite Series**

What does it mean for a series to converge absolutely?

What is the Absolute Convergence Test?

**Examples** Determine if the following series converge or diverge absolutely.

a. $\Sigma_{n=1}^{\infty}(-1)^n\left(\frac{1}{n^3}\right)$

b. $\Sigma_{n=1}^{\infty}\left(\frac{\cos n}{n^2}\right)$



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**Lesson: The Ratio Test**

State the Ratio Test

**Examples** Use the Ratio Test to determine the convergence of the series.

a. $\Sigma_{n=1}^{\infty}\left(\frac{(3n)!}{n!n!}\right)$

b. $\Sigma_{n=1}^{\infty}\left(\frac{n!n!}{(3n)!}\right)$



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**Lesson: The Root Test**

What is the Root Test?

**Example** Use the Root Test to determine if the series converge absolutely or diverge.

a. $\Sigma_{n=1}^{\infty}\left(\frac{5n-3}{4n+7}\right)^{n}$

b. $\Sigma_{n=1}^{\infty}\left(\sin^{n}\left(\frac{1}{\sqrt[3]{n}}\right)\right)$



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**Review: Factorials within a Series**

Back to basics: use $f(x) = x^2 - 2x + 1$ to compute the following

a. $f(3)$

b. $f(n+1)$

Now let's compute $\frac{u_{n+1}}{u_n}$ for the following:

a. $\frac{3^n}{(n+1)!}$

b. $\frac{(2n)!}{(3n+1)!}$



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Compute the ratio test for the following:

$$\sum_{n=1}^{\infty}\left(\frac{(n!)^2}{(2n)!}\right)$$



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**Examples: The Ratio Test**

Determine if the series converges absolutely or if it diverges.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{n+5}{5^n}$

b. $\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+4)^2}$



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**Examples: The Root Test**

Determine if the series converges absolutely or if it diverges.

a. $\sum_{n=1}^{\infty} \frac{5^n}{(4n)^n}$

b. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$



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**Examples: Series with Recursively Defined Terms**

Determine if the series $\sum_{n=1}^{\infty} a_n$ converge or diverge.

a. $a_1 = 2\ a_{n+1} = \frac{1+\cos n}{n}\ a_n$

b. $a_1 = 5, a_{n+1} = \frac{\sqrt[n]{n}}{2}\ a_n$



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**Lesson: Intro to Power Series**

What is a power series?

What is one power series that we are already familiar with?



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Describe the different terms in the power series:

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots$$

Describe some polynomial approximations of this power series, and then try to find some way to explain what these mean in the greater idea of a power series. You have some extra space here to draw our any graphs if you'd like.



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**Example** For what values of x do the power series converge?

a. $\Sigma_{n=1}^{\infty}(-1)^{n-1}\left(\frac{x^n}{n}\right)$



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b. $\Sigma_{n=0}^{\infty} n!\, x^{2n}$

c. $\Sigma_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)$



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State the Convergence Theorem for Power Series

What is a corollary to this theorem?



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Explain the steps for how to test a power series for convergence:



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**Example** For each of the following, i) find the series interval of convergence, ii) determine for what values the series converges absolutely, and iii) determine for what values the series converges conditionally.

a. $\Sigma_{n=0}^{\infty}(x+8)^n$



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b. $\Sigma_{n=1}^{\infty}\left(\frac{(x-3)^n}{n^4 4^n}\right)$



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**Lesson: Operations on Power Series**

Explain series multiplication for power series

Give an example of how this works:



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State the next theorem in this video

Explain term by term differentiation for power series:



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**Example** Find $f'(x)$ and $f''(x)$ if $f(x) = \frac{1}{x-1} = \Sigma_{n=0}^{\infty} x^n, -1 < x < 1$

Explain term by term integration for power series:



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**Example** Show how to differentiate and integrate term by term with the function: $\Sigma_{n=0}^{\infty}\left((-1)^n\left(\frac{x^{2n+1}}{3n+1}\right)\right)$



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**Examples: Intervals of Convergence**

Find the series' radius and interval of convergence.

a. $\sum_{n=1}^{\infty} \frac{x^n}{n*5^n*\sqrt{n}}$

b. $\sum_{n=1}^{\infty} x^n(\ln n)$



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c. $\sum_{n=2}^{\infty} \frac{x^n}{n\,(\ln n)^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{2*4*6*\ldots*2n}\ x^n$



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**Lesson: Introduction to Taylor and Maclaurin Series**

Recall our power series:

$\Sigma_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$

Consider term-by-term differentiation for this series:

It we plug in x=a, what is the result of our derivatives?

From here, what result do we get?



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Define a Taylor series generated by f(x) at x=a:

Explain what a Maclaurin series is:



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**Example** Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $x = 2$.



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Define a Taylor polynomial of order n:

**Example** Find the Taylor polynomials of orders 0, 1, 2, and 3 by: $f(x) = \frac{1}{x+1}$ at $x = 0$



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**Example** Find the Taylor series and Taylor polynomials generated by $f(x) = e^x$ at $x = 0$



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**Example** Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$



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**Examples: Finding Taylor Polynomials**

Find the Taylor polynomials of orders 0, 1, 2, and 3 for $f(x) = \ln(1 + x)$ at $a = 0$.



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**Examples: Finding Taylor Series at x=0 (Maclaurin Series)**

Find the Maclaurin series for the following:

a. $xe^x$



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b. $\sinh x$



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c. $x^4 - x^3 - 2x + 3$



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**Examples: Finding Taylor Series**

Find the Taylor series generated by each function at $x = a$

a. $f(x) = x^4 + 2x^2 + 1, a = -2$



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b. $f(x) = \frac{1}{x^2}, a = 1$



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**Examples: Maclaurin Series and Absolute Convergence**

Find the first three nonzero terms of the Maclaurin series for the function $f(x) = (\sin x)(\ln(1 + x))$ and the values of x for which the series converges absolutely.



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**Lesson: Convergence of Taylor Series**

State **Taylor's Theorem**

State **Taylor's Formula**



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What is another way to think of Taylor's formula?

What happens if $R_n(x) \to 0$ as $n \to \infty$?

**Example** Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for all values of x.



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State **The Remainder Estimation Theorem**

**Example** Show that the Taylor Series for $\sin x$ at $x = 0$ converges for every value of x.



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In the next examples, we will see how to use addition, subtraction, and multiplication with Taylor series. All of these operations are valid on the intersection of their intervals of convergence.

**Example** Use known series to find the first few terms of the Taylor series for the given function.

a. $\frac{1}{2}(3x + x\cos x)$

b. $x^2 \sin x$



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**Example** For what values of x can we replace $\sin x$ by $x - \frac{x^3}{3!}$ and obtain an error who magnitude is no greater than $3 - 10^{-4}$?



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**Lesson: Some Common Taylor Series**

List the common Taylor series that you should have memorized:



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**Examples: Finding Taylor Series with Substitution**

Use substitution to find the Taylor series at $x = 0$ of the functions below.

a. $e^{-3x}$

b. $\cos 7x^2$



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**Examples: Finding Taylor Series with Substitution**

Use power series operations to find the Taylor series at $x = 0$ of the functions below.

a. $\sin x - x + \frac{x^3}{3!}$



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b. $\sin x * \cos x$



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c. $\sin^2 x$



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**Lesson: The Binomial Series for Powers and Roots**

Determine the Taylor series generated by: $f(x) = (1 + x)^n$

What is this series called?



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When does the binomial series converge?

What is another way we can define the binomial series?

**Example** Determine the first 4 terms of the binomial series for the function $f(x) = \left(1 + \frac{x}{2}\right)^{-2}$



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**Lesson: Evaluating Nonelementary Integrals**

We can use a familiar Taylor series to find the sum of given power series. For instance, how could we express:

$\int \cos x^2 \ dx$ as a power series?

How can we estimate $\int_0^1 \cos x^2 \ dx$ with an error less than 0.001.



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**Example** Find a polynomial that will approximate $F(x)$ in the given interval with an error of magnitude less than $10^{-3}$:

$F(x) = \int_0^x t^2 e^{-t^2} dt, [0, 1]$



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**Lesson: Evaluating Indeterminate Forms**

We can evaluate indeterminate forms by expressing the functions involved as Taylor series.

**Example** Evaluate the limits:

a. $\lim_{x\to 0} \frac{\sin x - x + (\frac{x^3}{6})}{x^5}$



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b. $\lim_{x \to 2} \left( \frac{x^2 - 4}{\ln(x-1)} \right)$



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**Lesson: Euler's Identity**

Review the powers of i:

Explain what $e^{i\theta}$ is expanded:

What result does this give us?



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What is Euler's identity?

**Example** Write the power in the form $a + bi$: $e^{\frac{i\pi}{4}}$



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**Examples: Approximations and Nonelementary Integrals**

a. Use series to estimate the integrals' value with an error of magnitude less than $10^{-5}$:

$$\int_0^{0.4} \frac{e^{-x}-1}{x}\,dx$$

b. Use series to approximate the values of the integral with an error of magnitude less than $10^{-8}$:

$$\int_0^{0.1} e^{-x^2}\,dx$$



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**Examples: Indeterminate Forms**

Use a series to evaluate the limits.

a. $\lim_{x\to 0} \frac{(e^x - e^{-x})}{x}$

b. $\lim_{x\to \infty} (x+1)\cos\frac{1}{x+1}$



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**Lesson: Intro to Three-Dimensional Coordinate Systems**

Show how we can draw the three-dimensional coordinate system using the right-hand rule.

How can we consider graphing $f(x, y) = x^2 + y^2$?



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Using the $(x, y, z)$ coordinate, what do we have if we set …

- $x = 0$?

- $y = 0$?

- $z = 0$?

How do these 3 coordinate plans divide up the coordinate system?

**Example** What part of the Cartesian coordinates is each referring to?

a. $x \geq 0$

b. $y = 3$

c. $x \leq 0, y \leq 0, z \leq 0$



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**Example** What points satisfy $x^2 + z^2 = 9$ and $y = 2$?



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**Lesson: The Standard Form of a Sphere**

What is the standard equation for the sphere of radius a and center $(x_0, y_0, z_0)$?

**Example** Find the center and radius for the sphere: $4x^2 + 4y^2 + 4z^2 - 8x - 3y = 8$



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**Lesson: The Distance Between Two Points in 3 Dimensions**

State the distance formula.

**Example** Find the distance between $P_1(4, 5, 6)$ and $P_2(5, 7, 8)$



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**Examples: Describing the Regions Created by Different Equations**

Describe what each set of points look like that fit the following conditions:

a. $x = 4, y = 6$

b. $x^2 + y^2 = 9, z = x$

c. $x^2 + y^2 + z^2 = 9, y = z$

d. $z = y^2, x = 2$



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**Examples: Describing the Regions Created by Different Equations and Inequalities**

Describe what each set of points look like that fit the following conditions:

a. $x \geq 0, y \geq 0, z \geq 0$

b. $x \geq 0, y \geq 0, z = 0$

c. $x^2 + y^2 \leq 4, z = 0$



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d. $x^2 + y^2 \leq 4$, no restriction on z

f. $1 \leq x^2 + y^2 + z^2 \leq 9$



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**Examples: The Standard Form of a Sphere**

Find the center C and the radius a for the spheres:

a. $x^2 + y^2 + z^2 + 4x - 6y - 4z = 0$

b. $2x^2 + 2y^2 + 2z^2 - 8y + 6z = 0$

c. $(x - 2)^2 + (y - 1)^2 + (z + 1)^2 = 103 + 4x + 2y - 2z$



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**Examples: Finding the Equation that Fulfills Given Criteria**

a. Find the equation that describes the plane perpendicular to the x-axis at $(5, 0, 0)$

b. Find the equation that describes the plane through the point $(2, -1, 1)$ perpendicular to the y-axis.

c. Find the equation of a circle of radius 3 centered at $(2, -2, 1)$ and lying in a plane parallel to the yz-plane.



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**Lesson: Intro to Vectors**

What is the definition of a vector?

How do we describe and represent a vector with a line segment?

How do we represent vectors?

When are two vectors equal?



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How can we represent a vector created by a line segment between two points?

Define the magnitude of a vector:

What vector(s) have a length of 0?



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**Example** Find the component form and length of the vector with initial point $P(1, 2, 3)$ and terminal point $Q(-2, 1, 5)$.

Physics often discuss forces, which intuitively means the push or pull on a object and how that object moves. Forces are just vector quantities. Here is an example: a small cart is being pulled along a smooth horizontal floor with a 10-lb force $\vec{F}$**,** making a $45°$ angle to the floor. How can we describe the effective force moving the cart forward?



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**Lesson: Intro to Vector Operations**

Suppose we have three vectors: $\vec{u} =< u_1, u_2, u_3 >$, $\vec{v} =< v_1, v_2, v_3 >$, and $\vec{w} =< w_1, w_2, w_3 >$. Let $a$ and $b$ be scalars.

Describe vector addition:

Describe scalar multiplication:



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Write out the **Properties of Vector Operations**



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**Example** Let $\boldsymbol{u} =< 1, 2, 3 >$ and $\boldsymbol{v} =< -4, 3, -2 >$. Find the components of the following:

a. $2\boldsymbol{u} + 4\boldsymbol{v}$

b. $\boldsymbol{v} - \boldsymbol{u}$

**c.** $|\frac{1}{2}\boldsymbol{v}|$



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**Lesson: Intro to Unit Vectors**

What are the standard unit vectors?

How can any vector be written as a linear combination of the unit vectors?

How can we take any vector and rewrite it as a unit vector?



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**Example** Find a unit vector $\boldsymbol{u}$ in the direction of a vector from $P(0, 1, 2)$ to $Q(3, -1, 1)$.

In physics, speed in relation to a vector just refers to the vector's magnitude.

**Example** If $\boldsymbol{v} = 2\mathrm{i} - \mathbf{3}\mathrm{j}$ is a velocity vector, express $\boldsymbol{v}$ as a product of its speed times its direction of motion.

Summarize the results of this lesson:



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**Examples: Components and Magnitudes of a Vector**

Let $\boldsymbol{u} =< 2, 5 >$ and $\boldsymbol{v} =< 4, -1 >$. Find the component form and magnitude of the following vectors.

a. $4\boldsymbol{v}$

**b.** $\boldsymbol{u} - \boldsymbol{v}$

**c.** $\frac{3}{5}\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$



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**Examples: Finding the Component Form of a Vector**

Find the component form of each vector.

a. The vector $\overrightarrow{PQ}$ when $P = (3, 4)$ and $Q = (7, -1)$

b. The vector $\overrightarrow{OP}$ where O is the origin, P is the midpoint of the line segment $AB$, and $A = (-3, 1)$ and $B = (-1, 5)$

c. The unit vector obtained by rotating the vector $< 1, 0 > 135°$ counterclockwise about the origin.



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**Examples: Finding the Magnitude of Vector Forces**

Refer to the drawing in the video. A 50-N weight is suspended by two wires. IF the magnitude of vector $\overrightarrow{F_1}$ is 35N, find the angle $\alpha$ and the magnitude of the vector $\overrightarrow{F_2}$.



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**Examples: Geometric Representations of Vectors**

Copy the vector diagram in the video, and the perform the following operations geometrically.

a. $\boldsymbol{u} + \boldsymbol{v}$

b. $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}$

**c.** $2\boldsymbol{u} - \boldsymbol{v}$



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**Examples: Expressing Vectors as a Product of Length and Direction**

Express each vector as a product of its length and direction.

a. $3\boldsymbol{i} - 2\mathbf{j} + 5\mathbf{k}$

b. $\frac{1}{2}\boldsymbol{i} + \frac{1}{4}\boldsymbol{k}$



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**Lesson: The Dot Product**

How can we find the angle between two vectors?

Part 1: when we can utilize a right triangle

Part 2: when we cannot utilize a right triangle



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Define the Dot Product:

Summarize the angle between vectors:

What is another way to define the dot product?



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**Example** Find the dot product.
a. $\langle 3, -2, 5\rangle \cdot \langle -1, 3, 1\rangle$

b. $(\frac{1}{2}\boldsymbol{i} - \frac{1}{3}\boldsymbol{j} + \frac{1}{4}\boldsymbol{k}) \cdot (6\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k})$

**Example** Find the angle between the vectors $\boldsymbol{u} = \langle 2, 1, 3\rangle$ and $\boldsymbol{v} = \langle -1, 0, 3\rangle$



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Define what it means for two vectors to be orthogonal.

**Example** Determine if the following vectors are orthogonal: $\boldsymbol{u} = \langle 2, 3 \rangle$ and $\boldsymbol{v} = \langle -6, 4 \rangle$

What vector is orthogonal to every nonzero vector?



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List the Properties of the Dot Product



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**Lesson: Vector Projections**

Explain how vector projections work:

Summarize the vector projection of $\vec{u}$ onto $\vec{v}$:



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Summarize the scalar component of $\vec{u}$ in the direction of $\vec{v}$L

**Example** Let $\boldsymbol{u} = 3\boldsymbol{i} - 2\mathbf{j} + \mathbf{k}$ and $\boldsymbol{v} = 2\boldsymbol{i} + \mathbf{j} - 3\mathbf{k}$.

a. Find $\boldsymbol{u} \cdot \boldsymbol{v}$, $|\boldsymbol{u}|$, $|\boldsymbol{v}|$

**b.** Find the cosine of the angle between the vectors.

c. Find the scalar component of $\boldsymbol{u}$ in the direction $\boldsymbol{v}$**.**

d. Find the vector $proj_{\vec{v}}\vec{u}$



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**Lesson: Work and Vectors**

With vectors, we can study work done by a constant force when a force is applied in the direction other than the direction in motion.

What is the equation for work done by a constant force applied in the direction of motion? Explain this equation.

Explain how we can understand a force moving on an object through a displacement vector.



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**Example** If $|\boldsymbol{F}| = 20N$ and $|\boldsymbol{D}| = 2m$ and $\theta = 60°$, find the work done by $\boldsymbol{F}$.



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**Examples: Dot Products and Projections**

For the following vectors, find
i) $\boldsymbol{u} \cdot \boldsymbol{v}, |\boldsymbol{u}|, |\boldsymbol{v}|$
ii) the angle between the vectors
iii) the scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$
iv) The vector $proj_{\boldsymbol{v}}\boldsymbol{u}$

a. $\boldsymbol{u} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

**b.** $\boldsymbol{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}} \rangle, \boldsymbol{v} = \langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \rangle$



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**Examples: Working with Diagonals of Shapes**

Show that the diagonals of a rhombus are perpendicular.



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**Examples: Work**

How much work does it take to slide a crate 10m along a loading dock by pulling on with a 100N force at an angle of $30°$ from the horizontal?



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**Lesson: The Cross Product**

Define the cross product and where it comes from:



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When are two vectors parallel?

List the properties of the cross product:

Explain the significance of $|\vec{u} \times \vec{v}|$



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How can you use the determinant formula for calculate the cross product?

**Example** Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ If $\vec{u} = \langle 2, 1, 3 \rangle$ and $\vec{v} = \langle -1, 3, 1 \rangle$



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**Example** Find a vector perpendicular to the plane of $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

**Example** Find the area of the triangle with vertices $P(1,0,1), Q(2,1,-1), R(-1,1,2)$

**Example** Find a unit vector perpendicular to the plane of $P(1,0,1), Q(2,1,-1), R(-1,1,2)$



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Explain how torque relates to vectors:

**Example** Find the magnitude of the force **F** at the pivot point P in the picture.



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Explain the triple scalar or box product:



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Summarize how to calculate the triple scalar product as a determinant:

**Example** Find the volume of the box determined by $\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = 5\boldsymbol{i} + 2\boldsymbol{k}, \boldsymbol{w} = -2\boldsymbol{j} + \boldsymbol{k}$



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**Review: Determinants**

Find the following determinants.

a. $\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}$

b. $\begin{vmatrix} -1 & 6 \\ 3 & 5 \end{vmatrix}$

c. $\begin{vmatrix} 3 & 1 & -1 \\ 2 & 1 & 2 \\ 1 & 4 & 2 \end{vmatrix}$



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Compute $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{vmatrix}$ two ways.



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**Lesson: Triple Scalar or Box Product**

Explain the Triple Scalar or Box Product

Summarize how to calculate the triple scalar product as a determinant:



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**Example** Find the volume of the box determined by $\boldsymbol{u} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}, \boldsymbol{v} = 5\boldsymbol{i} + 2\boldsymbol{k}, \boldsymbol{w} = -2\boldsymbol{j} + \boldsymbol{k}$



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**Examples: Finding Area of Triangles with the Cross Product**

Find i) the area of the triangle determined by the points P, Q, and R. ii) Find a unit vector perpendicular to plane PQR.

$P(1,2,1), Q(1,0,-1), R(3,-1,1)$



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**Lesson: Lines and Line Segments in Space**

Explain the vector equation for a line.

What are the parametric equations for a line?



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**Example** Find the parametric equations for the line through $(-1, 0, 2)$ parallel to $\boldsymbol{v} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

Example Find the parametric equations for the line through $P(-2, 0, 1)$ and $Q(-1, 1, 2)$



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**Example** Parametrize the line segment joining the points $P(-2, 0, 1)$ and $Q(-1, 1, 2)$.

How do you find the distance from a point S to a line through P parallel to a vector $\boldsymbol{v}$**?**



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**Lesson: Planes in Space**

How can we find the equation for a plane in space?

**Example** Find an equation for the plane through $P_0(-1, 2, 3)$ perpendicular to $\boldsymbol{n} = 3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.



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**Example** Find an equation for the plane through $A(0,0,2), B(3,0,0), C(0,1,0)$

**Example** Find a vector parallel to the line of intersection of the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$.



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**Example** Find parametric equations for the line in which the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$ intersect.

**Example** Find the point where the line:

$$x = \frac{3}{2} + 2t, y = 2t, z = 1 - t$$

Intersects the plane $2x - 3y - 6z = 12$



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Describe the distance from a point S to a plane with normal **n** at point P.

**Example** Find the distance from $S(1, 2, 1)$ to the plane $2x - 3y + 6z = 12$.



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**Example** Find the angle between the planes $2x - 3y - 6z = 12$ and $2x + 2y - z = 6$.



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**Examples: Parametric Equations for Lines**

Find parametric equations for the lines.

a. The line through $P(1, 1, 2)$ and $Q(-1, 0, 1)$

b. The line through $(2, 4, 6)$ perpendicular to the plane $x - y + 4z = 8$.



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**Examples: Equations for Planes**

Find the equations for the planes.

a. The plane through $(1, 1, 2), (3, 0, 3), (0, -1, 1)$

b. The plane through $A(1, -1, 0)$ perpendicular to the vector from the origin to A.



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**Examples: Finding Planes Containing Intersecting Lines**

Find the plane containing the intersecting lines:

$$L_1: x = 1 - t, \quad y = 3 + t, \quad z = 1 + t, \quad -\infty < t < \infty$$
$$L_2: x = 1 - 2s, \quad y = 1 + 4s, \quad z = 3 - 3s, \quad -\infty < t < \infty$$



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**Examples: Distance from a Point to a Line**

Find the distance from the point $(0, 1, 1)$ to the line $x = 2 + 3t,\ \ y = 1 - t,\ \ z = -2 + 2t$



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**Examples: Sketching Quadric Surfraces**

Sketch the following.
a. $x^2 + z^2 = 9$

b. $z = 4 + x^2 + y^2$

c. $z = 1 - y^2 + x^2$



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**Lesson: Intro to Vector-Valued Functions**

What are component functions?

What is a vector-valued function?

Draw an example of a vector-valued function:



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**Examples: Determining a Space Curve from Its Function**

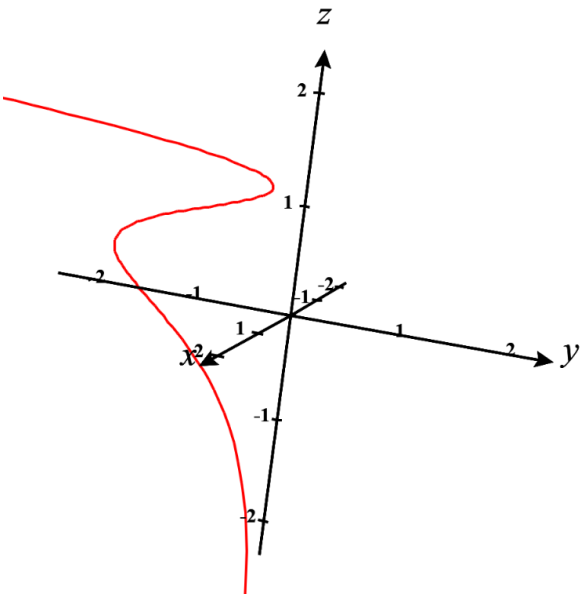
Determine which $\vec{r}(t)$ matches the space curves below.

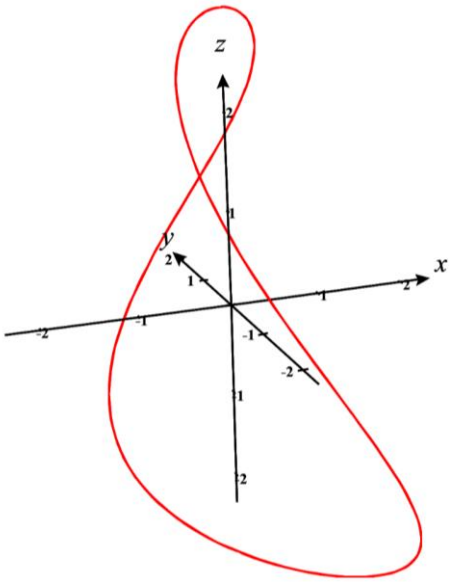
a. $\vec{r}(t) = t\vec{\imath} - (\sin t)\vec{j} + (\ln t)\vec{k}$

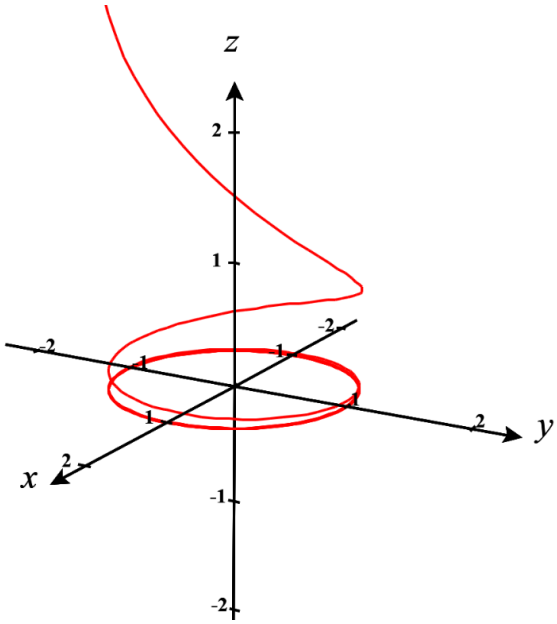
b. $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\vec{j} + (e^t)\vec{k}$

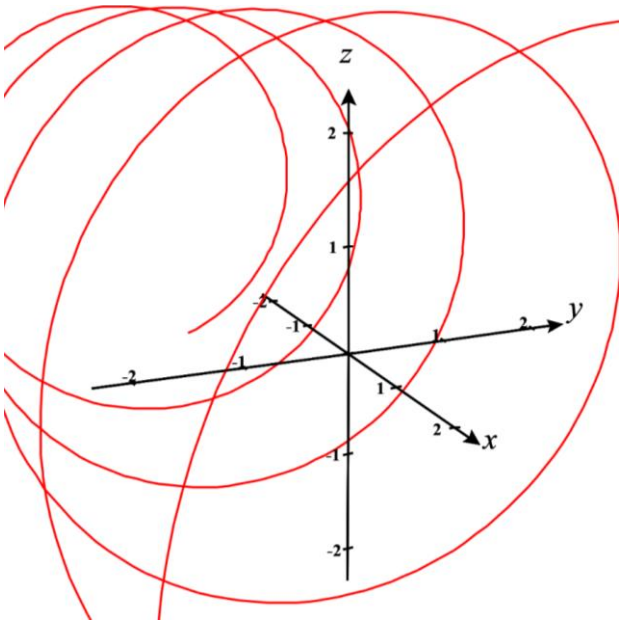
c. $\vec{r}(t) = \left(\frac{1}{2}t\right)\vec{\imath} + (3\cos t)\vec{j} + (3\sin t)\vec{k}$

d. $\vec{r}(t) = (\sin 2t)\vec{\imath} + (3\cos t)\vec{j} + (3\sin t)\vec{k}$



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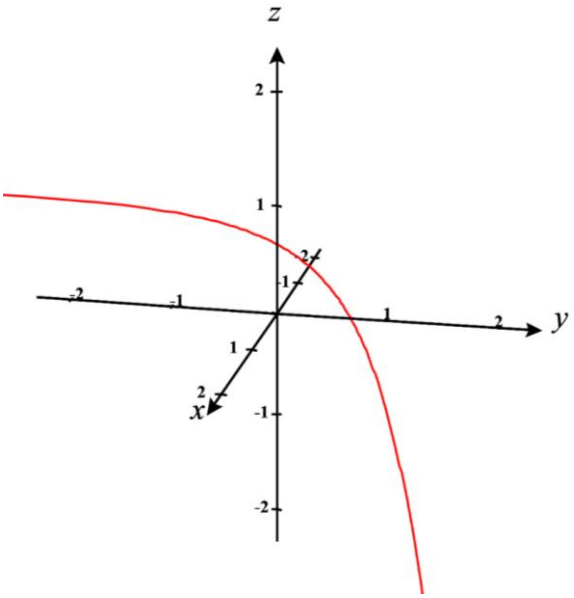
Determine which $\vec{r}(t)$ matches the space curves below.

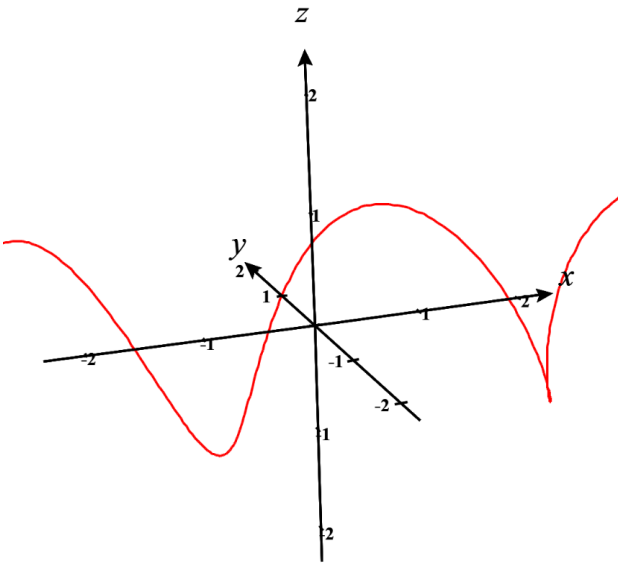
a. $\vec{r}(t) = (\sin t)\,\vec{\imath} + (\cos t)\vec{\jmath} + (\frac{1}{2}t)\vec{k}$

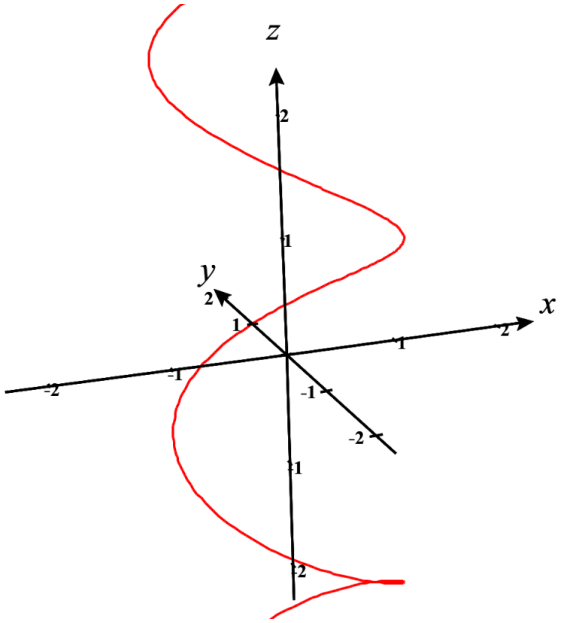
b. $\vec{r}(t) = \left(\frac{1}{2}t\right)\vec{\imath} + (\cos t)\vec{\jmath} + (\sin t)\vec{k}$

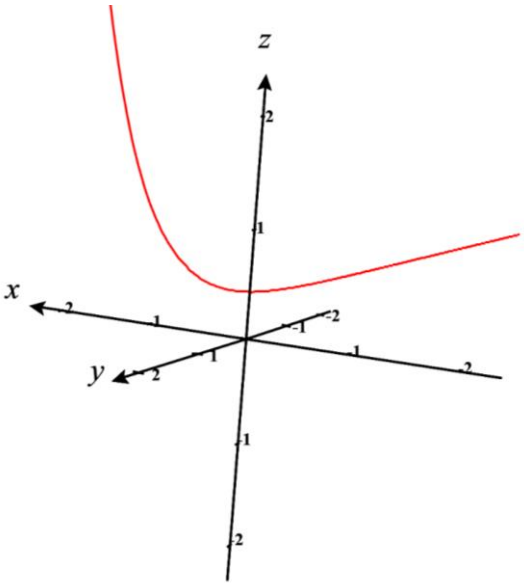
c. $\vec{r}(t) = (t)\vec{\imath} + (\ln t)\vec{\jmath} + (1)\vec{k}$

d. $\vec{r}(t) = (1)\vec{\imath} + (t)\vec{\jmath} + (e^t)\vec{k}$



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**Examples: Limits with Vectors**

Find the following limits.

a. $\lim\limits_{t\to\frac{\pi}{2}}[(\cos t)\vec{\imath} + (\sin 2t)\vec{\jmath} + \tan(4t)\vec{k}]$

b. $\lim\limits_{t\to 1}\left[\frac{t-1}{\sqrt{t}-1}\vec{\imath} + \frac{t^3-1}{\ln t}\vec{\jmath} - (\tan^{-1} t)\vec{k}\right]$



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**Lesson: The Basics of Tangent Lines for Space Curves**

Write out what is reviewed in the beginning of the video.

Show how to find the tangent line for $\vec{r}(t) = (\sin t)\vec{\imath} + (\cos t)\vec{\jmath} + (t)\vec{k}$ at $t = 0$



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**Examples: Differentiating Vector-Valued Functions**

Below $\vec{r}(t)$ represents the position of a particle in space at time time. Find i) the particle's velocity, ii) the particle's acceleration, iii) the speed of the particle at any moment, and iv) find a way to write the particle's velocity as a product of its speed and direction

a. $\vec{r}(t) = (t^2)\vec{\imath} + (\cos t)\vec{\jmath} - (e^{-t})\vec{k}; t = 0$

b. $\vec{r}(t) = (\sec t)\vec{\imath} - (\ln 4t)\vec{\jmath} + (\tan t)\vec{k}; t = \frac{\pi}{4}$



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Derivative Rules Challenge for Calculus III

| Name | Function Form | Derivative |
|---|---|---|
| Product Rule | $f(x) * g(x)$ | |
| Quotient Rule | $\frac{f(x)}{g(x)}\ (g(x) \neq 0)$ | |
| Sine | $\sin x$ | |
| Cosine | $\cos x$ | |
| Tangent | $\tan x$ | |
| Cotangent | $\cot x$ | |
| Secant | $\sec x$ | |
| Cosecant | $\csc x$ | |
| Chain Rule | $f\big(g(x)\big)$ | |
| Natural Log | $\ln x$ | |
| Natural Exponent | $e^x$ | |
| Hyperbolic sine | $\sinh x$ | |
| Hyperbolic cosine | $\cosh x$ | |

Find the derivatives:

a. $y = \cos 5t$

b. $y = e^{-t}$

c. $y = t \sin t$

d. $y = e^{t} \cos t$

e. $y = \frac{t^2 - 1}{\ln t}$

f. $y = \cos^3 t$

g. Find the second derivative of $y = \cos^3 t$

**Examples: Finding Tangent Lines of Vector-Valued Functions**

Find the tangent lines to the smooth curves at the given moment $t_0$. Write the line with parametric equations.

a. $\vec{r}(t) = (t^2)\vec{\imath} + (4t + 1)\vec{\jmath} + (t^4)\vec{k}, t_0 = 2$

b. $\vec{r}(t) = (t \ln t)\vec{\imath} - (\ln 2t)\vec{\jmath} + \left(\frac{t-1}{t+3}\right)\vec{k}, t_0 = 1$



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**Examples: Finding Tangent Lines Given a Point**

Find the value(s) of t so that the tangent line to the given curve contains the given point.

$\vec{r}(t) = (t)\vec{\imath} - \left(\frac{2}{3}t^{\frac{3}{2}}\right)\vec{\jmath} + (5)\vec{k}, (0, -\frac{8}{3}, 5)$



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**Examples: Integrating Vector-Valued Functions**

Integrate the following.

a. $\int_1^2 \left[ (4-4t)\vec{\imath} + \left(6\sqrt{t}\right)\vec{\jmath} + \left(\frac{4}{t}\right)\vec{k} \right] dt$

b. $\int_0^{\frac{\pi}{2}} \left[ (\cos t)\vec{\imath} + (t \sin t)\vec{\jmath} + (\sin^2 t)\vec{k} \right] dt$



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**Review: Some Common Integration Techniques and Tricks for Calc III**

Trig Identities

Pythagorean Identities

Half Angle Identities + Modifications



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a. $\int \tan^2 x \ dx$

b. $\int \sin^2 x \ dx$

Substitution

a. $\int 2x\sqrt{x^2+1}dx$

b. $\int t\, e^{t^2} dt$



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Integration by Parts

Formula:

a. $\int x \sin x \ dx$

b. $\int \ln x \ dx$



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**Examples: Solving Initial Value Problems with Vector-Valued Functions**

Solve the initial value problem.

$$\frac{d\vec{r}}{dt} = t\vec{\imath} + \frac{1}{2}t\vec{\jmath} + t^3\vec{k}$$

$$\vec{r}(0) = \vec{\imath} - \vec{\jmath}$$



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**Lesson: Arc Length Parameter**

We are going to take interest in different mathematical features of curves that will describe how they turn or twist.

First we will discuss this idea conceptually. Do your best to summarize the difference between arc length versus arc length parameter.



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Write the arc length formula for a smooth curve $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$, $a \leq t \leq b$ that is traced only once as t increases from a to b:

What else does this formula represent?

How could we rewrite this formula in short hand?

**Example** Find the length of the path from $t = 0$ to $t = \pi$ for $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\,\vec{\jmath} + (t)\vec{k}$



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Now write out how to compute arc length parameter:

**Example** Find the arc length parameter for the previous example from $t = 0$ to $t = \pi$ for $\vec{r}(t) = (\sin 2t)\vec{\imath} + (\cos 2t)\,\vec{j} + (t)\vec{k}$

What are the pros and cons of arc length versus arc length parameter?



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**Lesson: The Unit Tangent Vector**

How do we define the unit tangent vector?

Explain the derivation for an easier way to compute this formula.



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**Examples: Arc Length Parameter**

Find the arc length parameter along the curve from the point where t=0 by evaluating:

$$s = \int_0^t \{\vec{v}(\tau) d\tau$$

Then find the length of the indicated portion of the curve.

$$\vec{r}(t) = (1 + 2t)\vec{\imath} + (1 + 4t)\vec{\jmath} + (1 + 8t)\vec{k}, -1 \leq t \leq 0$$



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**Examples: Finding Tangent Vectors and Lengths**

Find the curve's unit tangent vector, then find the length of the indicated portion of the curve.

a. $\vec{r}(t) = (4\cos t)\vec{\imath} - (4\sin t)\vec{\jmath} + (3t)\vec{k}$, $0 \le t \le \pi$

b. $\vec{r}(t) = (t\sin t)\vec{\imath} - (t\cos t)\vec{\jmath}$, $0 \le t \le \pi$



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**Lesson: Curvature and Normal Vectors**

How do we define curvature?

Explain the idea of curvature conceptually:

What is one alternative way to derive an easier formula for curvature?



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**Example** Find the curvature $\kappa$ for the curve $\vec{r}(t) = (\cos t + t \sin t)\vec{\imath} + (\sin t - t \cos t)\vec{\jmath}, t > 0$

**Example** Find the curvature of a circle $\vec{r}(t) = (a \cos t)\vec{\imath} + (a \sin t)\vec{\jmath}$ for a circle with radius a



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Define a principal unit normal vector.

Explain the principal normal vector conceptually.

Show how to derive a simpler formula for calculating $\vec{N}$



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**Example** Find the normal vector $\vec{N}$ for the curve $\vec{r}(t) = (\cos t + t \sin t)\vec{\imath} + (\sin t - t \cos t)\vec{j}, t > 0$



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**Examples: Finding $\vec{T}$, $\kappa$, and $\vec{N}$ for Space Curves**

Find $\vec{T}$, $\kappa$, and $\vec{N}$ for the space curves:

a. $\vec{r}(t) = (e^t \cos t)\vec{\imath} + (e^t \sin t)\vec{\jmath} + 5\vec{k}$

b. $\vec{r}(t) = (\cos^3 t)\vec{\imath} + (\sin^3 t)\vec{\jmath}, 0 < t < \frac{\pi}{2}$



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**Examples: Maximum Curvature**

Find the maximum curvature for the function $f(x) = \ln x$



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**Examples: Writing Acceleration with Tangential and Normal Components**

Write $\vec{a} = a_T\vec{T} + a_N\vec{N}$ without finding $\vec{T}$ and $\vec{N}$ for

$$\vec{r}(t) = (t\cos t)\vec{\imath} + (t\sin t)\vec{\jmath} + t^2\vec{k}$$



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**Examples: Finding $\vec{B}$ and $\tau$**

In section 13.4, you found $\vec{T}$, $\kappa$, and $\vec{N}$ for the space curves. Now find $\vec{B}$ and $\tau$.

a. $\vec{r}(t) = (e^t \cos t)\vec{\imath} + (e^t \sin t)\vec{\jmath} + 5\vec{k}$

b. $\vec{r}(t) = (\cos^3 t)\vec{\imath} + (\sin^3 t)\vec{\jmath}, 0 < t < \frac{\pi}{2}$



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**Examples: Find and Sketch the Domain for Multivariable Functions**

Find and sketch the domain.

a. $f(x) = \ln(x^2 + y^2 - 4)$

b. $f(x) = \sqrt{(x^2 - 1)(x^2 - 4)}$



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**Examples: Sketching Level Curves and Level Surfaces**

Find the level curves or level surfaces for the following. Do your best to sketch or at least describe the region for surfaces.

a. $f(x,y) = \sqrt{x^2 + y^2}$

b. $f(x,y) = 1 - |x| - |y|$

c. $f(x,y,z) = \ln(x^2 + y^2 + z^2)$

d. $f(x,y,z) = z - x^2 - y^2$



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**Review: Algebra Tricks for Limits in 1 Variable vs Multivariables**

1. Factoring to Cancel

$\lim_{x\to 1} \frac{x-1}{x^2-1}$

Crashing Course Factoring Review:

Factor by the GCF: $\frac{3x^2-5x^3}{x^5-2x^4+3x^3+2x^2}$

Factor by Grouping:

a. $ax + ay + bx + by$

b. $\frac{x-1}{xy-y-2x+2}$



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Factor Trinomials:

a. $x^2 + 5x + 6$

b. $2x^2 + 5x + 3$

c. $15x^2 + 14x - 8$

Difference of Cubes Formulas

Difference of Cubes with Polynomial Long Division: $(x^3 + 1) \div (x + 1)$



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2. Using the Conjugate

What is the conjugate?

Show how to multiply by the conjugate in this one-variable example:

$\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$

3. Two Common Trig Tricks

Write the two common limits for trig:

$\lim_{x\to 0} \frac{\tan 3x}{x}$



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**Examples: Finding Limits of Multivariable Functions Using Algebra and Trig Tricks**

a. $\lim_{(x,y)\to(0,0)} \frac{e^x \sin y}{y}$

b. $\lim_{(x,y)\to(2,-5)} \frac{y+5}{x^2y-xy+5x^2-5x}$

c. $\lim_{(x,y)\to(3,0)} \frac{\sqrt{3x-y}-3}{3x-y-6}$



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**Examples: Limits that Do Not Exist**

Show that the limits do not exist.

a. $\lim_{(x,y)\to(0,0)} \frac{x+\sin y}{y+\sin x}$

b. $\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$



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**Examples: Second-Order Partial Derivatives**

Find all the second order partial derivatives.

a. $f(x,y) = xy^2 + \cos xy + y \sin x$

b. $f(x,y) = \ln(x+y)$

c. $z = xe^{\frac{y}{x^2}}$



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**Examples: Tangent Planes and Normal Lines to Surfaces**

Find equations for the tangent plane and normal line at the point $P$ on the surface:

$$x^2 + y^2 - 2xy - x + 3y - z = -4, P(2, -3, 18)$$



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**Examples: Tangent Lines to Intersecting Surfaces**

Find parametric equations for the line tangent to the curve of intersection of the surface at the given point:

Surfaces: $x^2 + 2y + 2z = 4; y = 1; P\left(1,1,\frac{1}{2}\right)$



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**Examples: The Chain Rule for Multivariable Functions**

Find the derivatives, and then determine the value at the given point.

a. $\frac{dw}{dt}$ for $w = \ln(x^2 + y^2 + z^2)$ where $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$ at the point $t = 3$



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b. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ for $u = e^{qp} \sin^{-1} r$ where $p = \frac{1}{z}, q = y^2 \ln z, r = \sin x$ at point $(x, y, z) = \left(\frac{\pi}{4}, 1, 1\right)$



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**Examples: Implicit Differentiation, Again**

Find $\frac{dy}{dx}$ at the given point: $ye^x + \sin xy + x - \ln 5 = 0, (\ln 5\,, 0)$



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**Examples: Finding Direction Derivatives**

Find the derivative of the function $P$ in the direction of $\vec{u}$

a. $f(x,y)=\frac{x-y}{xy+2}, P(1,-1), \vec{u}=12\vec{\imath}+5\vec{\jmath}$

b. $f(x,y,z)=\cos xy+e^{yz}+\ln zx\,, P\left(1,0,\frac{1}{2}\right), \vec{u}=\vec{\imath}+2\vec{\jmath}+2\vec{k}$



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**Examples: Directions Where a Function Increases or Decreases Most Rapidly**

Where does $f(x, y, z) = xe^y + z^2$ increase or decrease most rapidly at $P(4, 1, 1)$? Find the derivative at these directions.



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**Examples: Tangent Lines to Level Curves**

Sketch the curve with the gradient and the tangent line at the given point. Write the equation for the tangent line.

$$x^2 + y^2 = 9, \left(\sqrt{3}, \sqrt{3}\right)$$



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**Lesson: Extreme Values and Saddle Points**

Let $f(x, y)$ be defined on a range R containing the point $(a, b)$. Then:

- $f(a, b)$ is a **local maximum** if:

- $f(a, b)$ is a **local minimum** if:

State the First Derivative Test for Local Extreme Values:



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Define a **critical point** for $f(x, y)$

What is a saddle point?

What is the **discriminant** or **Hessian** of $f(x, y)$?



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**The Second Derivative Test for Local Extreme Values**

Let $f(x, y)$ be a function where the first and second derivatives are continuous throughout a disk centered at $(a, b)$ and that $f_x(a, b) = f_y(a, b) = 0$. Then:

i. $f$ has a local maximum at $(a, b)$ if:

ii. i. $f$ has a local minimum at $(a, b)$ if:

iii. $f$ has a saddle point at $(a, b)$ if:

iv. the test is inconclusive if:



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**Examples** Find the local extreme values of the following functions.

a. $f(x,y) = -5x^2 - 2y^2 + 2xy + 4x + 4y - 4$

b. $f(x,y) = e^x - xe^y$



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c. $f(x,y) = 1 - \sqrt[3]{x^2 + y^2}$



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**Examples: Finding Local Extrema**

Find all the local maxima, local minima, and saddle points

a. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$



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b. $f(x,y) = \ln(x+y) + x^2 - y$



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**Examples: Finding The Critical Points for Interesting Systems**

Find *only* the critical points for the following.

a. $f(x,y) = x^3 - y^3 + 3xy - 6$

b. $f(x,y) = 8xy - x^4 - y^4$



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c. $f(x,y) = e^{x^2+y^2-8x}$

d. $f(x,y) = \frac{1}{x^2+y^2-4}$



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**Examples: Finding Absolute Extrema**

Find the absolute maxima and minima for:

$f(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines:

$x = 0, y = 4, y = x$



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**Examples: Lagrange Multipliers with Two Independent Variables and One Constraint**

a. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$



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b. Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.



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**Examples: Lagrange Multipliers with Three Independent Variables and One Constraint**

Find the point on the sphere $x^2 + y^2 + z^2 = 9$ farthest from the point $(1, -1,\ 1)$



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b. $f(x,y) = \ln(x+y) + x^2 - y$



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**Examples: Evaluating Iterated Integrals**

Evaluate the following integrals.

a. $\int_0^2 \int_0^4 \left(\frac{x}{4} + \sqrt{y}\right) dx\ dy$

b. $\int_1^3 \int_1^{e^2} \frac{\ln x}{xy} dx\ dy$



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c. $\int_1^2 \int_1^2 x \ln y \ dy \ dx$



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**Examples: Evaluating Double Integrals Over Rectangles**

Evaluate the following integrals over the region R.

a. $\int \int_R y \sin(x+y)\, dA,\ R: 0 \le x \le \pi, -\pi \le y \le 0$

b. $\iint_R \frac{xy^3}{x^2+1} dA\,, R: 0 \le x \le 1, 0 \le y \le 1$



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**Examples: Sketching Regions of Integration**

Sketch the describe regions of integration.

a. $0 \leq x \leq 4, 0 \leq y \leq 3x$

b. $0 \leq x \leq 1, e^x \leq y \leq e$



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**Examples: Write an Integral Over a Region**

Write an iterated integral for the region R using vertical cross sections AND horizontal cross sections.

a. $y = x^2, y = 2x$

b. $y = 3x, x = 2$



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**Examples: Sketch the Region of Integration and Evaluate the Integral**

Sketch the region of integration and evaluate the integral.

a. $\int_0^{\pi} \int_0^{\cos x} y \, dy \, dx$

b. $\int_0^1 \int_0^{x^2} 3x^3 e^{xy} \, dy \, dx$



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**Examples: Sketch the Region of Integration and Reverse the Order of Integration**

Sketch the region of integration and write an equivalent double integral with the order of integration reversed.

a. $\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$

b. $\int_0^3 \int_1^{e^y} (x+y) dx\, dy$



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**Examples: Volume Beneath a Surface**

Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$



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**Examples: Integrating with Polar Coordinates**

Change the Cartesian integral into an equivalent polar integral, then integrate.

a. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx\, dy$

b. $\int_0^1 \int_x^{\sqrt{2-x^2}} (x + 2y)\ dx\ dy$



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**Examples: Area by Double Integrals**

Sketch the region bounded by the given lines and curves, Then express the region's area as a double integral, and integrate.

a. The parabola $x = y - y^2$ and the line $y = x$

b. The lines $y = x, y = \frac{x}{2}$, and $y = 2$



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**Examples: Evaluating Triple Integrated Integrals**

Integrate the following.

a. $\int_0^1 \int_0^{1-x^2} \int_3^{9-x^2-y^2} x \, dz \, dy \, dx$

b. $\int_0^1 \int_1^{\sqrt{e}} \int_1^e x \, e^x \ln y \frac{(\ln z)^2}{z} \, dz \, dy \, dx$



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**Examples: Reversing the Order for Triple Integrals**

Using the integral: $\int_{-1}^{1}\int_{x}^{2}\int_{0}^{1-x} dz\ dy\ dx$

Rewrite an equivalent integral in the following order:

a. $dy\ dz\ dx$

b. $dy\ dx\ dz$

c. $dz\ dx\ dy$



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**Examples: Volume with Triple Integrals**

Find the volume in the following situations.

a. The region between the cylinder $z = x^2$ and the xy-plane that is bounded by the planes $x = -1, x = 1, y = 0, y = 1$.

b. The region between the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y\text{^}2$



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**Examples: Sketching Graphs in Cylindrical Coordinates**

Sketch the graph describe by the cylindrical coordinates.

a. $z = r$

b. $r = \theta$

c. $0 \leq r \leq 2\sin\theta, 1 \leq z \leq 2$



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**Examples: Sketching Graphs in Spherical Coordinates**

Sketch the graph describe by the spherical coordinates.

a. $\rho = 4$

b. $\rho \cos\theta = 2$

c. $0 \leq \rho \leq 1, \frac{\pi}{2} \leq \phi \leq \pi, 0 \leq \theta \leq \pi$



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**Examples: Setting Up Triple Integrals Across Coordinate Systems**

Set up the triple integrals for the volume of the sphere $\rho = 4$ in:

a. Spherical coordinates

b. Cylindrical coordinates

c. Rectangular coordinates



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Now let D be a solid ball of radius 4 units that is cut by a plane that is 2 units from the center of the sphere. Express the volume of this situation in:

a. Cylindrical coordinates

b. Spherical coordinates

c. Rectangular coordinates



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**Examples: Double Integral Substitutions**

Evaluate $\int_0^2 \int_{\frac{y}{2}}^{\frac{y+4}{2}} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$ using the transformation $x = u + \frac{v}{2}, y = v$.



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Evaluate $\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dy\, dx$ using the transformation $x = u^2 - v^2, y = 2uv$.



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## Not included (no notes file):

112 (Lesson Introducing Where Calculus Starts)

823 (Field Trip on CalcPlot3D for Space Curves)