Review: Piecewise Functions for Calc Students

Piecewise functions look like this:

Explain how to graph the following examples. It would be best to actually write out directions for yourself in case you ever need to reference this again.

a. $f(x) = \begin{cases} 2, & x \geq 3 \\ -3, & x < 3 \end{cases}$

b. $f(x) = \begin{cases} 4, & x \geq 2 \\ x + 2, & -1 \leq x < 2 \\ -2x + 1, & x < -1 \end{cases}$

c. $f(x) = \begin{cases} 3, & x > 4 \\ 2, & -1 < x \leq 1 \\ x^2, & x < -1 \end{cases}$

d. $f(x) = \begin{cases} 3x + 1, & x \geq 3 \\ x + 1, & x < -2 \\ 3, & x = 0 \end{cases}$

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Review: Basic Graphs and Translations

Sketch the basic graphs from the video.

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Explain the basic translations.

Examples State what translations have occurred and sketch the graph.

a. $f(x) = (x + 2)^2 - 3$

b. $g(x) = -|x - 2| + 1$

c. $f(x) = \sqrt{x + 3}$

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Review: Zeros vs Factors vs Intercepts

Use the following functions. Find the zeros, factors, and x-intercepts of each function.

a. $f(x) = x^2 + 6x + 8$

b. $g(x) = x^2 - 9$

c. $f(x) = x^2 - 7x + 12$

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Lesson: A Quick Review of Some Things About Graphing

Remind yourself how to graph the following points. Draw your own graph.

Point 1: $(-3, 4)$ Point 2: $(0, -2)$

Draw another rectangular coordinate system below and label the quadrants and origin.

Often when graphing, we use equations in two variables.

Example Find the ordered pair associated with the equation $3x + y^2 = 7$ when $x = 1$

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Example Find at least 3 ordered pairs for the following equations.

a. $y = \sqrt{x+2}$

b. $2x + 3y = 6$

Finding 3 points can be a strategy for graphing.

Example Sketch a graph for the previous 2 examples.

What is the x-intercept? And how do you find it?

What is the y-intercept? And how do you find it?

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Examples Find the x- and y-intercepts for the following equations.

a. $y = 3x - 1$

b. $x = \sqrt{y - 1}$

c. $x^2 - 3x + 2 = y$

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Lesson: The Distance Formula

Graph the points $(2, 4)$ and $(-4, -6)$. Explain how you can find the distance between these points by summarizing the explanation given in the video (make a decent effort to explain this – you will have an easier time remembering the formula if you understand WHY it works).

State the distance formula:

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Example Find the distance between the following points: $(4, -1)$ and $(7, 3)$

There are 2 ways we can use the distance formula.

Example Show how to use the distance formula to determine whether the following points are the vertices of a right triangle.

a. $(0, -2), (-4, 3)$ and $(5, 1)$

b. $(-2, -8), (0, -4)$ and $(-4, -7)$

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What does it mean for 3 points to be collinear?

Example Show how to use the distance formula to determine if the 3 points are collinear.

a. $(-1, 4), (-2, -1), (1,14)$

b. $(-2, -4), (-6, 11), (0, -12)$

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Lesson: The Midpoint Formula

What is the midpoint formula?

So what is the midpoint formula effectively doing?

Example Find the midpoint between the points $(-5, -6)$ and $(-3, 2)$

Example The midpoint of a line segment is $(5, 8)$ and the endpoint of that line segment is $(11, 10)$. What is the OTHER endpoint on this line segment?

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Example The midpoint of a line segment is $(3, 1)$ and the endpoint of that line segment is $(6, 8)$. What is the OTHER endpoint on this line segment?

Example The midpoint of a line segment is $(2a, 5b)$ and the endpoint of that line segment is $(-1a, 9b)$. What is the OTHER endpoint on this line segment?

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Lesson: Review of Functions

How do you represent a function?

What is a function often interchangeable with? How can you rewrite $f(x) = 3x + 8$?

So what IS a function? What does it actually represent?

What's another handy feature of functions?

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What are the domain and range?

Does domain and range only work for functions?

Examples Are the following functions, and what is the domain and range?

a. $\{(2,3), (5, -1), (-3, 7)\}$

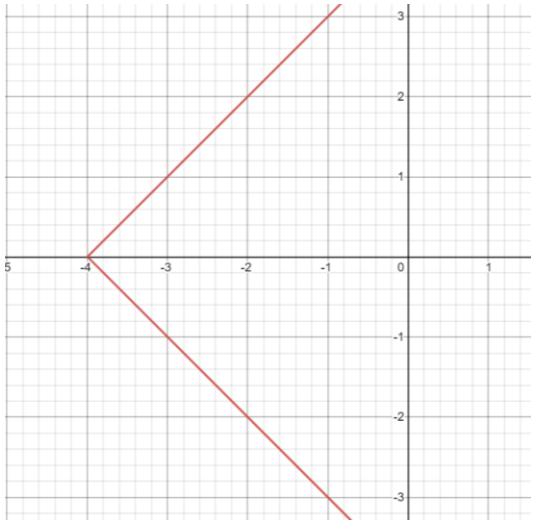
b. $\{(-1, 5), (-8, 3), (-1, 6)\}$

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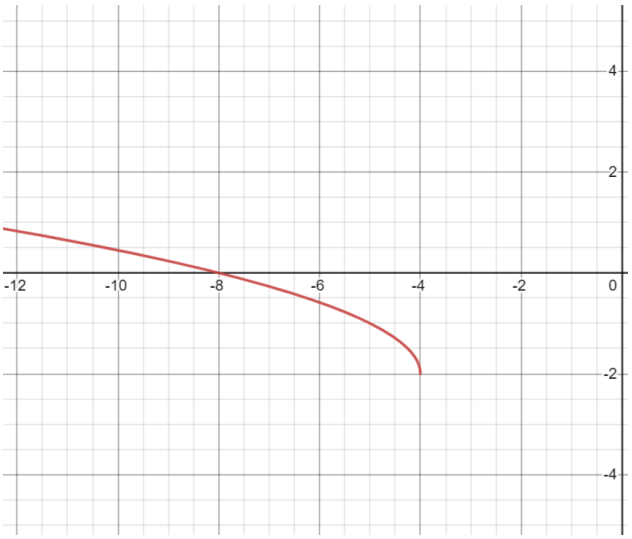
What is the Vertical Line Test, and how does it help you determine if you have a function?

Example State the domain and range of the following, and use the vertical line test to determine if the graph represents a function.

a.



b.



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Examples Determine if the following represent functions.

a. $y = 3x^2$

b. $x = y^2$

c. $y = \frac{5}{x^2}$

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Example Write the following in function notation: $3x + 2y = 6$

Example Suppose $f(x) = 3x - 5$ and $g(x) = x^2 + 4x - 1$ and $h(x) = \{(2, 5), (6, -3), (-1, -4)\}$. Find the following:

a. $f(7) =$

b. $g(3) =$

c. $f(k) =$

d. $g(x + h) =$

e. $h(6) =$

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Lesson: Increasing, Decreasing, and Constant Functions

Below are the "formal math" definitions of whether a function is increasing, decreasing, or constant. Let I be an open interval and let x_1 and x_2 be two points in that interval.

- A function is **increasing** over I if when $x_1 < x_2, f(x)_1 < f(x_2)$
- A function is **decreasing** over I if when $x_1 < x_2, f(x)_1 > f(x_2)$
- A function is **constant** over I if when $x_1 < x_2, f(x)_1 = f(x_2)$

While these are the formal definitions, what we really want to know are the intuitive meanings of these definition.

Sketch a graph that is increasing:

Decreasing:

And constant:

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Example Sketch the graph in the video, and then identify where the graph is increasing, decreasing, or constant.

a.

b.

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Lesson: Graphs of Basic Functions

In this section, we will review some basic graphs that are vital to know for this class and calculus. Knowing the "basic shapes" of certain graphs will make your life much easier and give you greater intuition when graphing.

Show hot to graph the following and highlight the **key points** and **key patterns.** This will come in handy for the next lesson. Take your time to sketch your own NEAT CLEAN CLEAR graphs.

$f(x) = x$

$f(x) = x^2$

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$f(x) = x^3$

$f(x) = \sqrt{x}$

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$f(x) = \sqrt[3]{x}$

$f(x) = |x|$

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Lesson: Piecewise Functions

What is a piecewise function? Draw an example of what they look like.

Example Given the function: $f(x) = \begin{cases} 3x - 1, & x \geq 2 \\ \frac{1}{2}x - 4, & x < 2 \end{cases}$, evaluate the following:

a. $f(3) =$

b. $f(-1) =$

c. $f(2) =$

Example Given the function: $f(x) = \begin{cases} 4, & x > 3 \\ 4x - 2, & -2 < x \leq 3 \\ x^2 - 5, & x \leq -2 \end{cases}$ evaluate the following:

a. $f(4) =$

b. $f(-5) =$

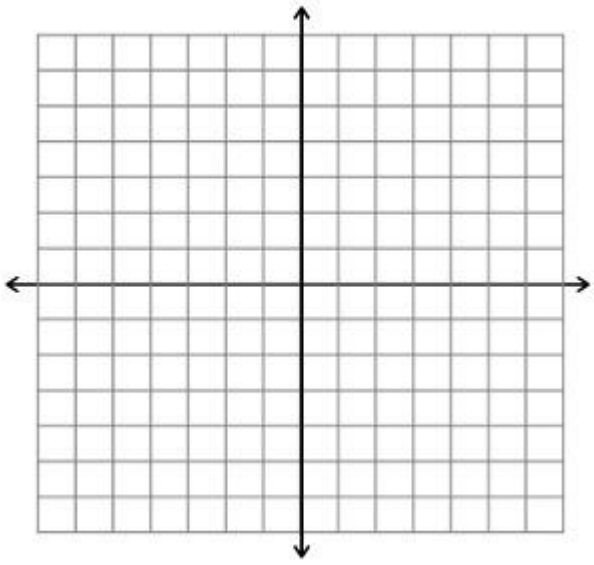
c. $f(-2) =$

d. $f(3) =$

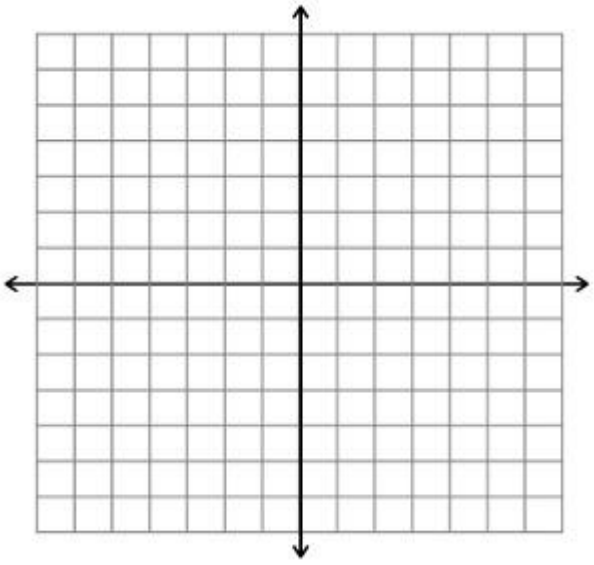
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Example Show how to graph the following piecewise functions.

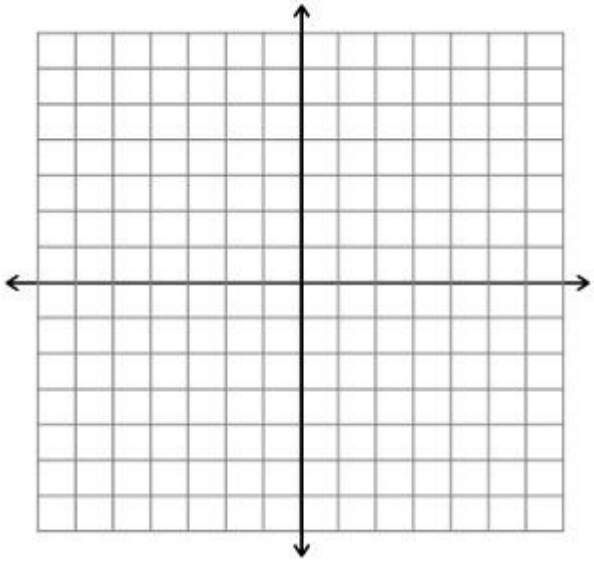
a. $f(x) = \begin{cases} 4, & x > 5 \\ -1, & x \leq 5 \end{cases}$



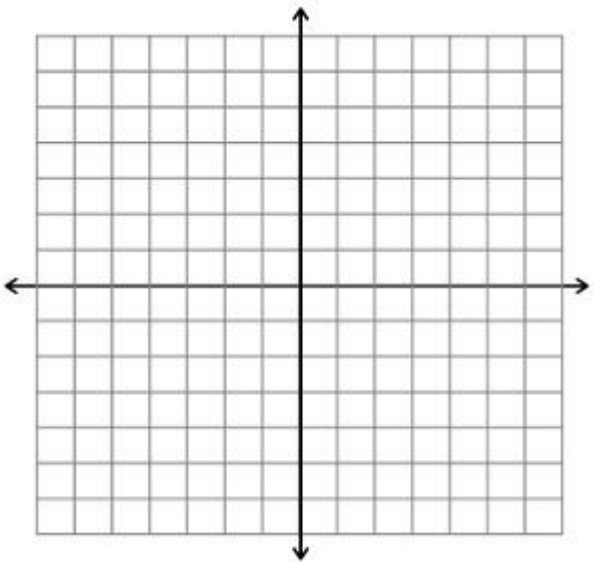
b. $f(x) = \begin{cases} 3x + 1, & x > -3 \\ -2x, & x \leq -3 \end{cases}$



c. $f(x) = \begin{cases} 3, & x > 2 \\ 2x, & -1 < x \leq 2 \\ -x + 3, & x \leq -1 \end{cases}$



d. $f(x) = \begin{cases} 3x + 1, & x \geq 1 \\ 6, & -1 \leq x < 1 \\ x^2, & x < -1 \end{cases}$



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Lesson: The Greatest Integer Function

Show how to graph the greatest integer function $f(x) = [\![x]\!]$

Example Graph $f(x) = \left[\!\left[\frac{1}{2}x - 2\right]\!\right]$

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Review: Finding Slopes

Explain how to find the slope of a line from the equation:

Example Find the slope of each line.

a. $y = 2x - 1$

b. $y - 4 = \frac{1}{2}(x + 2)$

c. $3x - 2y = 6$

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State the formula to find the slope between 2 points.

Examples Find the slope between the 2 points.

a. $(3,1)\ (2,4)$

b. $(2,1)\ (2,5)$

c. $(5,2)\ \ (3,2)$

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Draw the graphs from the video and explain how to find the slope for each.

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Lesson: Translations and Reflections of Functions

Using the graph $f(x) = |x|$ as the "base function," sketch how vertical translations work (summarize what was discussed in the video).

How do you describe vertical translations in function notation?

Using the graph $f(x) = x^2$ as the "base function," sketch how horizontal translations work (summarize what was discussed in the video).

How do you describe horizontal translations in function notation?

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Use the graph $f(x) = \sqrt{x}$ to show how reflections over the x- and y-axis work (summarize what was discussed in the video).

How do you describe reflections over the x-axis and y-axis using function notation.

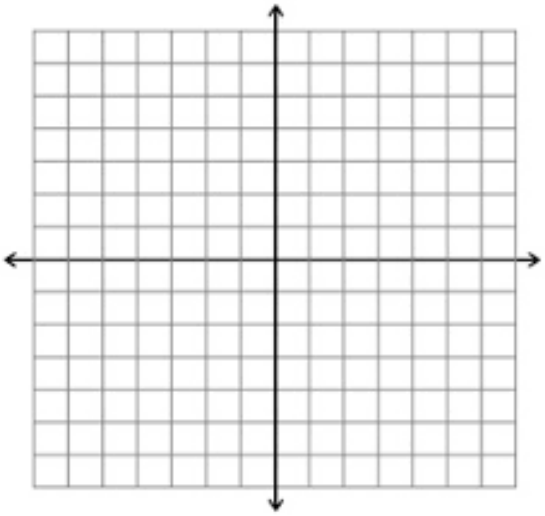
Use the graph $f(x) = |x|$ to show how stretching and shrinking works (summarize what was discussed in the video).

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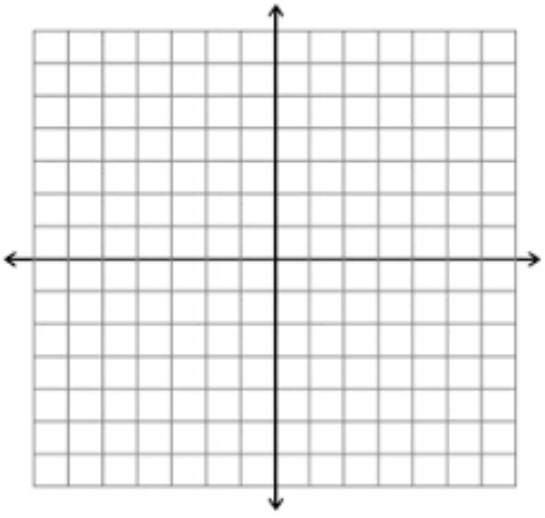
Explain how stretching and shrinking works with function notation.

Example Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $f(x) = -x^2 + 3$



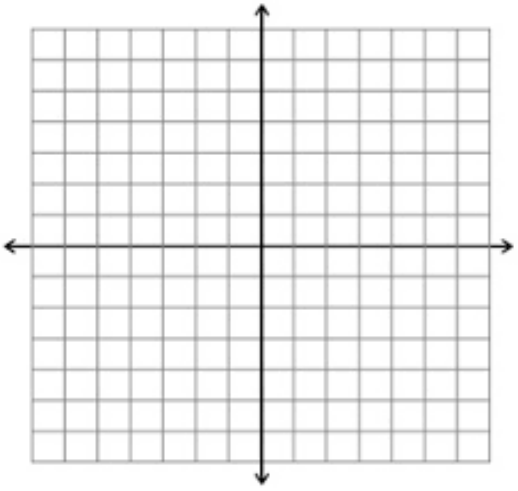
b. $g(x) = (x + 2)^3 - 3$



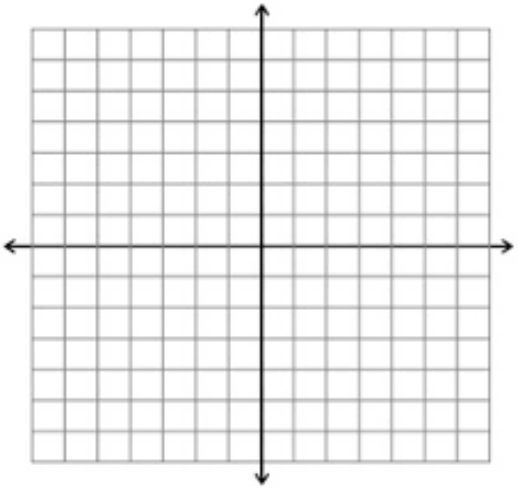
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Example Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $g(x) = -2(x+1)^2 - 5$



b. $h(x) = 2\sqrt{-x} + 1$



Examples Write the equation of the given functions with the following characteristics.

a. The function $f(x) = x^2$ moved 3 units left and 5 units down.

b. The function $g(x) = \sqrt[3]{x}$ reflected over the y-axis, translated 2.5 units right and 1.75 units up.

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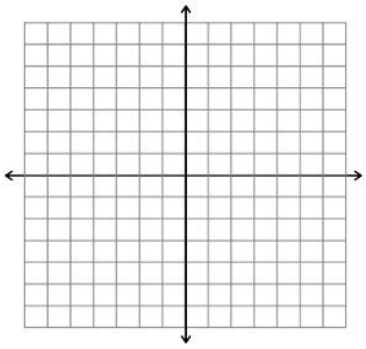
Transformations of Functions

These are looking at graphs of things like:

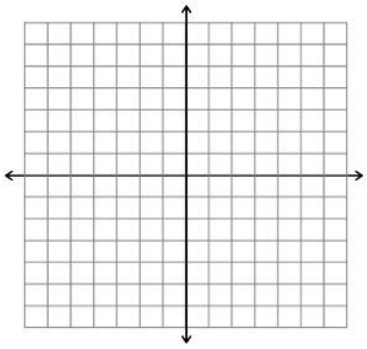
$$f(x) = x^2 - 5 \quad \text{or} \quad f(x) = |x + 3| - 2 \quad \text{or} \quad f(x) = \sqrt{x - 3}$$

First we need to discuss the shape of a few basic functions. Write these down and note the important points.

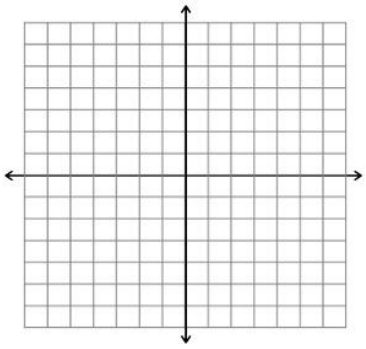
a. $f(x) = x^2$



b. $f(x) = |x|$



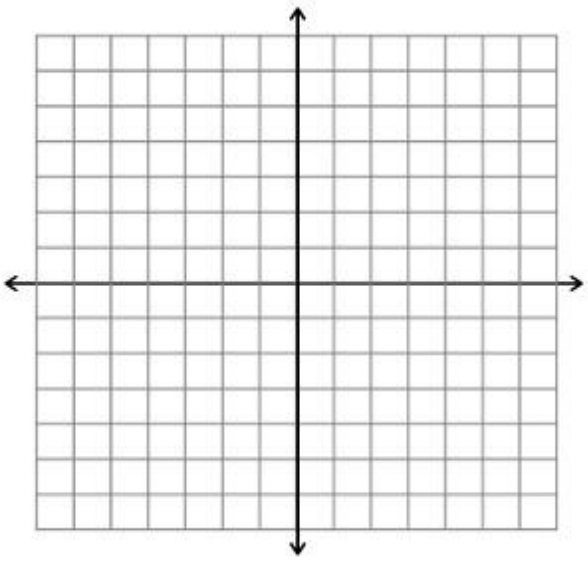
c. $f(x) = \sqrt{x}$



Using a few different colors, sketch out what the graphs of

a. $f(x) = x^2$ versus b. $g(x) = x^2 + 3$ versus c. $h(x) = x^2 - 1$

look like all on the same graph below:



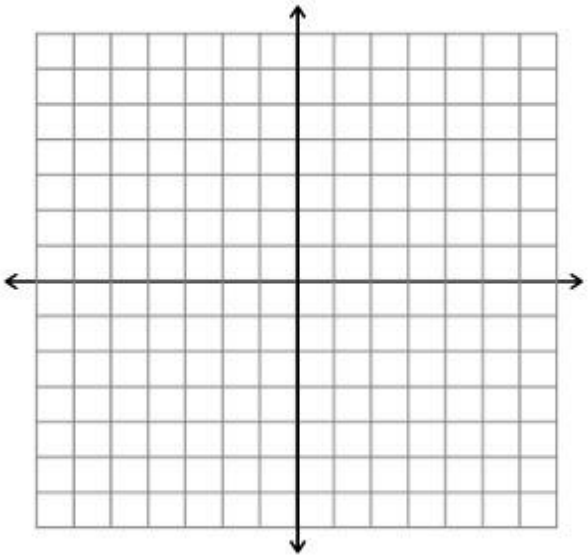
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This is an example of a vertical shift. Explain what this means in mathematical terms:

Using a few different colors, sketch out what the graphs of

a. $f(x) = |x|$ versus b. $g(x) = |x - 1|$ versus c. $h(x) = |x + 2|$

look like all on the same graph below:



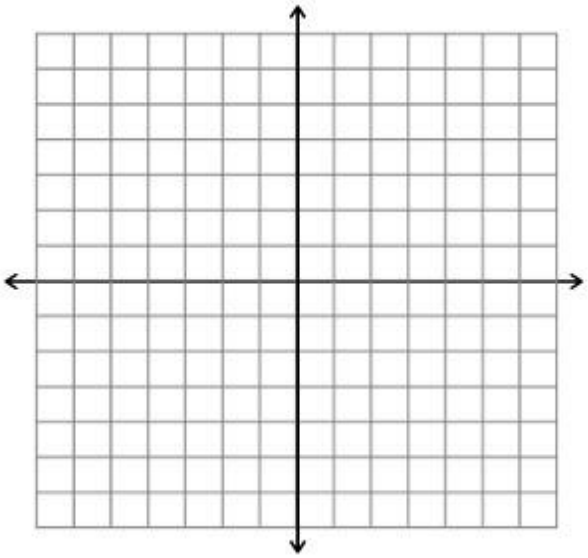
This is an example of a horizontal shift. Explain what this means in mathematical terms:

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Using a few different colors, sketch out what the graphs of

$$\text{a. } f(x) = |x| \text{ versus b. } g(x) = -|x|$$

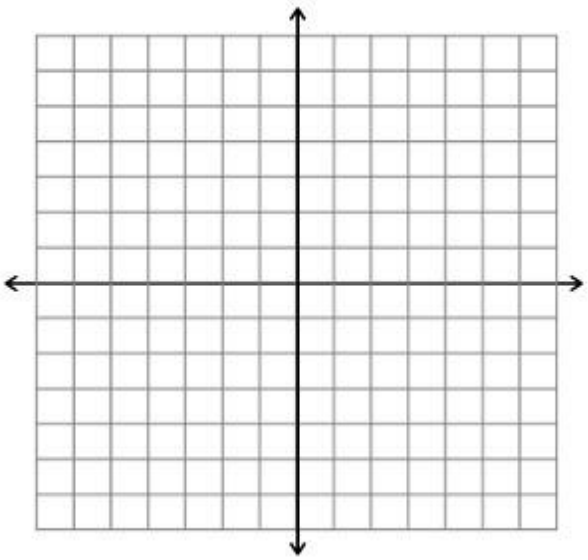
look like all on the same graph below:



This is an example of a reflection over the x-axis. Explain what this means in mathematical terms:

Example Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

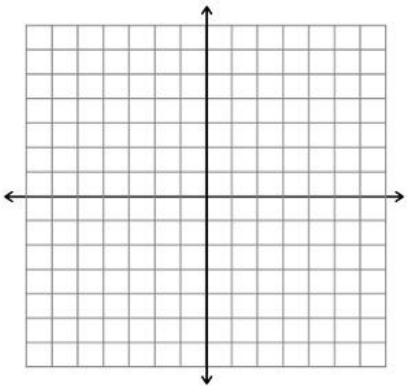
$$f(x) = |x| + 3$$



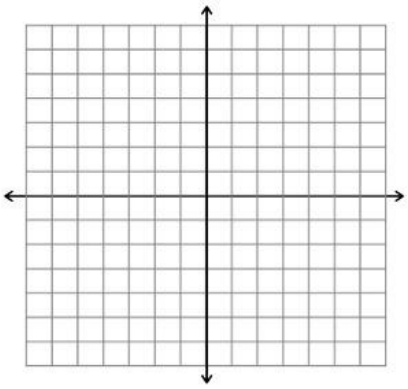
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Examples Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

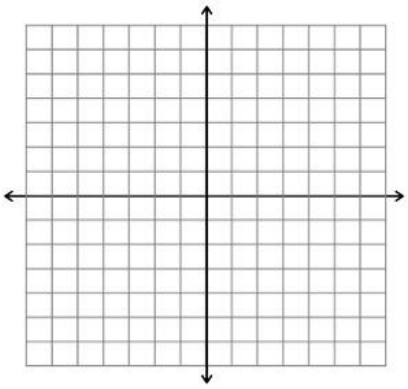
a. $f(x) = (x-1)^2$



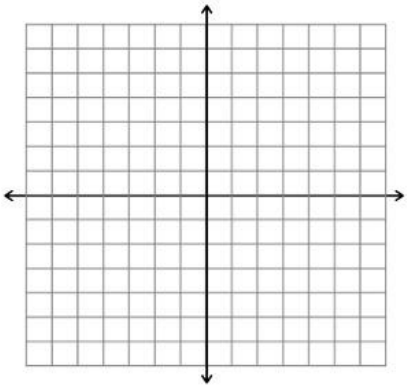
b. $g(x) = \sqrt{x} - 2$



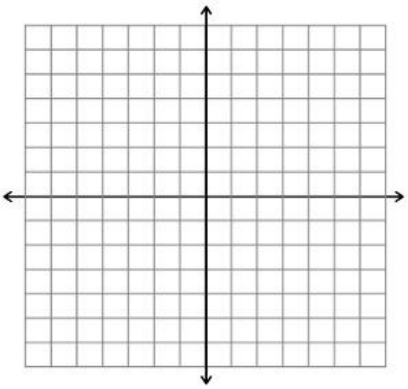
c. $h(x) = |x+1| - 2$



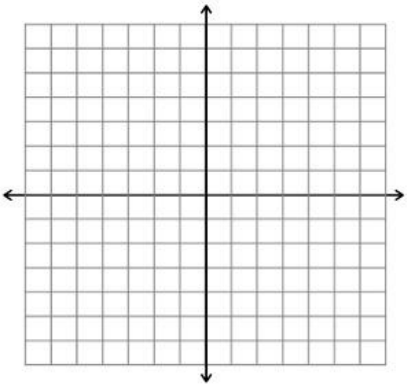
d. $k(x) = (x-2)^2 + 4$



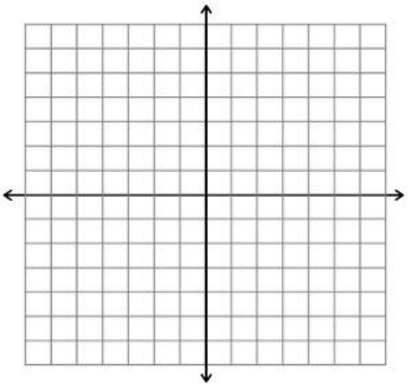
e. $f(x) = -(x+3)^2 - 2$



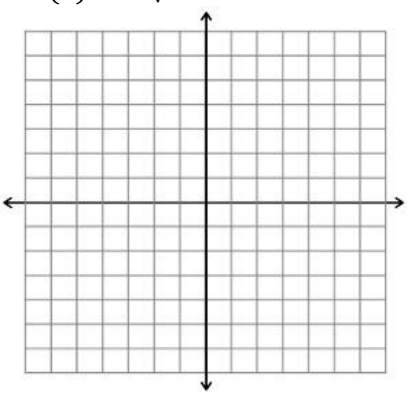
f. $g(x) = -|x+3| - 1$



g. $h(x) = -\sqrt{x-2}$



h. $k(x) = -\sqrt{x+1} - 3$



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Examples Write the equations that represent the listed transformations.

a. A function $f(x) = \sqrt{x}$ is shifted 7 units down.

b. A function $g(x) = x^3$ is shifted 3 units left and 2 units up.

c. A function $h(x) = |x|$ is shifted 2.5 units right and reflected over the x-axis.

d. A function $f(x) = e^x$ is shifted 4 units up and reflected over the x-axis.

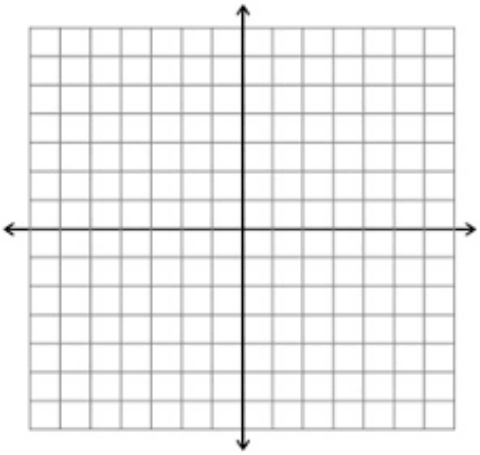
e. A function $g(x) = \sqrt[3]{x}$ is shifted 3.2 units right, 5.7 units up, and reflected over the x-axis

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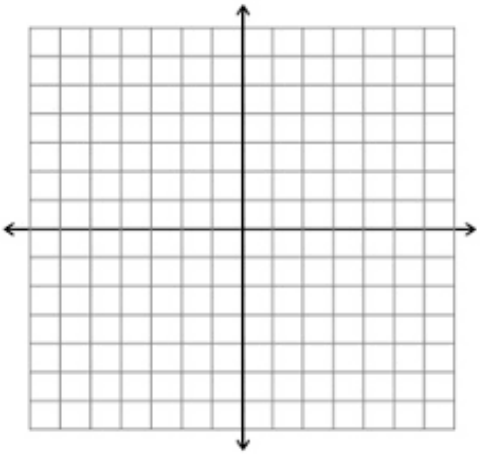
Examples: Translations and Reflections of Functions

Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

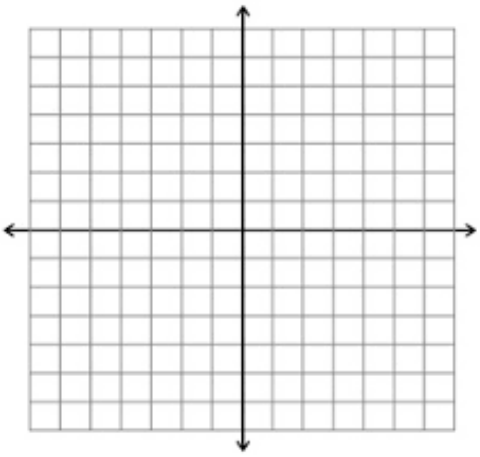
a. $f(x) = -(x+2)^2 - 3$



b. $g(x) = \frac{1}{2}(x-1)^3 + 2$



c. $h(x) = -\frac{1}{3}\sqrt{x+1} + 1$



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Lesson: Symmetry and Even or Odd Functions

Explain when a graph is symmetric with respect to the y-axis:

Explain when a graph is symmetric with respect to the x-axis:

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Explain when a graph is symmetric with respect to the origin:

What makes a function odd?

What makes a function even?

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So what is the test for even or odd functions?

Examples Determine what type of symmetry the following functions have:

a. $y = x^2 + 4$

b. $x^2 + y^2 = 25$

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Examples Determine what type of symmetry the following functions have:

a. $y = x^3 - 5$

b. $y = 7x^2 + 2x - 4$

c. $y = 2x^{\frac{1}{3}} - 3x$

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Lesson: Review of Function Operations

Given that f and g are functions, how do we define the function operations?

$(f+g)(x) =$

$(f-g)(x) =$

$(fg)(x) =$

$\left(\frac{f}{g}\right)(x) =$

Example Let $f(x) = 3x+1, g(x) = x^2-4, h(x) = 3x^2+7x+2$, and $k(x) = x^2+6x+9$. Find each of the following:

a. $(f+g)(x) =$

b. $(f+g)(3) =$

c. $(g-k)(1) =$

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Example Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = 3x^2 + 7x + 2$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $\left(\frac{h}{f}\right)(x) =$

b. $(gh)(x) =$

c. $\left(\frac{k}{h}\right)(1) =$

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Lesson: Review of Composition of Functions

For two functions f and g, what is the composition $(f \circ g)(x)$?

Example Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(f \circ g)(x) =$

b. $(g \circ f)(x) =$

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Example Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(k \circ f)(1) =$

b. $(g \circ h)(4) =$

c. $(h \circ k)(x) =$

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Lesson: The Difference Quotient

Explain what the difference quotient is and draw some sketches to explain how it works. In other words, watch the video explanation and then explain it in your own words on this page.

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State the formula for the difference quotient here:

Example Calculate the different quotient for the following.

a. $f(x) = 4x + 5$

b. $f(x) = x^2 - 1$

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Lesson: The Difference Quotient

Find the difference quotient.

a. $f(x) = \frac{1}{x+3}$

b. $f(x) = x^2 - 4x + 1$

c. $f(x) = -x^2 + 3$

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Lesson: Review of Triangles and Similar Triangles

The sum of angle measures of any triangle is:

Draw the different types of triangles:

Examples Draw the triangles in the example, then classify the triangles as acute, right, or obtuse. Also classify as equilateral, isosceles, or scalene.

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What makes 2 triangles similar?

What makes 2 triangles congruent?

What is the key relationship between similar and congruent triangles?

What are the 2 conditions for similar triangles?

In other words, triangle ABC is similar to triangle DEF if:

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Example Draw the similar triangles in the video, and then find any missing angle measures.

Example Draw the similar triangles in the video, and then find any missing side lengths.

Example Draw the similar triangles in the video, and then find any missing measures.

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Lesson: The Pythagorean Theorem, The Distance Formula, and Intro to Trig Functions

State the Pythagorean theorem:

State the distance formula:

How does the Pythagorean theorem relate to the distance formula?

Now we will introduce **trigonometric functions!** Explain what makes trig functions unique:

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How do trig functions relate to the distance formula? Draw the picture associated with this.

Explain the 6 trig functions and how they relate to what we discussed above.

Sometimes you will see expressions such as $\sin^2 90°$. What does this mean?

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Example The terminal side of an angle θ in standard position passed through the point (-12, -5). Find the values of the 6 trig functions of angle θ.

Example Find the 6 trig function values of an angle θ in standard position if the terminal side of θ is defined by

$3x + 5y = 0, x \geq 0$

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Example Suppose the point (x, y) is in the indicated quadrant. Determine whether given ratio is positive or negative.

$$III, \frac{y}{r}$$

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Lesson: Right-Triangle Based Definitions of Trigonometric Functions and Cofunctions

Instead of using standard position to describe trig functions, we can alternatively use right triangles.

Draw a right triangle to orient yourself.

What does A represent?

Show how to define the 6 trig functions with this right triangle.

1.

2.

3.

4.

5.

6.

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What mnemonic device can help you remember the trig functions?

Example Draw the triangle in the video, and then show how to find the values of sine, cosine, and tangent.

Example Using the same example above, show how to find the values of cosecant, secant, and cotangent.

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Draw a new right triangle, and show an alternative angle measure we can find from this picture:

These relationships are called the **cofunction identities,** summarize these below:

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Examples Write each function in terms of its cofunction

a. $\sin 10°$

b. $\cot 65°$

c. $\csc 34°$

Example Find one solution for each equation. Assume all angles involved are acute angles.

a. $\cos\theta = \sin(2\theta - 30°)$

b. $\sec(\theta + 15°) = \csc(2\theta + 25°)$

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Lesson: Radian Measure

Radian measure is another way to describe angle measure, as opposed to degrees.

What does radian measure allow us to do?

What is the definition of a radian?

Are radians a unit of measure like feet or meters? Circle: Yes No

What is the formula for the distance around a circle?

What does this formula show?

Write a way to relate radians to degrees:

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So what is π radians?

Show a conversion rate between radians and degrees:

Examples Convert each degree to radians:

a. $35°$

b. $-175°$

c. $27.8°$

Convert each radian measure to degrees:

a. $\frac{7}{4}\pi$

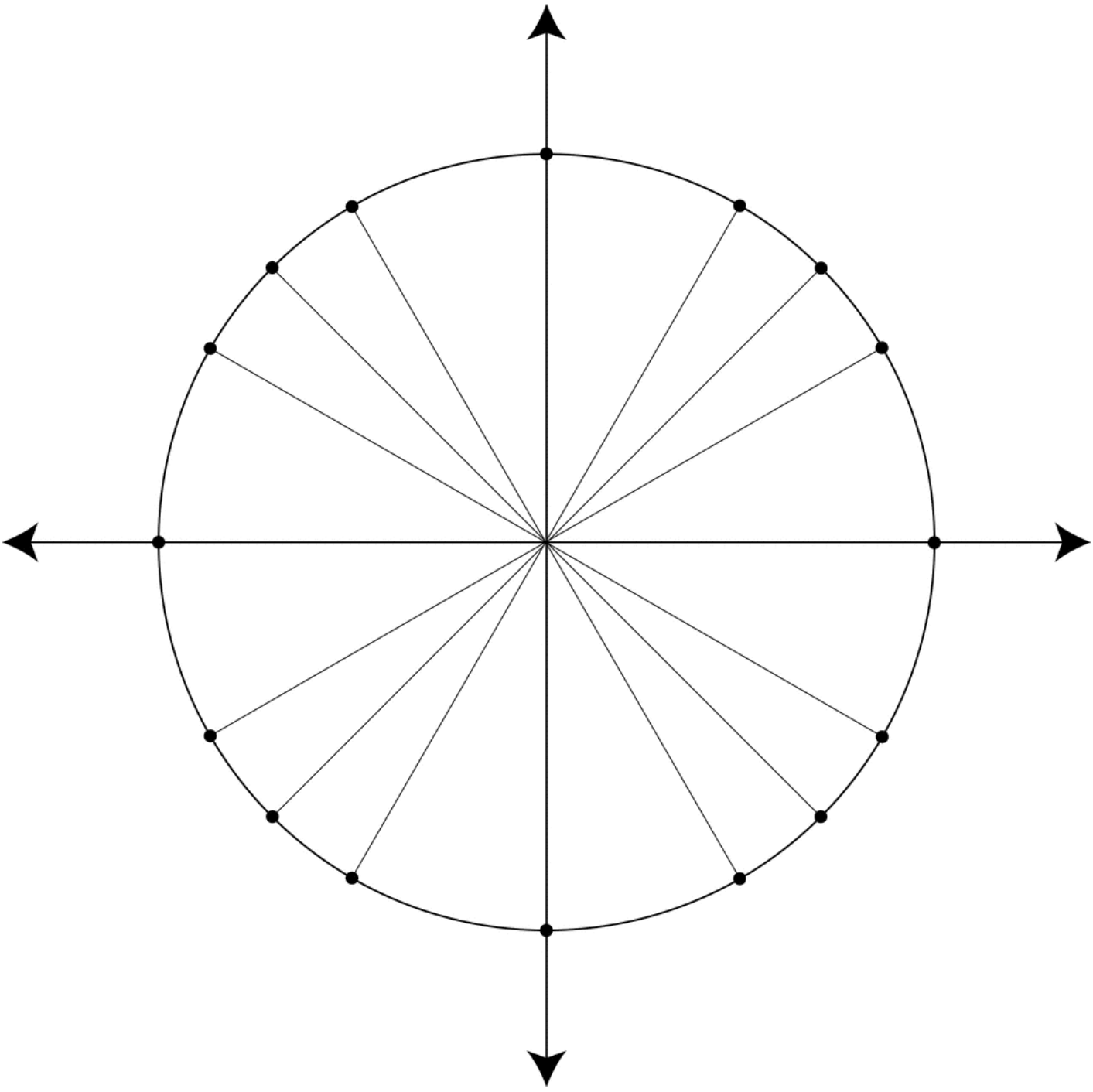
b. $-\frac{2\pi}{3}$

c. 7

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If no unit of angle measure is specified, what should you assume?

Below we will construct an illustration of all the common degree measures we have discussed, and then write their radian equivalent. Take your time to copy this perfectly!



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You will need to know all of those common degrees AND radian measures for the exam!

You may want to reference cheat sheets to help you with the next few problems.

Examples Find the value of each function.

a. $\sin\frac{5\pi}{6}$

b. $\tan\frac{3\pi}{4}$

c. $\csc\frac{5\pi}{3}$

d. $\cos\left(-\frac{2\pi}{3}\right)$

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Lesson: The Unit Circle and Circular Functions

Explain how we create the unit circle:

What is the equation for the unit circle?

Recall the formula for arc length is $s = r\theta$. When the radius of the circle is 1, what does this formula become?

Why does this matter / what is cool about this?

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Since r=1 on the unit circle, what can we say about our previously studied trig functions? Just summarize what we discussed in the video as best as you can.

Write out the **circular functions**

1.

2.

3.

4.

5.

6.

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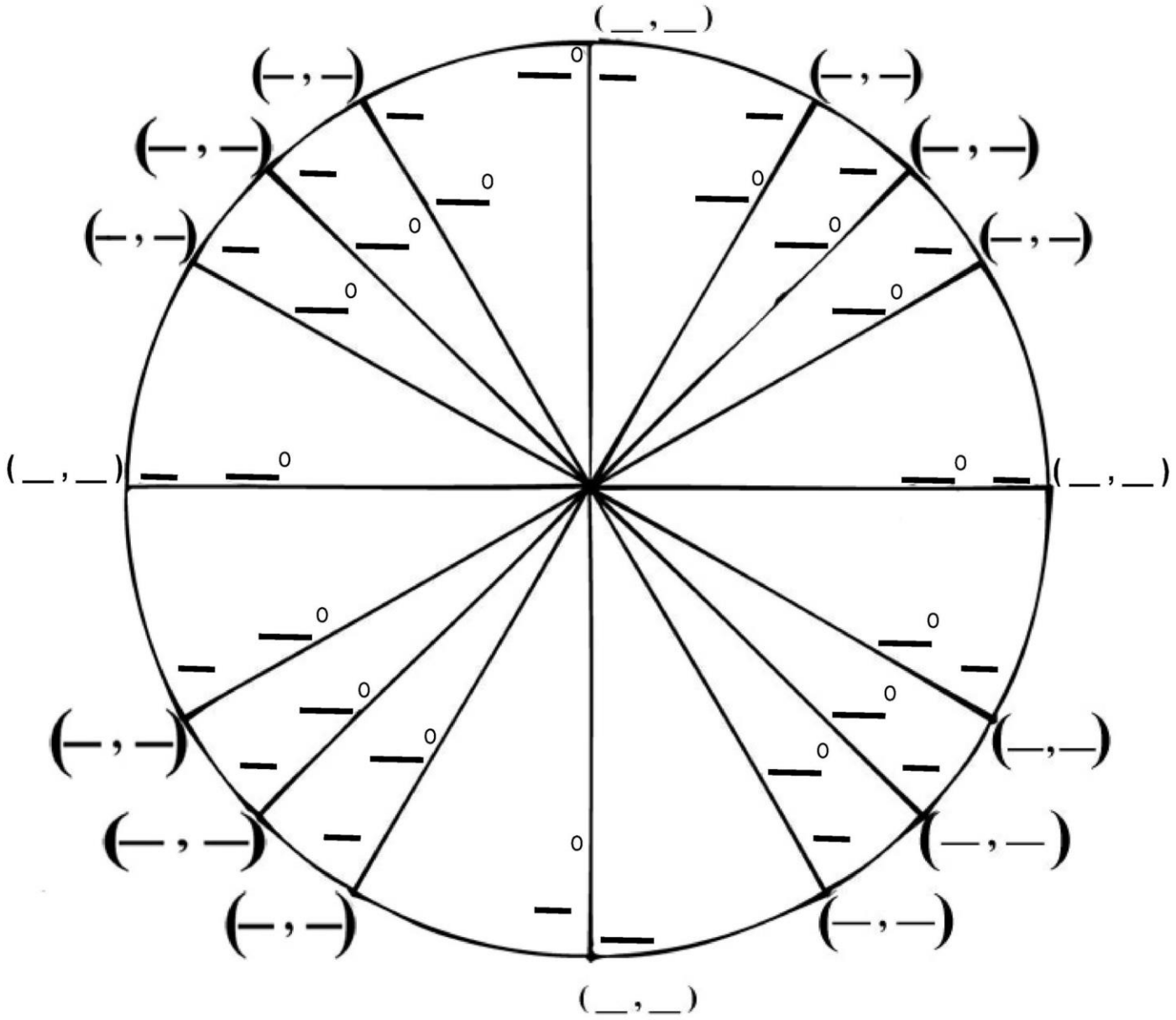
Explain the symmetry of the unit circle:

If a point (a, b) in on the unit circle, what other points are guaranteed to be on the unit circle?

How can we use symmetry with drawing the unit circle? How is it handy?

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Label the unit circle. Note: you will have to have this memorized! These are the standard angle measures that you'd be expected to do without a calculator!



Now, going back to the circular functions we discussed, how do you view the x-coordinates?

How do you view the y-coordinates?

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Examples Use the unit circle to find the values of the following:

a. $\sin\frac{\pi}{3}$

b. $\cos\frac{5\pi}{4}$

c. $\sin\frac{11\pi}{6}$

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Lesson: Arc Length on a Circle and Some Applications

Using the unit circle, we can also determine the domains of our circular functions.

What is the domain of sine and cosine, and why?

What is the domain of tangent and secant, and why?

What is the domain of cotangent and cosecant, and why?

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Lesson: Values of Circular Functions

We have previously discussed the unit circle (you will want to have it handy). One the most important things about the unit circle is that it will quickly tell you the value of the sine and cosine functions.

On the unit circle, what does the y-coordinate represent?

On the unit circle, what does the x-coordinate represent?

When we think about the values of these functions, what do we assume?

What does this mean for your calculator?

Examples Use the unit circle to find the values of:

a. $\sin\frac{\pi}{2}$ b. $\cos\frac{\pi}{2}$ c. $\tan\frac{\pi}{2}$

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Examples Use the unit circle to find the values of:

a. $\sin\frac{5\pi}{4}$ b. $\cos\frac{5\pi}{4}$ c. $\tan\frac{5\pi}{4}$

Example Compare methods. Use reference angles and degrees to find the exact value of $\sin\frac{2\pi}{3}$, then confirm your answer with the unit circle.

For approximate values, we still must use a calculator. Remember, when you are not specifically told you are using degrees, then you assume you are using radians.

What must you ensure with your calculator?

Examples Use a calculator to approximate the function values.

a. $\cos(1.72)$

b. $\sin(-3.54)$

c. $\cot(1.438)$

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Lesson: Values of Inverse Circular Functions

What do the functions $\sin^{-1} x$, $\cos^{-1} x$, $\cot^{-1} x$, etc. represent?

Your calculator makes a few key assumptions if it uses these functions (the reasons why are explained later):

1. For all x in $[-1, 1]$, using $\sin^{-1} x$ with a calculator will return a value between:

2. For all x in $[-1, 1]$, using $\cos^{-1} x$ with a calculator will return a value between:

3. For all real numbers x, using $\tan^{-1} x$ with a calculator will return a value between:

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Sometimes you will be asked to find answers in different quadrants than what your calculator will return!

Example Solve for s, given that s is in the interval $\left[0, \frac{\pi}{2}\right]$: $\sin s = 0.3311$

Example Solve for s, given that s is in the interval $\left[0, \frac{\pi}{2}\right]$: $\tan s = 0.2155$

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Example Solve for s, given that s is in the interval $\left[\pi, \frac{3\pi}{2}\right]$: $\tan s = \sqrt{3}$

Example Solve for s, given that s is in the interval $[0, 2\pi)$: $\cos^2 s = \frac{1}{2}$

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Lesson: The Unit Circle and Circular Functions as Line Segments

The unit circle is really just an extension of how we have previously thought about trig functions as line segments / right triangles.

Draw the illustration in the video, then show how these sides relate to our trig functions.

From your illustration, how do we define the following?

$\cos\theta =$ $\tan\theta =$ $\csc\theta =$

$\sin\theta =$ $\cot\theta =$ $\sec\theta =$

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Example Using your illustration from the previous page, suppose the measurement of TVU is $45°$. Find the lengths of the following line segments:

OQ

OU

OV

VR

US

PQ

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Lesson: Fundamental Identities of Trig

Review the test for symmetry (and the results):

We know that the sine function is odd, show what this means:

We know that the cosine function is even, show what this means:

These are called **identities.**

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Using these identities, what other identity can we determine about the tangent function:

These are called the **even-odd identities.**

We have already seen a handful of identities, and we call these the fundamental identities. Summarize them below.

Reciprocal Identities

Quotient Identities

Pythagorean Identities

Even-Odd Identities

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Example If $\cos\theta = \frac{3}{4}$ and θ is in quadrant I, find $\sin\theta$.

Example If $\tan\theta = -\frac{1}{4}$ and θ is in quadrant IV, find the values of all the other 5 trig functions.

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We use the identities to work on problems called transformations.

Example Write $\tan x$ in terms of $\sec x$.

Example Write the following in terms of sine and cosine, and then simplify until no quotients appear and all functions are of θ only.

a. $\csc\theta \cos\theta \tan\theta$

b. $\cos\ \theta(\cos\theta - \sec\theta)$

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Examples: Using Fundamental Trig Identities to Find Values of Other Trig Functions

Find the five remaining trig functions given the following information.

a. $\csc\theta = -\frac{5}{2}$, θ in quadrant III.

b. $\sec\theta = \frac{7}{2}$, $\tan\theta < 0$

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Examples: Writing Trig Functions in Terms of Other Trig Functions (Fundamental Identities)

a. Write $\cot x$ in terms of $\sin x$

b. Write $\csc x$ in terms of $\cos x$

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Examples: Using Fundamental Identities to Write Expressions in Terms of Sine and Cosine

Write each expression in terms of sine and cosine, then simplify so that no quotients appear and all functions are in θ only.

a. $\cos\theta \csc\theta$

b. $\frac{(1+\cot\theta)}{\cot\theta}$

c. $\frac{\sec^2\theta-1}{csc^2\theta-1}$

d. $\frac{\csc\theta}{\cot(-\theta)}$

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Lesson: Proofs of the Sum and Difference Identities for Cosine

Write out the proof for $\cos(A - B) = \cos A \cos B + \sin A \sin B$. Do your best to understand the proof and then explain it in a way that you will understand.

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Write out the identity for $\cos(A - B)$ below (yes I know I stated it above, but YOU should write it so you have a better chance of remembering it):

Write out the proof for $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Write out the identity below (yes, write it out so you don't forget!):

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Lesson: Using Sum and Difference Identities for Cosine

Examples Show how to use the cosine of a sum or difference to find the exact value of the expressions:

a. $\cos 105°$

b. $\cos \frac{\pi}{12}$

c. $\cos 40° \cos 50° - \sin 40° \sin 50°$

Show how the cofunction identities are an extension of the cosine of a difference:

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Write out all the **cofunction identities**

Examples Find one value of θ that satisfies the following:

a. $\sec\theta = \csc 72°$

b. $\tan\theta = \cot(-44°)$

c. $\cos\theta = \sin\frac{5\pi}{6}$

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Example Write $\cos(180° - \theta)$ as a trigonometric function of θ alone.

Example Suppose $\sin s = \frac{3}{5}$ and $\sin t = -\frac{12}{13}$, where s is in quadrant I and t is in quadrant III. Find $\cos(s + t)$ and $\cos(s - t)$.

Example Verify that $\sec(\pi + x) = -\sec x$

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Lesson: Sum and Difference Identities for Cosine

Write out the proof for $\sin(A + B)$

Write out the proof for $\sin(A - B)$

Write out the identities for the sum and difference of sine below:

Write out the proof for $\tan(A + B)$

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Write all identities for the sum and difference of tangent below:

Examples Find the exact value of each expression.

a. $\sin(-15°)$

b. $\tan\left(-\frac{5\pi}{12}\right)$

c. $\sin\left(\frac{5\pi}{9}\right)\cos\left(\frac{\pi}{18}\right)+\cos\left(\frac{5\pi}{9}\right)\sin\left(\frac{\pi}{18}\right)$

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Examples Write each function as an expression involving functions of θ alone.

a. $\cos(30° + \theta)$

b. $\sin(45° + \theta)$

c. $\tan\left(\frac{\pi}{4} + x\right)$

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Examples Suppose A and B are angles in standard position such that $\sin A = -\frac{3}{5}\ \frac{3\pi}{2} < A < 2\pi$ and

$\cos B = -\frac{7}{25},\ \pi < B < \frac{3\pi}{2}$. Find the following.

a. $\sin(A + B)$

b. $\tan(A - B)$

c. the quadrant of A-B

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Examples Verify the identities.

a. $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

b. $\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$

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Examples: Sine of a Sum and Tangent of a Sum

Let $\cos s = \frac{5}{12}, \sin t = \frac{4}{5}$ where s and t are in quadrant I. Find:

$\sin(s + t)$

$\tan(s + t)$

The quadrant of $(s + t)$

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Lesson: Double-Angle Identities

Show how we derive the first identity for $\cos 2A$:

Show how we derive the second identity for $\cos 2A$:

Show how we derive the third identity for $\cos 2A$:

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Show how we derive the identity for $\sin 2A$:

Show how we derive the identity for $\tan 2A$:

Summarize the Double-Angle Identities Below:

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Example Give $\sin\theta = \frac{5}{8}$ and $\cos\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

Example Find the values of the six trig functions give $\cos 2\theta = -\frac{11}{13}$ and $180° < \theta < 270°$.

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Example Verify the following identities:

a. $(\sin x + \cos x)^2 = \sin 2x + 1$

b. $\frac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$

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Examples Simplify the expressions.

a. $\sin^2 15° - \cos^2 15°$

b. $\sin^2 \frac{3\pi}{5} - \cos^2 \frac{3\pi}{5}$

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Example Write $\cos 3x$ in terms of $\cos x$

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Lesson: Product-to-Sum Identities

Show what happens when we add the identities $\cos(A+B)$ to $\cos(A-B)$

Now show what happens when we subtract them:

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Examples Write the following as the sum or difference of 2 trig functions.

a. $2 \sin 48° \cos 104°$

b. $8 \sin 9x \sin 5x$

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Examples Write the expression as a product of trigonometric functions.

a. $\cos 6x + \cos 9x$

b. $\sin 106° - \sin 96°$

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Lesson: Half-Angle Identities

Show how we derive the identity for $\sin\frac{A}{2}$:

Show how we derive the identity for $\cos\frac{A}{2}$:

Show the first way we can derive an identity for $\tan\frac{A}{2}$:

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Show the second way we can derive an identity for $\tan\frac{A}{2}$:

Show the third way we can derive an identity for $\tan\frac{A}{2}$:

Summarize the half-angle identities below:

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Example Show how to use the half-angle identity to evaluate: $\sin 12.5°$

Example Find the exact value of $\cos 165°$

Example Find the exact value of $\tan 75°$ using the identity $\tan A = \frac{\sin A}{1+\cos A}$

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Example Given $\cos s = \frac{2}{7}$ with $0 < s < \frac{\pi}{2}$, find $\sin\frac{s}{2}$, $\cos\frac{s}{2}$, and $\tan\frac{s}{2}$

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Example Simplify each expression.

a. $\sqrt{\frac{1-\cos 40^\circ}{2}}$

b. $\frac{1-\cos 64.72^\circ}{\sin(64.72^\circ)}$

c. $\pm\sqrt{\frac{1+\cos 10x}{2}}$

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Example Verify the identities.

a. $\cot^2\frac{x}{2} = \frac{(1+\cos x)^2}{\sin^2 x}$

b. $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

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Lesson: Summarizing All of the Identities

Using all the trig identities that we've discussed, verify the following:

a. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

b. $1 - \tan^2 \frac{x}{2} = \frac{2\cos x}{1+\cos x}$

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c. $\tan\left(\frac{a}{2}+\frac{\pi}{4}\right)=\sec a+\tan a$

d. $\sin 7x+\sin 5x+\sin 3x+\sin x=4\sin 4x\cos 2x\cos x$

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e. $\tan 4x = \frac{2\tan 2x}{2-\sec^2 2x}$

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Lesson: Determining if Functions are One-to-One

Before we begin, you should already be familiar with the concept of functions and the vertical line test. First let's do some review.

Review Determine whether the following relations are functions, then determine the domain and range.

a. $\{(2,3),(5,7),(2,4)\}$

b.

Now we will discuss a specific type of function: a one-to-one function. Define this below:

How can we visualize this?

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Example Determine if any of the following are i) functions and ii) one-to-one functions

a. $\{(2,4),(-1,7),(3,7)\}$

b. $\{(3,6),(-1,2),(-3,-4)\}$

Similar to how we have a test to determine if a graph represents a function, we have another test to determine if a function is one-to-one.

Describe the horizontal line test below:

Examples Determine whether the following graphs represent i) functions and ii) one-to-one functions (draw your own):

a. b. c.

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Lesson: Finding the Inverse of Functions

One-to-one functions have inverses. Describe what the inverse of a function is:

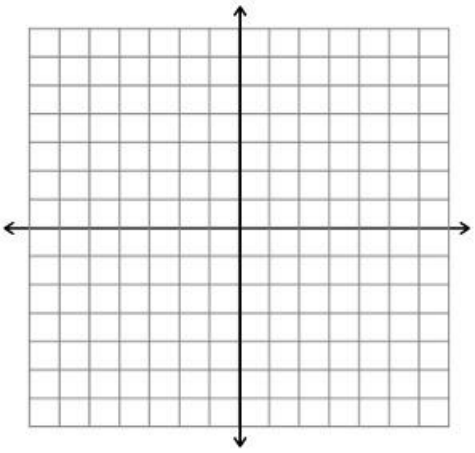
Example Find the inverse of the following function: $\{(1,4),(7,8),(3,-2)\}$

Describe the steps to find the inverse of a function:

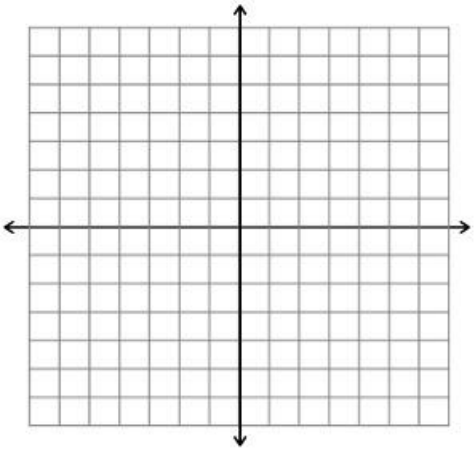
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Examples Find the inverse of the following one-to-one functions. Then plot the original function on the graph with its inverse.

a. $f(x) = 3x + 1$



b. $g(x) = -\frac{1}{2}x + 4$



What is the relationship of a graph with its inverse?

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Examples Find the inverse of the following one-to-one functions.

a. $h(x) = \sqrt[3]{x}$

b. $f(x) = x^2, x \geq 0$

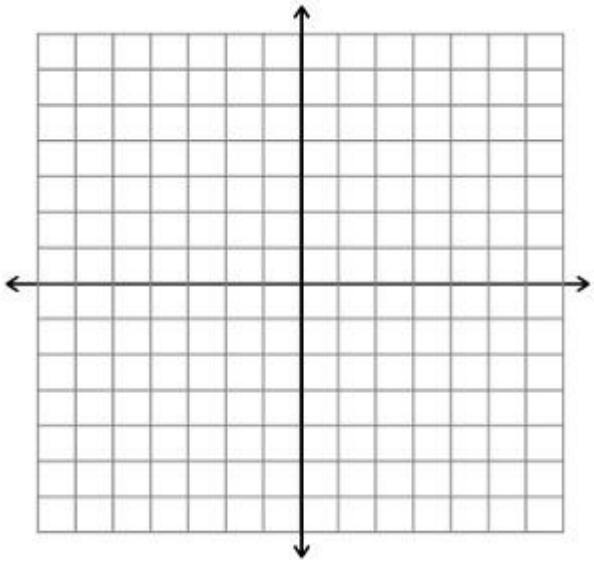
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Lesson: Exponential Functions

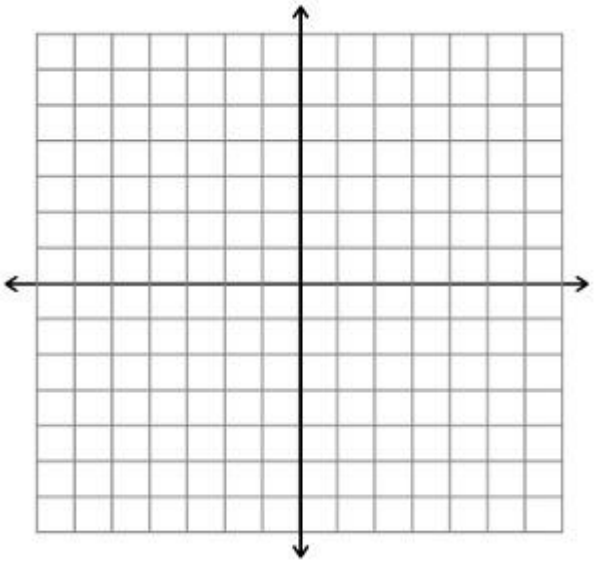
An exponential function is a function of the form _____ where $a > 0, a \neq 1$.

Examples Graph the following functions, then determine their domain and range.

a. $f(x) = 2^x$



b. $g(x) = \left(\frac{1}{2}\right)^x$

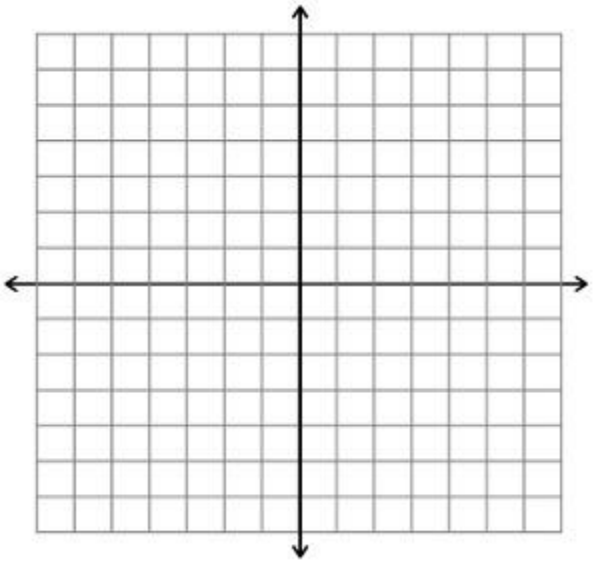


Using the graphs above, summarize some of the characteristics of these graphs:

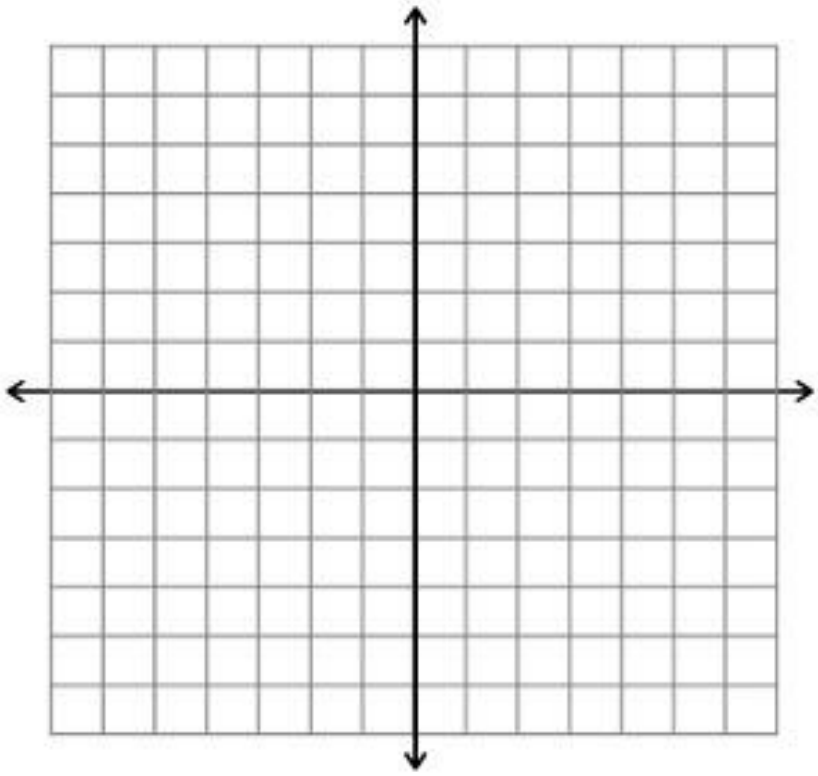
The number e is an irrational number that is frequently used in math. What is an approximation of e?

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Example Graph $f(x) = 2^x, g(x) = e^x, h(x) = 3^x$ all on the same graph below for comparison:



Translations work similarly to other graphs we have discussed. Using the graph of $f(x) = 2^x$ as a "base graph", draw the following graphs together on the same graph: $f(x) = 2^x, g(x) = 2^x + 3, h(x) = 2^{x-1}$



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Lesson: Solving Exponential Equations

In this lesson, we will solve equations that look like this:

$$3^x = 27 \qquad \text{or} \qquad 4^{x+3} = 8^{x-2}$$

To do this, we will write directions around the following example: $2^{3x} = 8^{x+1}$

Examples Solve:

a. $49^{x-3} = 7^{2x}$

b. $9^{4x} = 27^{x+4}$

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Examples Solve:

a. $5^{x-2} = \frac{1}{25}$

b. $\left(\frac{3}{5}\right)^{2x} = \left(\frac{27}{125}\right)^{x+2}$

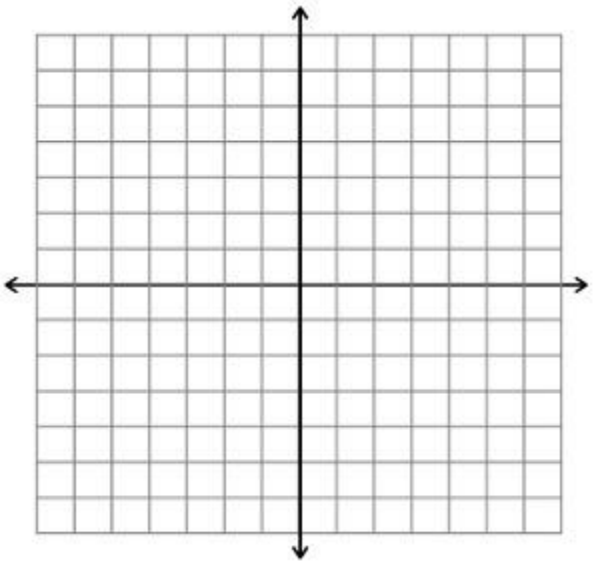
c. $27^{3x+1} = 9^{4x+2}$

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Lesson: Intro to Logarithmic Functions

First, we want to review how to function inverses of functions. Find the inverse of: $f(x) = 2x + 8$

Graph the exponential function $f(x) = 3^x$ below:



Recall the exponential functions are one-to-one, which means that they have inverses. Attempt to find the inverse of $f(x) = 3^x$ below:

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As we can see, there is an issue with finding the inverse – we have no way to solve for y! As such, we need new mathematics to help solve this.

Write down the definition of a logarithm:

Examples Write the following in exponential form:

a. $\log_3 81 = 4$

b. $\log_2 \frac{1}{8} = -3$

c. $\log_5 125 = 3$

d. $\log_9 9 = 1$

Examples Write the following in logarithmic form:

a. $2^4 = 16$

b. $3^0 = 1$

c. $6^{-2} = \frac{1}{36}$

d. $8^{\frac{1}{3}} = 2$

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Now we can finally determine what is the inverse of $f(x) = 3^x$:

There are a few common logarithms:

- $\log x$ means:

- $\ln x$ means:

There are also 2 key properties of logarithms:

1. $\log_a a = 1$: explain why this makes sense

2. $\log_a 1 = 0$: explain why this makes sense:

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Lesson: Basics of Solving Logarithmic Equations and Evaluating Logarithms

We want to solve equations that look like:

$$\log_3 x = 4 \qquad \text{or} \qquad \log_x 36 = 2$$

Before you begin, you should already be able to do the following:

a. Put 3^5 in logarithmic form:

b. Put $\log_5 5 = 1$ in exponential form:

Now we can explain how to solve logarithmic equations using the example $\log_3(2x + 1) = 3$

Example Solve: $\log_x 16 = 4$

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Example Solve:

a. $\log_9 x = 2$

b. $\log_6(2x + 4) = 2$

Now we can also discuss how to evaluate logarithms. Explain how to evaluate: $\log_3 27$:

Examples Evaluate:

a. $\log_5 \frac{1}{25}$

b. $\log 100$

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Examples Evaluate:

a. $\ln e$

b. $\log_3 27$

c. $\log_{25} 5$

d. $\log \frac{1}{10}$

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Lesson: How to Graph a Logarithm

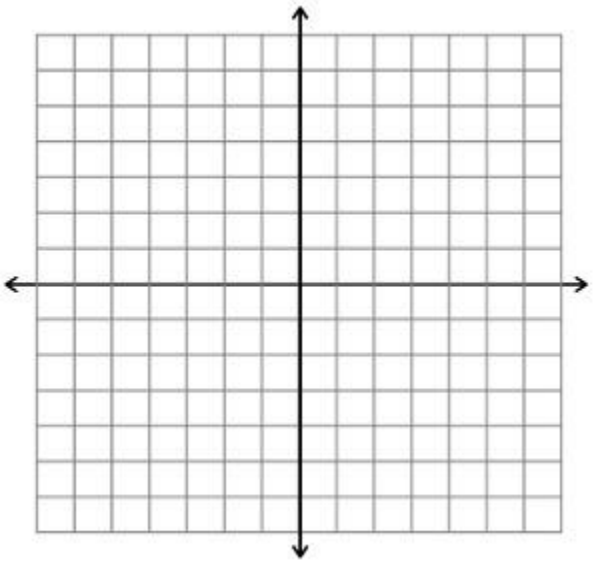
Before we begin, you should be able to do the following:

a. Convert to logarithmic form: $25 = 5^2$

b. Convert to exponential form: $\log_2 8 = 3$

c. Find the inverse of $y = 2^{\mathrm{x}}$

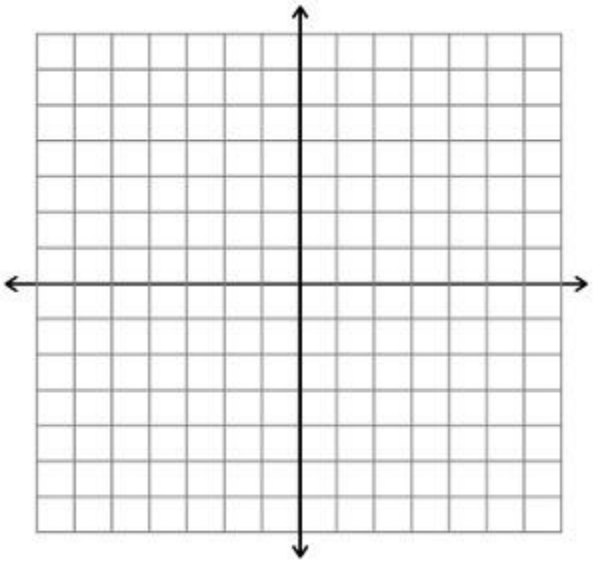
Now we can explain how to graph $y = \log_3 x$



For comparison, also graph $f(x) = 3^x$ on the same graph above.

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Example Graph $y = \log_{\frac{1}{2}} x$ below:



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Review: Exponents for Calc/Pre-calc Students

The 3 Main Rules:

Examples

a. $(3x^4y^7)^2 * (x^7y^3)$

b. $\frac{(-2x^3y^6)^3}{(3xy^2)^4}$

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Examples

a. $(4x^3y^7z^{10})^{-3}$

b. $\left(\frac{3x^{-4}y^{-7}}{x^3y^{-10}}\right)^{-2}$

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Examples

a. $\sqrt[3]{x^2} * \sqrt{x}$

b. $\frac{x^{\frac{1}{2}}x^{\frac{1}{3}}}{x^{\frac{1}{4}}}$

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Review: Logarithms and Properties of Logarithms for Precalc/Calc Students

Write down the definition of the logarithm:

Example Write as a logarithm:

a. $2^3 = 8$ b. $25^{\frac{1}{2}} = 5$

Example Write in exponential form:

a. $\log_7 49 = 2$ b. $\log_3 1 = 0$

Write down the common logarithms and what they mean:

Write down all the properties of logarithms:

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Example Expand as much as possible:

a. $\log 3xy^7$

b. $\log \frac{10x^3}{y^{10}z}$

c. $\ln \sqrt[3]{2xy^4}$

Example Write as a single logarithm:

a. $\ln 3 - 2\ln x$

b. $4\ln 2 + 5\ln y - 3\ln x$

c. $\frac{1}{2}\log 4 + \frac{1}{2}\log y^3 - 7\log y$

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Review: How to Solve Logarithmic Equations

What are the 5 main properties of logarithms?

When solving logarithmic equations, what else should you remember about logarithms?

Examples

a. $\log x + \log x^2 = 3$

b. $\log_5(3x + 5) + \log_5(x + 1) = 1$

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c. $\log(5x-6)-\log(x+1)=\log\ 3$

d. $\log_2(5+4x)+\log_2(x+3)=\log_2 3$

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Examples: Finding Average Rates of Change

Calculate the average rate of change of the functions over the given intervals.

a. $f(x) = x^2 + 3x; [0, 3]$

b. $f(x) = 2 + \sin x; \left[\frac{\pi}{2}, \ \pi\right]$

c. $f(x) = \sqrt{4x + 1}; [0, 2]$

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Examples: Determining the Slope at a Point and the Tangent Line

Find the slope of the curve at the given point and find the equation of the tangent line at that point. Do this only using the formula for the slope of the secant line.

a. $f(x) = x^2 - 1, (2, 3)$

b. $f(x) = x^2 - 4x; (2, -4)$

c. $y = \frac{1}{x}; (2, \frac{1}{2})$

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Lesson: An Intro to Limits

Graph $f(x) = \frac{(x^2-4)}{x-2}$ below. What does it look like this graph should equal when x=2?

Explain this previous question now in mathematical terms

Example Find $\lim_{x\to 1} f(x)$ when $f(x) = \begin{cases} x^2+1 & x \neq 1 \\ -2 & x = 1 \end{cases}$. Draw a picture of this situation.

What is the value of $f(1)$?

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Example What is $\lim_{x \to 2} f(x)$ when $f(x) = x$? Draw a graph to accompany your answer.

What do we call the function $f(x) = x$?

What can we say about the limit of these types of functions?

Example What is $\lim_{x \to -3} f(x)$ when $f(x) = 4$? Draw a graph to accompany your answer.

What do we call the function $f(x) = c$, where c is a constant?

What can we say about the limit of these types of functions?

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Write out all the limit laws on this space:

Write out the other 2 handy theorems:

Theorem 1: Limits of Polynomials

Theorem 2: Limits of Rational Functions

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Example Using the limit laws, calculate the following limits:

a. $\lim_{x\to 3}(2x+1)$

b. $\lim_{x\to 1}\frac{5x+2}{x-4}$

c. $\lim_{x\to 2}\sqrt{5x+4}$

d. $\lim_{x\to -3}5\sqrt{2-3x}$

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Examples: Intro to Calculating Limits

Calculate the following limits:

a. $\lim_{x \to 3}(x^2 + 2x + 1)$

b. $\lim_{x \to 0}(8 - 2x)^{\frac{2}{3}}$

c. $\lim_{x \to \frac{2}{5}}(5x + 2)(3x - 1)$

d. $\lim_{x \to \frac{1}{3}} \sqrt[3]{3x + 7}$

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Lesson: Limits that Do Not Exist

List the types of limits that do not exist, as well as an example and graph

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Lesson: One-Sided Limits

Use the following function to describe one-sided limits: $f(x) = \begin{cases} 3, & x > 2 \\ 1, & -1 \leq x \leq 2 \\ -2, & x < -1 \end{cases}$

Evaluate the following:

a. $\lim_{x \to 2^+} f(x)$

b. $\lim_{x \to 2^-} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to -1^+} f(x)$

e. $\lim_{x \to -1^-} f(x)$

f. $\lim_{x \to -1} f(x)$

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Draw the graph presented in the video then evaluate the following:

a. $\lim_{x\to 3^+} f(x)$

b. $\lim_{x\to 3^-} f(x)$

c. $f(3)$

d. $\lim_{x\to -2^+} f(x)$

e. $\lim_{x\to 4^-} f(x)$

f. $\lim_{x\to 4^+} f(x)$

g. $\lim_{x\to 4} f(x)$

h. $f(4)$

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Write down the theorem that explains when $\lim_{x\to c} f(x)$ exists:

Examples Evaluate the following:

a. $\lim_{x\to 2^+} \sqrt{\frac{x-2}{x+1}}$

b. $\lim_{x\to 2^-} \left(\frac{1}{x+1}\right)\left(\frac{x-3}{x}\right)\left(\frac{4}{4-x}\right)$

c. $\lim_{x\to -3^-} \frac{(x+1)|x+3|}{x+3}$

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Examples: Basic One-Sided Limits

a. $\lim_{x\to 2^+} \sqrt{x^2-4}$

b. $\lim_{x\to 0^+} \frac{\sqrt{2x^2+3x+1}-1}{x}$

c. $\lim_{x\to 3^-} \begin{cases} 4, & x > 3 \\ -2, & x \leq 3 \end{cases}$

d. $\lim_{x\to 5^-} \frac{x-5}{|x-5|}$

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Examples: One-Sided Limits with Absolute Value

a. $\lim_{x\to 2^+} \frac{2-x}{|x-2|}$

b. $\lim_{x\to 5^-} \frac{(x-4)|x-5|}{x-5}$

c. $\lim_{x\to 2^+} \frac{\sqrt{3x}(x-2)}{|x-2|}$

d. $\lim_{x\to 0^-} \frac{\sin x}{|\sin x|}$

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Lesson: Troubleshooting Limits With Algebraic Tricks (When You Get 0/0)

Trick #1 (name this trick here):

Note what happens if we try to evaluate $\lim_{x\to 2} \frac{x^2-4}{x-2}$

Draw what this situation looks like graphically.

How can we solve this problem?

Example Evaluate $\lim_{x\to 3} \frac{(x^3-27)}{x-3}$

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Trick #2 (name it):

Examples Evaluate:

a. $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$

b. $\lim_{x\to 1} \frac{2-\sqrt{x^2+3}}{x-1}$

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Trick #3 (name it):

Examples Evaluate

a. $\lim_{x\to 3} \frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{x-3}$

b. $\lim_{x\to 1} \frac{\left(1-x^{-1}\right)}{1-x}$

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Examples: Algebraic Tricks for Limits

Evaluate the following:

a. $\lim_{x\to -4} \frac{\left(\frac{1}{\sqrt{12+x}}-\frac{1}{3}\right)}{x+4}$

b. $\lim_{x\to 0} \frac{\left(\frac{1}{3+x}-\frac{1}{3}\right)}{x}$

c. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

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Lesson: The Sandwich Theorem and Basic Trig Limits

Explain the idea of the Sandwich Theorem:

State the Sandwich Theorem

Example Suppose there is a function $f(x)$ where $3 - x^2 \leq f(x) \leq 3\cos x$. Find $\lim_{x \to 0} f(x)$.

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Show how to use the Sandwich Theorem to evaluate the following limits:

a. $\lim_{x \to 0} \sin x$

b. $\lim_{x \to 0} \cos x$

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Examples: Basic Trig Limits

Evaluate the following:

a. $\lim_{x\to 0} \frac{3+x+\sin x}{4\cos x}$

b. $\lim_{x\to \frac{\pi}{3}} \sec x$

c. $\lim_{x\to \pi} \sin^2 x$

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Lesson: The $\boldsymbol{lim_{x\to 0}\frac{sin\, x}{x} = 1}$

This lesson is about the tricky but very important limit:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Or equivalently:

$$\lim_{x\to 0}\frac{x}{\sin x} = 1$$

The proof is interesting and worth noting, you can write it down if you wish but you should minimally attempt to understand the reasoning behind it. It's difficult, but difficult things will expand your mind!

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Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\, x}{x}=1}$

a. $\lim_{x\to 0}\frac{\sin 3x}{5x}$

b. $\lim_{x\to 0} 12x^2(\cot 2x)\,(\csc 3x)$

c. $\lim_{x\to 0}\frac{x^3-x+\sin 5x}{3x}$

d. $\lim_{x\to 0}\frac{\sin x}{\sin 3x}$

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Step-by-Step Examples: Limits using $\boldsymbol{lim_{x\to 0}\frac{sinx}{x}=1}$

a. $\lim\limits_{x\to 0}\frac{\sin 7x}{x}$

b. $\lim\limits_{x\to 0} x\cot 3x$

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More Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\, x}{x} = 1}$

a. $\lim_{x\to 0}\frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)$

b. $\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}$

c. $\lim_{x\to 0}\frac{1-\cos 2x}{x}$

d. $\lim_{x\to 0}\frac{\sin 3x}{\mathrm{x}\cos 5x}$

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Examples: Troubleshooting with $\lim_{x\to 0}\frac{\sin x}{x}$ problems – examples of what not to do

a. $\lim_{x\to 0}\frac{\tan 5x}{x}$

b. $\lim_{x\to 0}\frac{\sin 3x}{\sin 7x}$

Lesson: Continuity

The idea of continuity is probably intuitive at the point. Draw the example in the video to demonstrate continuity.

For a point c, define when a function is:

1. Continuous

2. Right – continuous

3. Left – continuous

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Draw the graph in the video to demonstrate these ideas.

Example Draw the graph in the video, then describe the continuity at the following points:

x = 1 x = 3 x = -4 x = -2

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A function is continuous at a point c is it meets these 3 conditions:

1. f(c) exists

2. the limit $\lim_{x \to c} f(x)$ exists

3. $\lim_{x \to c} f(x) = f(c)$

Describe the different kinds of discontinuities below:

What does it mean for a function to be continuous?

If a function is not continuous, then it is _____

If $f(x)$ and $g(x)$ are continuous functions at the point x=c, then the following are continuous at c:

1. sum/differences:

2. constant multiples (for any number k):

3. products:

4. quotients:

5. powers:

6. roots:

7. compositions:

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Example Describe where the following functions are continuous:

a. $f(x) = \frac{x+3}{x-2}$

b. $f(x) = \sqrt{2x}$

What can we say about the limits of a composition of continuous functions $f(x)$ and $g(x)$?

Example Find the limit $\lim_{x \to 0} (x + 2)e^{\sin x}$

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Examples: Continuity

Determine where the following functions are continuous:

a. $f(x) = \frac{1}{(x+3)^2} - 2$

b. $g(x) = \begin{cases} 1 + x, & x \geq 3 \\ 4, & -2 \leq 0 < 3 \\ 2 - x, & x < -2 \end{cases}$

c. $h(x) = \begin{cases} |x + 1|, & x \geq -2 \\ -5, & x < -2 \end{cases}$

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Examples: Using Continuity with Limits

Find the limits.

a. $\lim_{x\to 0} \cos\left(x + \cos(\frac{\pi}{2} + x)\right)$

b. $\lim_{x\to \frac{\pi}{2}} \sin(\cos(\tan x))$

c. $\lim_{x\to 0} \sin^{-1}\left(e^{\cos\left(\frac{\pi}{2}+x\right)}\right)$

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Lesson: The Intermediate Value Property and Continuous Functions

What is the Intermediate Value Property?

There are 2 consequences of this:

1. Connectedness:

2. Root Finding:

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Example Determine the value of a that would make the function continuous at every x:

$$f(x) = \begin{cases} x^2 + 1, & x > 2 \\ ax, & x \leq 2 \end{cases}$$

Example Show that $x^3 - 14x + 2 = 0$ has 3 solutions in the interval $[-4, 4]$

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Examples: Continuous Extensions

Determine the value of a that would make the following functions continuous:

a. $f(x) = \begin{cases} 2x, & x > 1 \\ ax^2, & x \leq 1 \end{cases}$

b. $g(x) = \begin{cases} \frac{2x+a}{a+2}, & x > 0 \\ x^2 + a, & x \leq 0 \end{cases}$

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Step-by-Step Examples: Continuous Extensions

Use the space below to find a (or a and b) to make the following functions continuous:

a. $f(x) = \begin{cases} x + a, & x \geq 1 \\ 3, & x < 1 \end{cases}$

b. $f(x) = \begin{cases} 3, & x \geq 1 \\ ax + b, & -2 < x < 1 \\ -3, & x \leq -2 \end{cases}$

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Lesson: The Precise Definition of a Limit

What is the issue with the idea of limits?

Explain the example that went with this (it's better to watch this then write this out after):

State the Precise Definition of a Limit

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Examples: The Precise Definition of a Limit with a Specific Epsilon

Given the function $f(x)$, a point c, and a specific ϵ, find L and the specific δ that fulfills the definition of a limit.

a. $f(x) = 3x + 1, c = 2, \epsilon = 0.01$

b. $g(x) = x^2 + 1, c = 1, \epsilon = 0.1$

c. $f(x) = \sqrt{1 - 4x}, c = -2, \epsilon = 0.02$

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Examples: The Precise Definition of a Limit with a Specific Epsilon

Given the function $f(x)$, a point c, and a specific ϵ, find L and the specific δ that fulfills the definition of a limit.

$$f(x) = \frac{1}{x}, c = 1, \epsilon = 0.01$$

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Examples: The Precise Definition of a Limit with a No Specific Epsilon

Find the specific δ for each limit that would correspond to the precise definition of a limit.

a. $\lim_{x\to 1} \sqrt{5-x} = 2$

b. $\lim_{x\to 1} f(x) = 3$ when $f(x) = \begin{cases} 2x+1, & x \neq 1 \\ 6, & x = 1 \end{cases}$

c. $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

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Lesson: The Formal Definition of the Derivative

One of the things we learn in pre-calculus is about secant lines. You can write a review of this below:

When the distance between two points for a secant line goes to zero, what do we get?

What is the formal definition of the derivative?

What is the alternate formula for the derivative?

Which formula should you use?

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How do you find the derivative at a specific point?

What 4 things does the derivative at a point represent?

1.

2.

3.

4.

Example Calculate the derivative of $f(x) = x^2 + 2$

Example What is the value of the derivative of $f(x) = x^2 + 2$ at $(1, 3)$?

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Example Find the equation of the tangent line for $f(x) = x^2 + 2$ at $(1, 3)$

List all the different ways to represent a derivative below:

Write out the theorem that discusses differentiability and continuity

What types of functions will not have a derivative at a point?

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Examples: Calculating Derivatives Using the Formal Definition Only and Finding Tangent Lines

Use the formal definition of a derivative to calculate the follow derivatives of each function. Then determine the tangent line at the given point.

a. $f(x) = 4 - x^2; (-1, 3)$

b. $y = \frac{1-x}{2x}, (1, 0)$

c. $g(x) = x^{\frac{3}{2}}, (4, 8)$

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Examples: Using the Alternative Formula for Derivatives

Use the alternative formula for derivatives to calculate the derivative of the following function. Compare this calculation to the original formula.

a. $y = x^3 - x^2$

b. $y = \frac{1}{x+3}$

c. $f(x) = 2 + \sqrt{x}$

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Step-by-Step Examples: Comparing the Definition of the Derivative to the Alternative Formula

a. $f(x) = x + \frac{3}{x}$

b. $g(x) = x^3 - 5x^2 + 1$

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Example: Finding the Derivative of a Square Root with the Definition of the Derivative

Find the tangent line at (6,5) for $y = \frac{5}{\sqrt{7-x}}$ using only the definition of the derivative.

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Lesson: Other Notes about the Derivative

Define right-hand derivatives

Define left-hand derivatives

Explain how $y = |x|$ is differentiable on each side

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The derivative of $f(x) = x^2$ is $f'(x) = 2x$. Compare their graphs below:

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Lesson: Basic Differentiation Rules

Now we will introduce the basic differentiation rules, which will make life much easier. Let $f(x)$ and $g(x)$ be differentiable functions. Then:

1. Derivative of a Constant Function

Example Find the derivative of $f(x) = 6$

2. Power Rule

Example Find the derivative of $f(x) = x^{\frac{3}{2}}$

3. Constant Multiple Rule

Example Find the derivative of $f(x) = 4x^7$

4. Sum Rule

Example Find the derivative of $f(x) + 5x^2 + 2$

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5. The Natural Exponential Function

Example Find the derivative of $f(x) = 5e^x - 7\sqrt{x}$

6. Product Rule

Example Find the derivative of $f(x) = (3x^2 - 5)e^x$

7. Quotient Rule

Example Find the derivative of $f(x) = \frac{5x}{e^x}$

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Examples: Finding Tangent Lines

a. Find the line tangent to $y = 4x^2 + 3$ that has a slope of 16. State the point that this occurs at on the graph, and also state the equation of the tangent line.

b. Find the equation of any tangent lines perpendicular to the x-axis for $f(x) = x^3 - 4x^2 + 4$

c. Find the equation of the line that is tangent to the curve $f(x) = e^x$ and goes through the origin.

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Examples Finding a Point Tangent to One Graph, Perpendicular to Another Line

Find all the points on $f(x) = \frac{x}{x-2}$ where the tangent lines are perpendicular to $f(x) = 2x + 7$

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Lesson: Higher Derivatives

How do you find the second derivative?

Notation for the second derivative:

How do you find and notate higher derivatives?

Examples Find the 4th derivatives of the following:

a. $y = x^7 - 5x^3 + 2x$

b. $f(x) = e^x - 7x^2 + x^3$

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Example: The Most Basic Derivative Examples: Constant, Constant Multiple, Power Rule, and Sum Rule

Take the derivative of the following only using the constant, constant multiple, power, or sum rules:

a. $f(x) = 5x^7 - 3x^5 + \pi x^2$

b. $g(x) = 3x^{-5} - 2\sqrt{x} + \frac{1}{2}x^{\frac{9}{5}} + x^{\pi}$

c. $y = \frac{x^5 - 3x^2 - 5}{x^2}$

d. $y = (2x + 1)(5x - 4)$

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Example: The Product and Quotient Rules

Take the derivative of the following only the product and quotient rules. Do not simplify these expressions, the point is to ues the rules.

a. $f(x) = (3x^2 - 6x)(2x + 1)$

b. $g(x) = \frac{3x^7}{2x^5}$

c. $y = \left(5\sqrt{x} + 3x^{\frac{4}{5}}\right)(7e^x)$

d. $y = \frac{(\sqrt{x}-1)(x+1)}{(\sqrt{x}+1)(x-3)}$

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Step-by-Step Examples: The Quotient Rule

Find the derivatives:

a. $y = \frac{1+x+6\sqrt{x}}{x}$

b. $y = \frac{\sqrt{x}+1}{\sqrt{x}-1}$

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Examples: The Quotient Rule for Derivatives Onlyl

Take the derivatives:

a. $f(x) = \frac{\sin x}{\sqrt{x}}$

b. $y = \frac{9\cos x - 2\tan x}{e^x + \sqrt[3]{x} - 8\sin x}$

c. $f(x) = \frac{(5x^7 - 4x^3 + 3x)}{\sin x - x^{\frac{3}{4}}}$

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Lesson: The Derivative as a Rate of Change

We've talked about that the derivative at a point x_0 represents:

- the slope of the curve at x_0
- the derivative at x_0
- the slope of the tangent line at x_0
- the instantaneous rate of change at x_0

We will focus on the derivative as a rate of change in this lesson.

Often times, we omit the word "instantaneous" when we talk about this.

What represents position?

What is displacement?

What is average velocity?

How do we find velocity (instantaneous velocity)?

How is velocity different from speed?

How do you find speed?

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How do you find acceleration?

What comes after acceleration?

If I move at a constant speed, what would the function and graphs that model this situation from a position, velocity, and acceleration standpoint?

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If I ride down a hill, what would the function and graphs that model this situation from a position, velocity, and acceleration standpoint?

Now let's model the situation where I drop a tennis ball from a height of 6ft (1.8m). The equation that models this situation is $s(t) = -4.9t^2 + 1.8$

a. When do the ball hit the ground?

b. When the ball hits the ground, what was its velocity? What about its speed?

c. What is the acceleration of the ball at any time?

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Examples: Position, Velocity, and Acceleration

Let $s = -t^3 + 9t^2 - 24t$ represent the position of a moving object, in feet, at time t, in seconds.

a. Find the acceleration of the object when the velocity is 0.

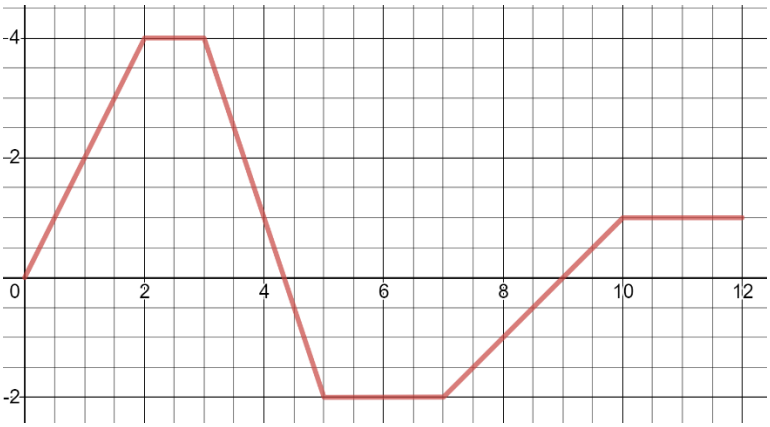
b. What is the speed of the object when the acceleration is 0?

c. What is the total distance traveled from t=0 to t=7?

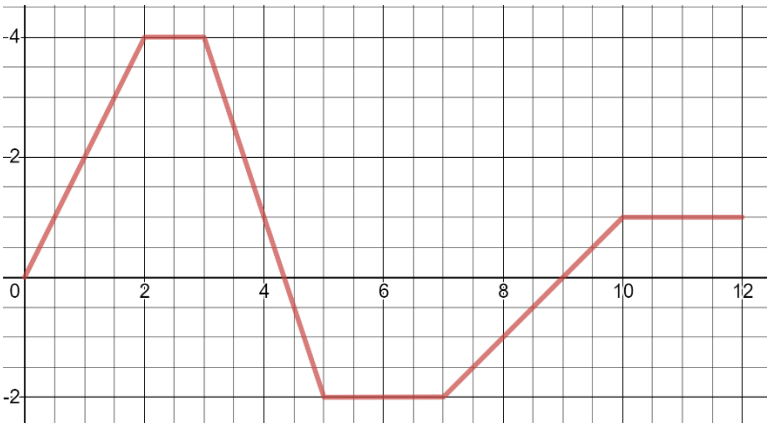
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Examples: Position, Velocity, and Acceleration from a Graph

This is the graph of velocity for an object. The y-axis is m/s, the x-axis is seconds; the graph is from desmos.com. I have reprinted the graph for each question is case you want to mark it up.

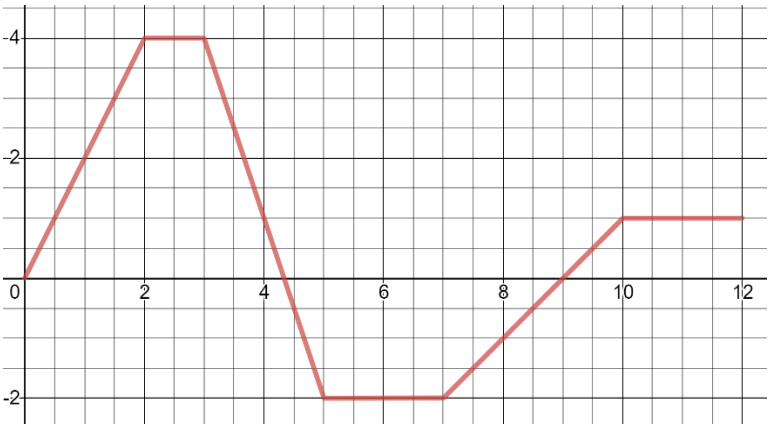


When does the object reverse its direction?

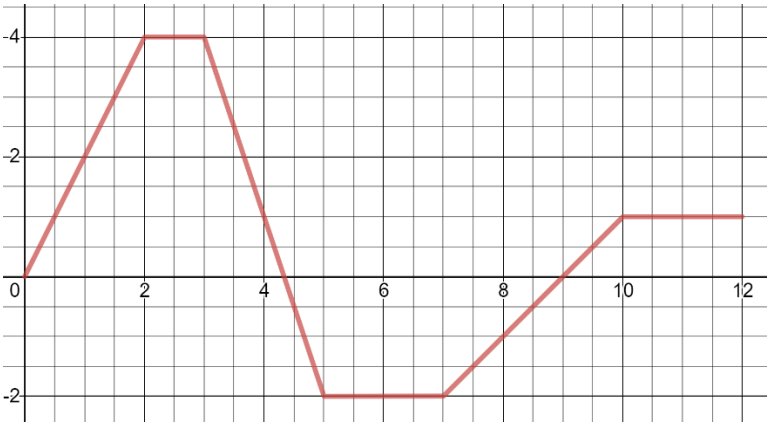


When is the object going a constant speed?

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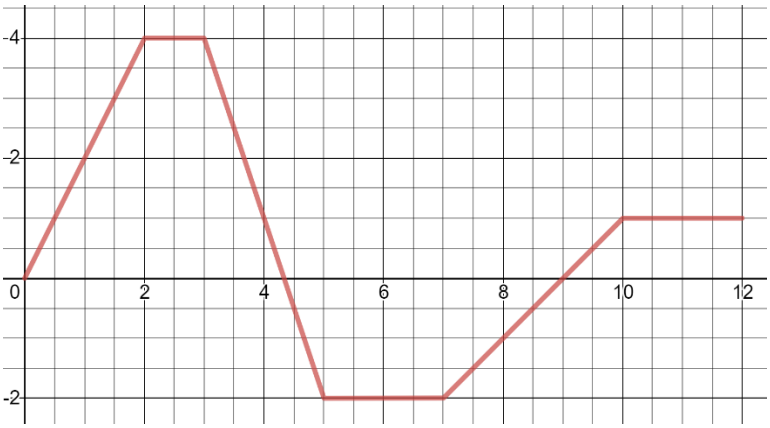


What would the graph of the object's speed look like?



What would the graph of the object's acceleration look like?

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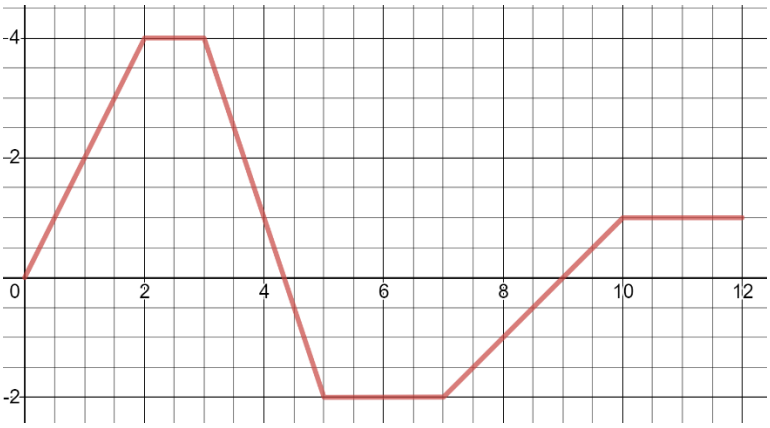


When is the object's acceleration …

Positive?

Negative?

Constant?



When does the object attain its greatest speed?

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Lesson: Derivatives of Trigonometric Functions

Name of Function	Function Form	Derivative
Sine	$f(x) = \sin x$	
Cosine	$f(x) = \cos x$	
Tangent	$f(x) = \tan x$	
Cotangent	$f(x) = \cot x$	
Secant	$f(x) = \sec x$	
Cosecant	$f(x) = \csc x$	

Examples Find the following derivatives:

a. $f(x) = 3x^7 - \cos x$

b. $g(x) = e^x \sin x$

c. $y = \frac{5}{\cos x} - \frac{3}{\tan x}$

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Examples: Derivatives of Trig Functions

Note: these examples assume you do not know the chain rule, you will strictly use the rules for trig derivatives here.

Find the derivatives of the following:

a. $f(x) = \sqrt{x}\csc x - \frac{1}{x^2}$

b. $g(x) = \sin^2 x$

c. $y = \frac{\sin x}{\cos^2 x}$

d. $y = x^2 \sin x$

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Lesson: The Chain Rule

The Chain Rule focuses on what types of functions?

State the Chain Rule:

Examples Take the derivatives of the following functions:

a. $y = (x^2 - 8)^{12}$

b. $f(x) = e^{\sin(5x)}$

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The key to the chain rule is understanding how functions are being composed with one another. So before we start taking any derivatives, we will first focus on understanding compositions of functions.

The basic rules that we have for derivatives are (you should have this memorized, quiz yourself!):

Constant Rule $(k)' =$	Sine Derivative $(\sin x)' =$
Power Rule $(x^n)' =$	Cosine Derivative $(\cos x)' =$
Constant Multiple Rule $\big(kf(x)\big)' =$	Tangent $(\tan x)' =$
Sum Rule $\big(f(x) + g(x)\big)' =$	Cotangent $(\cot x)' =$
Natural Exponential Rule $(e^x)' =$	Secant $(\sec x)' =$
Product Rule $\big(f(x) * g(x)\big)' =$	Cosecant $(\csc x)' =$
Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' =$	

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The first thing we must do is identify when we actually need the chain rule.

Example Determine if the chain rule is needed for each function (is it a composition of the derivatives in the previous list?), and if so, then identify how many compositions of functions there are.

Function	Chain Rule Needed?	How Many/Which Functions are Being Composed?
a. $f(x) = \sin(3x)$	Yes No	
b. $y = e^{\sin 4x}$	Yes No	
c. $g(x) = (x^4 - 5)^7$	Yes No	
d. $y = \sin x \tan x$	Yes No	
e. $f(x) = \sin^2(\cos(5x))$	Yes No	
f. $g(x) = (x^2 + 7)^5(3x + 1)^{-2}$	Yes No	
g. $y = \frac{1}{3}\cos x\, e^x$	Yes No	

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Examples Now take the derivative of each function:

a. $f(x) = \sin(3x)$

b. $y = e^{\sin 4x}$

c. $g(x) = (x^4 - 5)^7$

e. $f(x) = \sin^2(\cos(5x))$

f. $g(x) = (x^2 + 7)^5(3x + 1)^{-2}$

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Examples: The Chain Rule

Find the derivatives of the following functions:

a. $y = \left(\frac{1}{3}x^3 - 7x^2 + 1\right)^8$

b. $f(x) = (4x + 1)e^{3x+1}$

c. $y = \cos\left(\tan\left(e^{\sin 4x}\right)\right)$

d. $y = \sin\left(\frac{x}{\sqrt{x+2}}\right)$

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e. $f(x) = x^3 e^{-2x} \sin 5x$

f. $y = 5x(e^{-x} - 4x + 2)^5$

g. $y = \left(\frac{\sqrt{x}-7}{e^{\sin x}}\right)^{-3}$

h. $f(x) = \frac{e^{\sin(4x)}}{x^2 \cos(3x-1)}$

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Lesson: Help! The Chain Rule is messing with my head!

The key to the chain rule is understanding how functions are being composed with one another. So before we start taking any derivatives, we will first focus on understanding compositions of functions.

The basic rules that we have for derivatives are (you should have this memorized, quiz yourself!):

Chain Rule $(f(g(x))' =$	Quotient Rule $\left(\frac{f(x)}{g(x)}\right)' =$
Constant Rule $(k)'$ =	Sine Derivative $(\sin x)' =$
Power Rule $(x^n)' =$	Cosine Derivative $(\cos x)' =$
Constant Multiple Rule $(kf(x))' =$	Tangent $(\tan x)' =$
Sum Rule $(f(x) + g(x))' =$	Cotangent $(\cot x)' =$
Natural Exponential Rule $(e^x)' =$	Secant $(\sec x)' =$
Product Rule $(f(x) * g(x))' =$	Cosecant $(\csc x)' =$

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Example Take the derivatives of the following:

a. $f(x) = \sin(7x)$

b. $g(x) = e^{x^2}$

c. $h(x) = (3x + 7)^{21}$

Example Take the derivatives of the following:

a. $f(x) = x\sin(7x)$

b. $g(x) = e^{x^2}\tan x + 77$

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Example Take the derivatives of the following:

c. $h(x) = \frac{(3x+7)^{21}}{9x}$

d. $y = e^{x^2}\sin(7x)$

e. $f(x) = \frac{e^{x^2}}{(3x+7)^{21}}$

f. $f(x) = \sin\left(7x + e^{x^2}\right)$

g. $f(x) = \tan(3x)\, e^{4x-\cos(3x)}$

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Examples: The Chain Rule Mixed with Quotient Rule or Product Rule (messy functions)

Find the following derivatives.

a. $y = \left(\frac{\sin 4x - e^{\tan x}}{\sec(3x-1) - x^7 + 9}\right)^8$

b. $f(x) = \sin(5x+1)\, e^{4x - \csc\left(x^3 - 2x\right)}$

c. $y = \left(\frac{\sin(4x)e^{5x} + x^3}{(x-7)^4 \tan^2 x}\right)^9$

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Step-by-Step Example: The Chain Rule and the Quotient Rule

Take the derivative: $y = \frac{3x}{(x^3+5)^6}$

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Examples: Trig Functions in the Chain Rule

a. $y = \sin\left(\frac{1}{\sqrt{x+7}}\right)$

b. $y = e^{\sin^2(7x+2)}$

c. $y = \frac{(x+3)^7}{\ln(\sin x)}$

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Examples: The Chain Rule and Second Derivatives

Find the first and second derivatives of the following functions:

a. $y = \sin 6x^2$

b. $f(x) = xe^{-x}$

c. $y = x(3x + 5)^3$

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Lesson: Understanding the Derivatives of Inverse Trig Functions

Show how to find the derivative of $y = \sin^{-1} x$

Show how to find the derivative of $y = \tan^{-1} x$

Show how to find the derivative of $y = \sec^{-1} x$

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State the Inverse Function-Inverse Cofunction Identities

Explain how these relate to finding the derivatives of $y = \cos^{-1} x,\ y = \cot^{-1} x,\ y = \csc^{-1} x$

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Examples: The Derivatives of Inverse Trig Functions

Find the derivatives of the following:

a. $y = \cos^{-1}(x^5)$

b. $f(x) = \cot^{-1}\sqrt{3x+1}$

c. $y = \ln\left(\sec^{-1}(2x-1)\right)$

d. $f(x) = \sin^{-1}(e^{-x})$

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Lesson: Implicit Differentiation

Not all equations are modeled with y on one side and x's on the other. When you have an equation like:

$$x = y^2,$$

How do you take the derivative?

But there's another way we can take the derivative: implicit differentiation. Show how this works below.

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Let's look at a few other examples of how to do this:

a. $2x + 3y - y^2 = 5$

b. $x^3y^5 = 5$

c. $\sqrt{x + 3y} = 8$

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Examples: Implicit Differentiation

Use implicit differentiation to find $\frac{dy}{dx}$:

a. $x^2 = \frac{3-y}{4x+5y}$

b. $x^6 - \cos(xy) = x^2y^5$

c. $y\cos(3y) = 1 - 4xy$

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Step-by-Step Examples: Implicit Differentiation

Find $\frac{dy}{dx}$: $5x - 7y = 3x^2y^5$

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Lesson: Implicit Differentiation, Tangent Lines, and Normal Lines

Explain where we might encounter a normal line:

Show how to find the tangent and normal line to the curve at the following point:

$x^2 + y^2 = 100$ at $(8, -6)$

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Examples: Implicit Differentiation, Tangent Lines, and Normal Lines

Find the tangent and normal line to the curve at the following point:

a. $x^2y^2 = 25, (25, -1)$

b. $x^2 - x + y - y^2 = 4, (-2, 2)$

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Examples: Implicit Differentiation and Finding Tangent/Normal Lines With Different Characteristics

a. Find 2 points where $x^2 - xy + y^2 = 5$ crosses the x-axis. Are the tangent lines at these points parallel?

b. Find any normals to the curve $3xy + 4x - y = 0$ that are parallel to $y = -4x + 2$.

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Lesson: Second Derivatives with Implicit Differentiation

Find the first and second derivatives of the following:

a. $x^2 + y^2 = 9$

b. $y^2 = x + 3$

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Examples: Second Derivatives with Implicit Differentiation

Find the first and second derivatives of the following:

a. $y^2 = e^{5x} - 5x$

b. $3 - \cos y = 4x + x^2$

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Examples: Implicit Differentiation and Simplifying with a Fraction within a Fraction

Find y'': $3x^2 - 2y^2 = 5$

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Lesson: Derivatives of Inverse Functions and Logarithms

Find the inverse of each function:

a. $y = x^2, x > 0$

b. $y = x^3 + 2$

Select a random point for each function. How does that point relate to each function's inverse? Write down your choice of point here and how it would appear in the inverse below:

a. Choice:

b. Choice:

Now take the derivative of each function and the derivative of its inverse below:

a. Derivatives:

b. Derivatives:

What is the relationship between the points you found when you plug them into the derivatives?

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Write down the derivative rule for inverses below:

Explain how to leverage this rule to find the derivative of $f(x) = \ln x$

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Lesson: How to Find the Derivatives of a^x and $log_a x$

Explain how to find the derivative of a^x:

Explain how to find the derivative of $\log_a x$:

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Example Find the derivatives of the following:

a. $f(x) = 3^{x^2}$

b. $y = \log_5(3x + 2)$

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Examples: Derivatives with $e^x, ln\ x, log_a x, a^x$

Find the derivatives of the following functions:

a. $f(x) = \ln(x^2)$

b. $g(x) = e^{\sin x + \ln x}$

c. $y = 5^{\ln x}$

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d. $f(x) = \log_3(x) + \log_3(x^4)$

e. $g(x) = \ln(\ln(\ln(\ln(4x))))$

f. $y = 7^{\log_2 x}$

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Examples: Using Properties of Logarithms to Simplify Derivatives

Take the derivatives of the following:

a. $y = \log_2 \left(\frac{3x+1}{x-1}\right)^{\ln 2}$

b. $y = \log_7 \sqrt[3]{\left(\frac{x+1}{4x-6}\right)^{\ln 7}}$

c. $y = x \log_4 e^{(\tan 2x)(\ln 4)}$

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Lesson: Logarithmic Differentiation

Logarithmic differentiation is another way to take derivatives.

Show how to use logarithmic differentiation to take the following derivatives:

a. $y = \frac{x\sqrt{5x+1}}{(x+9)^7}$

b. $f(x) = (x+1)^x$

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Examples: Logarithmic Differentiation

Use logarithmic differentiation to find the following derivatives:

a. $y = x(x-4)(x+5)(x+2)^7$

b. $f(x) = \sqrt[3]{\frac{\sin x}{(x^2+5)^3}}$

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c. $x^y = y^x$

d. $y = x^{\sin x}$

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Lesson: Understanding Related Rates with an Example

A 12-ft ladder is leaning against a wall when the base starts to slide. By the time the base is 10 feet from the house, the base is moving at the rate of 4 ft/sec.

a. How fast is the top of the ladder sliding down the wall?

b. What is the rate that the area is changing for the triangle formed between the wall, ladder, and floor?

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Example: Related Rates – Volume

A conical tank is being filled with water at a rate of $45 m^3$/min. The tank is 40 m tall and 10 m wide. How fast is the water level rising when the water is 15 ft deep?

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Example: Related Rates – Two Moving Entities

A hot air balloon is riding straight up from a point at a constant rate of 2 ft / sec. When the balloon is 50 ft about the ground, a motorcycle passes underneath is traveling at a constant speed of 30 ft / sec. How fast is the distance between the balloon and the motorcycle increasing 3 seconds later?

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Example: Related Rates – A Cone and a Cylinder

Este is making tea by pouring hot water into a conical filter with tea leaves and allowing them to drain into cylindrical pot. The water is draining into the pot at a rate of $11\frac{in^3}{min}$.

a. How fast is the level in the pot rising when the water in the cone is 5 inches deep?

b. How fast is the level in the cone falling?

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Example: Related Rates – Problems with Angles

a. A particle is moving along the curve $y = 3x^2$ in the first quadrant of a graph. The x-coordinate is increasing by a rate of 15 m/sec. How fast is the angle of inclination θ of the line joining the origin to the particle changing when x=1?

b. A sun is shining over a building creating a shadow. The building is 88 feet tall and the shadow behind the building is 66 feet long. The angle θ that the sun makes with the ground is changing at a rate of 0.00436 radians per min. What is the rate that the shadow is decreasing?

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Lesson: Linearization

Linearization is a way to estimate certain difficult calculations without a calculator. Explain the basic idea behind this below:

How do we define the linearization of a function f at a point a?

Example Find the linearization of an integer near the point a at which the given function and its derivative are easy to calculate. Then compare the actual value of the function at a to the approximated value.

$$f(x) = x^{-1}, a = -0.9$$

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Example Find the linearization of an integer near the point a at which the given function and its derivative are easy to calculate. Then compare the actual value of the function at a to the approximated value.

$$f(x) = \sqrt[3]{x}, a = 28.5$$

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Lesson: Differentials

Leibniz notation $\frac{dy}{dx}$ represents the derivative and does not represent a ratio, however we will now introduce some very similar notation dx and dy, and when their ratio exists then it is equal to the derivative.

Define the differential below:

Example Find

a. dy if $y = x^3 - 5x^2$

b. the value of dy if $x = 2$ and $dx = 0.1$

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Differentials are related to linearization and can also be used to do estimation.

Example Use differentials to estimate

a. $8.1^{\frac{1}{3}}$

b. $\sqrt{6}$

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Extreme Values of Functions on Closed Intervals

The general idea for closed intervals:

To begin, we need to understand the following definitions

Let $f(x)$ be a function. We say that:

- $f(x)$ has an absolute **maximum value** at a point c if $f(x) \leq f(c)$ for all x in the domain
- $f(x)$ has an absolute **minimum value** at a point c if $f(x) \geq f(c)$ for all x in the domain

What are some other words for maximum and minimum values?

The Extreme Value Theorem

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Here are some ways that extreme values can occur on closed intervals:

Example Sketch each graph and determine whether the function has any absolute extrema. Does this contradict the extreme value theorem?

$$f(x) = |x + 1|, 0 < x < 2$$

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Example Sketch each graph and determine whether the function has any absolute extrema. Does this contradict the extreme value theorem?

a. $f(x) = \begin{cases} \frac{1}{x^2}, & -1 \le x < 0 \\ x + 3, & 0 \le x \le 2 \end{cases}$

b. $y = 2\cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

If we focus on maximum or minimum values that occur within the interior of the interval, what do we notice about the tangent lines at these points? How can we think of these points?

A function $f(x)$ has a **local maximum value** at a point c with its domain if:

A function $f(x)$ has a **local minimum value** at a point c with its domain if:

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What is another word for local maxima/minima?

Critical Points

How to Find Absolute Extrema of a Continuous Function $f(x)$ on a Finite Closed Interval

1.

2.

3.

Example Find the critical points for the function, and then determine the absolute maximum and minimum on the given interval.

$$f(x) = \frac{2}{5}x - 2, \qquad -5 \leq x < 0$$

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Example Find the critical points for the function, and then determine the absolute maximum and minimum on the given interval.

a. $f(x) = -\frac{1}{x}, 1 \leq x \leq 3$

b. $f(x) = \sec x\,, -\frac{\pi}{3} \leq x \leq \frac{\pi}{6}$

c. $f(x) = x^{\frac{4}{3}}, -1 \leq x \leq 27$

d. $f(x) = 3x^{\frac{2}{3}},\ 1 \leq x \leq 8$

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Examples: Absolute Extrema on Closed Intervals

Find the absolute extrema for the following functions on the given closed intervals.

a. $f(x) = 4 - x^2, -1 \leq x \leq 2$

b. $f(x) = \sin x\,, 0 \leq x \leq \frac{\pi}{2}$

c. $f(x) = \frac{1}{x} + \ln x\,, 2 \leq x \leq 4$

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Review: Factoring with Rational Exponents (Trick for Critical Points)

Find the critical points and domain:

a. $y = x^{\frac{4}{5}}(x + 3)$

b. $y = x^{\frac{3}{4}}(x - 1)$

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Review: Factoring Out Fractions and Rational Exponents from Expressions

Factor the following out of each expression.

a. $\frac{1}{4}x^{-\frac{1}{3}}$ from $\frac{7}{4}x^{\frac{5}{3}}+\frac{3}{4}x^{-\frac{1}{3}}$

b. $\frac{2}{3}x^{-\frac{2}{5}}$ from $\frac{4}{3}x^{\frac{3}{5}}-\frac{2}{3}x^{-\frac{2}{5}}$

c. $\frac{1}{2}x^{\frac{4}{7}}$ from $6x^{\frac{11}{7}}+5x^{\frac{4}{7}}$

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Rolle's Theorem

State Rolle's Theorem

Justify why this makes sense:

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Example Show that $x^3 + 4x + 2 = 0$ has exactly 1 real solution.

Example Show $f(x) = x^4 + 4x + 1$ has exactly one zero in the interval $[-2, -1]$.

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The Mean Value Theorem

State the Mean Value Theorem

Explain what this means:

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Examples Find the value or values of c that satisfy the equation:

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

a. $f(x)=\sqrt{x+1}, [0,3]$

b. $f(x)=3x^2+6x-5, [-2,1]$

Examples Determine whether the functions satisfy the hypotheses of the mean value theorem on the given interval, with reasoning.

a. $f(x)=x^{\frac{4}{5}}, [0,1]$

b. $f(x)=3x^{\frac{2}{3}}, [-1,1]$

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Consequences of the Mean Value Theorem

Write the 2 corollaries associated with the mean value theorem

Examples Find a function with the following derivative that goes through the given point.

a. $f'(x) = 4x; f(0) = 5$

b. $f(x) = \frac{1}{x^2} + 2x; (-1, 1)$

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Examples Given the velocity function and initial position of an object, determine the object's position at time t.
a. $v = 9.8t + 5; s(0) = 10$

b. $v = \sin t,\ s(0) = 1$

Examples Find all possible functions with the given derivatives:
a. $y' = \cos x$

b. $y' = \frac{1}{\sqrt{x}}$

c. $y' = \sin\frac{x}{2}$

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Increasing and Decreasing Functions and the First Derivative Test
Part 1: Theory and An Example

Before we begin, let's first make sure we have an intuitive understanding of what increasing and decreasing functions look like:

There is a third corollary to the Mean Value Theorem regarding whether functions are increasing or decreasing:

What does monotonic mean?

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Review: What is a critical point? How do you find critical points?

Now we can state the **First Derivative Test for Local Extrema:**
Let $f(x)$ be a continuous differentiable function at every point in some interval, except possibly at a critical point c itself. If we examine this interval from left to right:

1.

2.

3.

We can visualize this:

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Examples Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this first set of examples, you will be given the derivative:

$$f'(x) = (x-1)(x+2)(x-3)$$

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Increasing and Decreasing Functions and the First Derivative Test
Examples 1

Examples Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this first set of examples, you will be given the derivative:
a. $f'(x) = \frac{(x+1)(x-4)}{(x+3)(x+6)}$

b. $f'(x) = x^{-\frac{1}{3}}(x+3)$

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Increasing and Decreasing Functions and the First Derivative Test
Examples 2

Examples Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this set of examples, you will have to find the derivative:
a. $f(x) = 4x^2 + 8x + 13$

b. $f(x) = x^2 e^{-\frac{x}{3}}$

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Increasing and Decreasing Functions and the First Derivative Test
Examples 3

Examples Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

In this set of examples, you will have to find the derivative:
a. $f(x) = x^2 - 4x - 8\ln x$

b. $f(x) = \sin x - \cos x, \quad 0 \le x \le 2\pi$

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Increasing and Decreasing Functions and the First Derivative Test
Examples 2

Examples Identify: i) the critical points of $f(x)$ ii) what intervals the function is increasing/decreasing iii) at what points there are local/global maximum or minimum values

$$f(x) = -6x^{\frac{6}{5}} + 3x^{\frac{2}{5}}$$

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Lesson: Concavity and the Second Derivative Test for Concavity

Explain concavity in non-technical terms:

For a differentiable function $f(x)$, the function is:
1. **concave up** on an open interval if I if:

Another word for concave up is: _____

2. **concave down** on an open interval I if:

The Second Derivative Test for Concavity:
Let $f(x)$ be a twice-differentiable function on some interval. Then:

1. If $f''(x) > 0$ on that interval, the graph is concave up on that interval
2. If $f''(x) < 0$ on that interval, the graph is concave down on that interval

When concavity changes from up to down (or vice versa), what do we have? Draw a picture of this idea:

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How do you find points of inflection?

Examples Estimate the points of inflection from the graphs (draw your own graphs, sorry!)

a.

b.

Examples Determine i) where the graph is concave up or concave down and ii) any points of inflection

$$f(x) = 2x^3 - 3x^2 - 12x + 3$$

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Examples: Concavity and the Second Derivative Test for Concavity

Examples Determine i) where the graph is concave up or concave down and ii) any potential points of inflection

a. $f(x) = 1 + x^{\frac{1}{3}}$

b. $f(x) = 5 + \cos 2x,\ [0, \pi]$

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Second Derivative Test for Local Extrema

How can concavity indicate maximums or minimums? Draw some graphs that illustrate this idea.

The Second Derivative Test for Local Extrema

Suppose $f''(x)$ is continuous on an open interval that contains $x = c$.

1.

2.

3.

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Examples Use the second derivative test for local extrema to find local maxima and minima. Then compare this to the first derivative test.

a. $f(x) = x^2 - 4x + 3$

b. $f(x) = x^4 - 8x^2$

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Examples: Second Derivative Test for Local Extrema

Examples Use the second derivative test for local extrema to find local maxima and minima.

a. $f(x) = x^2(x-1)^2$

b. $f(x) = \cos^2 x, -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

c. $f(x) = (x^2 - x)e^{-x}$

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Curve Sketching

Demonstrate all the steps for sketching a curve while sketching $f(x) = \frac{x^2+2}{2x}$

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Curve Sketching Examples 1

Example Sketch $f(x) = 6 + 8x^2 - x^4$, showing all the relevant steps

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Curve Sketching Examples 2

Example Sketch $f(x) = \cos x + \sqrt{2}\sin x,\ 0 \leq x \leq 2\pi$, showing all the relevant steps

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Curve Sketching Examples 3

Example Sketch $x^{\frac{2}{3}}(5-2x)$, showing all the relevant steps

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Curve Sketching Examples 4

Example Sketch $x^{\frac{2}{3}}(5-2x)$, showing all the relevant steps

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Curve Sketching when you only know y'

In another video, we discussed how to sketch curves. Given the following derivatives, what can you confidently sketch about the curves?

a. $f'(x) = x^2 + 5x + 6$

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b. $f'(x) = (x-2)^2(2x+1)$

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Lesson: Finite Limits as $x \to \infty$ and Limits That Go To Infinity, the Conceptual Idea

Recreate the graph below, then find the following limits:

a. $\lim_{x\to\infty} f(x)$

b. $\lim_{x\to-\infty} f(x)$

c. $\lim_{x\to3^+} f(x)$

d. $\lim_{x\to2^-} f(x)$

e. $f(2)$

f. $\lim_{x\to2^+} f(x)$

g. $\lim_{x\to3} f(x)$

h. $\lim_{x\to-4} f(x)$

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Lesson: Finite Limits as $x \to \infty$ and Rational Functions

Do the limit laws apply to limits as $x \to \infty$? Yes No

How do you evaluate limits at infinity for rational functions?

In the following examples, we will see all the different ways this trick might be carried out:

Examples Evaluate the following:

a. $\lim_{x\to\infty} \frac{4x+3}{x^2+4x+3}$

b. $\lim_{x\to-\infty} \frac{5x^2+3x+1}{7x^2+5x-4}$

c. $\lim_{x\to\infty} \frac{6x^2+3x+1}{x+2}$

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This trick can be extended.

Examples

a. $\lim_{x\to\infty} \sqrt{\frac{x^2+5}{3x^2-6}}$

b. $\lim_{x\to-\infty} \frac{x^{\frac{2}{3}}-x^{\frac{1}{3}}+2}{x^2+3}$

c. $\lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+3}}$

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Examples: Limits as $x \to \infty$

Find the horizontal asymptotes / evaluate the following:

a. $\lim_{x\to\infty} \frac{\sin x}{x^2}$

b. $\lim_{x\to\infty} \frac{7x^4-2x^3+8}{9x^4+2x^2+x-1}$

c. $\lim_{x\to-\infty} \frac{\sqrt{25x^2+x}}{x}$

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d. $\lim_{x\to\infty}\left(x-\sqrt{x^4+2x^2}\right)$

e. $\lim_{x\to-\infty}(5x^7+2x^3+8)$

f. $\lim_{x\to-\infty}\frac{x^3+2x+5}{x^2+3x+1}$

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Example: Limits as x approaches infinity

a. $\lim_{x\to-\infty} \frac{2x^6+3x^2-1}{6x^3-2x+1}$

b. $\lim_{x\to\infty} \frac{5x^{\frac{8}{3}}-3x^{\frac{5}{3}}+2}{x^{\frac{13}{5}}+2x-x^{\frac{1}{2}}}$

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Example: Comparing the Conjugate Trick as $x \to \infty$

a. $\lim_{x\to\infty}\left(x^2 - \sqrt{x^4 + 3}\right)$

b. $\lim_{x\to\infty}(x^2 - \sqrt{x^6 + 3x^4})$

c. $\lim_{x\to-\infty}(5x + \sqrt{25x^2 + 5x - 9})$

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Lesson: Horizontal Asymptotes and Limits as $x \to \infty$

Define a Horizontal Asymptote in terms of limits:

Examples Find the horizontal asymptotes of the following:

a. $f(x) = \frac{|x|^3+2}{x^3-1}$

b. $f(x) = 3^x$

c. $f(x) = \frac{3x+1}{5x+2}$

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Sometimes we can have horizontal asymptotes in surprising places.

Example Find the horizontal asymptotes of $x - \sqrt{x^2 + 4}$

Example Use the Sandwich Theorem to find any horizontal asymptotes of $y = 3 + \frac{\sin x}{x}$

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Lesson: Infinite Limits

To begin this conversation, we investigate the graph of $f(x) = \frac{1}{x}$. Graph this below then evaluate the following:

a. $\lim_{x \to 0^+} f(x) =$

b. $\lim_{x \to 0^-} f(x) =$

c. $\lim_{x \to 0} f(x) =$

Show how to use a table to prove this result:

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Now compare this to $g(x) = \frac{1}{x^2}$. Graph this below:

a. $\lim_{x \to 0^+} g(x) =$ b. $\lim_{x \to 0^-} g(x) =$ c. $\lim_{x \to 0} g(x) =$

Show how to use a table to prove this result:

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Define vertical asymptotes using limits:

Example Evaluate the following and state any vertical asymptotes.

a. $\lim_{x\to 2^+} \frac{1}{x-2}$

b. $\lim_{x\to 4} \frac{3}{(x-4)^2}$

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Examples: Infinite Limits

Evaluate the following:

a. $\lim_{x\to 0^+} \frac{1}{x^{\frac{1}{5}}}$

b. $\lim_{x\to 0^-} \csc x$

c. $\lim_{x\to 5} \frac{1}{x^2(x-5)}$

d. $\lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}}$

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Lesson: Using Substitution with Limits as $x \to \infty$

In this lesson, we are going to look at a few limits that end up using clever substitutions.

Examples Evaluate the following:

a. $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$

b. $\lim_{x\to 0^-} e^{\frac{1}{x}}$

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Examples: Horizontal and Vertical Asymptotes

Find any horizontal, vertical, or oblique asymptotes

a. $f(x) = \frac{x^2-5x+6}{x-2}$

b. $f(x) = \frac{\sqrt{25x^2+100}+x^2}{5x^2-25}$

c. $f(x) = \frac{x^2}{x-1}$

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Example: Horizontal, Oblique, and Vertical Asymptotes

Find any horizontal, oblique, or vertical asymptotes.

a. $f(x) = \frac{\sqrt{x^2+1}}{x}$

b. $g(x) = \frac{x^2-4}{x-3}$

c. $y = \frac{3x}{x^2-1}$

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Optimization Problems

Example 1: A piece of sheet metal is rectangular, 3 ft wide and 6 ft long. Congruent squares are to be cut from its 4 corners. The resulting piece of metal is to be folded and welded to form an open-top box. How should this be done to get the maximum volume?

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Optimization Problems

Example 2: What is the maximum possible area of a rectangle with a base that lies on the x-axis with two upper vertices that lie on the graph of the equation $y = 4 - x^2$?

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Optimization Problems

Example 3: Find the dimensions of the largest right circular cylinder of greatest volume that can be inscribed in a right circular cone of radius R and height H.

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Summary: Derivative Theorems About Min/Max/Inflection Points

In this summary, we will go through summarizing all of the theory and theorems about finding maximums and minimums with functions. We will then relate this to graphing and optimization problems.

First, we will review all of the vocabulary and theorems that usually come up in these concepts.

Absolute minimums/maximums versus local minimum/maximums

Critical point

Monotonic functions

Definition of an increasing function

Definition of a decreasing function

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Definition of concave up

Definition of concave down

Inflection point

Extreme Value Theorem

Example Does the extreme value theorem guarantee that $f(x) = \frac{1}{x}$ will have an absolute minimum or maximum point on the interval $[-3, 3]$?

Example Does the extreme value theorem guarantee that $f(x) = \frac{1}{x}$ will have an absolute minimum or maximum point on the interval $[1, 5]$?

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The First Derivative Test for Local Extrema

The Second Derivative Test for Local Extrema

Examples For each of the following functions, use a) the first derivative test for extrema to find all local/absolute extrema, and then compare that result with b) the second derivative test for extrema. Then c) find the intervals of increasing/decreasing.

a. $f(x) = x^3 - 3x + 3$

b. $f(x) = x^{\frac{2}{3}}(x - 2)$

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What are the steps for graphing a function?

Examples For each of the following functions, use a) the second derivative test for concavity to determine the intervals of concavity. Then determine b) if there are any inflection points. Then c) complete the graph of the function.

$$f(x) = x^3 - 3x + 3$$

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Examples For each of the following functions, use a) the second derivative test for concavity to determine the intervals of concavity. Then determine b) if there are any inflection points. Then c) complete the graph of the function using the steps above.

a. $f(x) = x^{\frac{2}{3}}(x-2)$

b. $f(x) = x + \cos x, 0 \leq x \leq 2\pi$

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One other application with min/max theorems are optimization problems.

Example A rectangular garden next to a barn needs to be fenced in. Since the garden is next to the barn, the side on the barn does not need fencing. If there is 600 feet of fencing, what is the largest area that can be enclosed?

Example Determine the rectangle of largest area that can be inscribed in a semicircle of radius 4.

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Lesson: L'Hopitals Rule

Does $\frac{0}{0} = 1$?

List all of the indeterminate forms:

Write down L'Hopital's Rule

Examples Determine the following limits.

a. $\lim_{x\to 0} \frac{5x+\sin x}{4x}$

b. $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

c. $\lim_{x\to 0} \frac{x-\sin x}{2x^3}$

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What must be true to use L'Hopitals Rule?

Example Evaluate

$\lim_{x\to 0} \frac{(x+\cos x)}{x+x^3}$

You can also combine L'Hopital's Rule with other limit knowledge

$\lim_{x\to 0} \frac{3\sin x}{x^2}$

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Lesson: L'Hopital's Rule for $\frac{\infty}{\infty}, \infty \times \mathbf{0}, \infty - \infty$ Forms

L'Hopitals Rule can also be applied when we find these types of indeterminate forms: $\frac{\infty}{\infty}, \infty \times 0, \infty - \infty$

a. $\lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1+\tan x}$

b. $\lim_{x \to \infty} \frac{e^x}{x^3}$

c. $\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$

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a. $\lim_{x\to 0^+} \sqrt{x}\ln x$

b. $\lim_{x\to\infty} (\ln 2x - \ln(x+2))$

c. $\lim_{x\to 0^+} \left(\frac{2x+1}{x} - \frac{1}{\sin x}\right)$

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Lesson: L'Hopital's Rule for Indeterminate Powers

L'Hopitals Rule can also be applied to indeterminate powers of these forms:

In this case, L'Hopital's Rule works a little differently. We use logarithms to help evaluate. Write down how this works below

Examples Evaluate.

a. $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}$

b. $\lim_{x\to\infty} x^{\frac{1}{x}}$

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Examples

a. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$

b. $\lim_{x\to\infty}\left(\frac{x^2+2}{x+1}\right)^{\frac{1}{x}}$

c. $\lim_{x\to e^+}(\ln x)^{\frac{1}{x-e}}$

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Lesson: What Happens when L'Hopitals Rule Doesn't Work?

In some cases, L'Hopitals Rule may not work and you will have to think of other ways to evaluate the limit. Let's look at a few cases.

a. $\lim_{x\to\infty} \frac{\sqrt{4x+1}}{\sqrt{x+3}}$

b. $\lim_{x\to\infty} \frac{2^x-5^x}{4^x+3^x}$

c. $\lim_{x\to 0^+} \frac{x}{e^{-\frac{1}{x}}}$

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Examples: L'Hopitals Rule

Evaluate the following limits using L'Hopitals Rule, if possible.

a. $\lim_{x\to 0} \frac{\sin x^2}{x}$

b. $\lim_{x\to \frac{\pi}{3}} \frac{3x+\pi}{\sin\left(x+\frac{\pi}{3}\right)}$

c. $\lim_{x\to 0} \frac{\left(\left(\frac{1}{3}\right)^x - 1\right)}{x}$

d. $\lim_{x\to \infty} x^2 e^{-x}$

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e. $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)$

f. $\lim_{x\to\infty}(\ln x)^{\frac{1}{x}}$

g. $\lim_{x\to 0^+} x^x$

h. $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{\sin x}}$

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Summary: L'Hopital's Rule

State L'Hopital's Rule

What are all the indeterminate forms that apply to L'Hopital's Rule?

For which type of indeterminate forms can you just take the derivative of the top and bottom?

Examples Find the following limits.

a. $\lim_{x\to 0} \frac{4x^2}{2\cos x - 2}$

b. $\lim_{x\to \infty} \frac{\ln x}{\sqrt[3]{x}}$

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For the forms $0 * \infty$ or $\infty - \infty$, what must you try?

Examples Find the following limits.

a. $\lim_{x \to \infty} 3x\, e^{-x}$

b. $\lim_{x \to 0^+} (\ln 5x - \ln(\sin x))$

What must you do with indeterminate powers?

Example Find the limit $\lim_{x \to 0^+} x^{\frac{-1}{\ln x}}$

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What happens if L'hopitals rule continues to cycle?

Example Find the limit $\lim_{x\to 0^+} \frac{\cot x}{\csc x}$

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Newton's Method

What is the idea and derivation behind Newton's Method?

Write the iterative formula for Newton's Method here:

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Example Use Newton's method to approximate a root for the following function. Stop when the first 5 decimal places do not change over 2 iterations.

$$f(x) = x^3 + x^2 + 1; x_0 = -1.5$$

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Example Use Newton's Method to find a solution to $4x - \sin x = 4$ on $[1, 2]$.

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Antiderivatives versus Indefinite Integrals versus Differential Equations

Define an antiderivative and any special notes about it:

Define an indefinite integral and any special notes about it:

Define a differential equation and any special notes about it:

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Example Find the antiderivative to the function: $2x$

Example Find the indefinite integral: $\int 2x\, dx$

Example Find the general solution to: $\frac{dy}{dx} = 2x$

Example Solve the initial value problem: $\frac{dy}{dx} = 2x; y(0) = 7$

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List all of the relevant antiderivatives you know here:

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Examples of Basic Antiderivatives / Indefinite Integrals

Find the antiderivatives of the following functions:

a. $\frac{1}{x^2}$

b. $3x^2$

c. $x^{-\frac{1}{2}}$

d. $\frac{2}{3}x^{-\frac{1}{3}}$

Find the indefinite integrals:

a. $\int 2\sin x \ dx$

b. $\int \sec^2 x \ dx$

c. $\int \frac{\pi}{2}\cos\frac{\pi}{2}x \ dx$

d. $\int -\csc 3x \cot 3x \ dx$

e. $\int e^{-2x}dx$

f. $\int \frac{2}{\sqrt{1-x^2}}dx$

g. $\int 3^x dx$

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Examples of Basic Antiderivatives / Indefinite Integrals

Integrate the following:

a. $\int \frac{5}{x} dx$

b. $\int 5x^{\sqrt{5}} dx$

c. $\int \frac{1+\cos 3x}{2} dx$

d. $\int \cot^2 x \, dx$

e. $\int \cos x \, (\tan x + \sec x) dx$

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Examples of Basic Differential Equations

Solve the initial value problems:

a. $\frac{dy}{dx} = 5 - 3x^2; y(0) = 4$

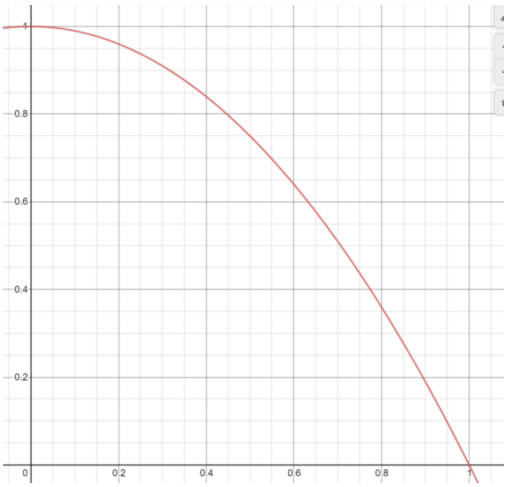
b. $\frac{dy}{dx} = 3x^{-\frac{2}{3}}; y(8) = 1$

c. $\frac{dy}{dx} = -\pi \sin \pi x \, ; y(0) = 1$

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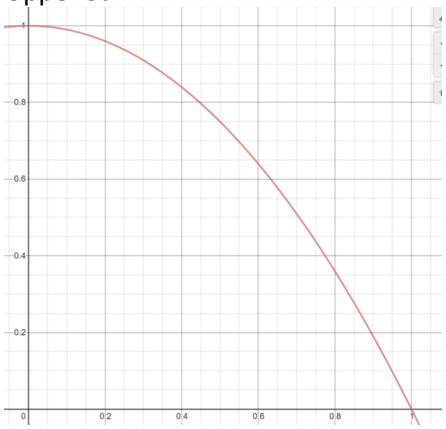
Area and Estimating with Finite Sums

First we will try to understand the idea behind this. Draw out one way to find the area of the curve $y = 1 - x^2$ below.

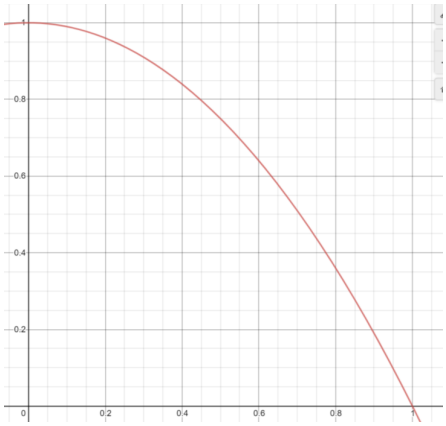


Now we need more formality around this idea. Show the idea of how to create sums using a more algorithmic approach. Detail what each sum means:

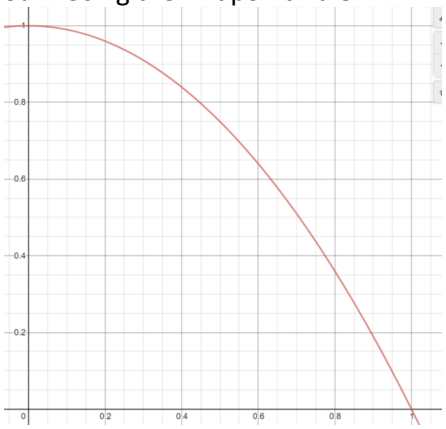
Upper Sum



Lower Sum

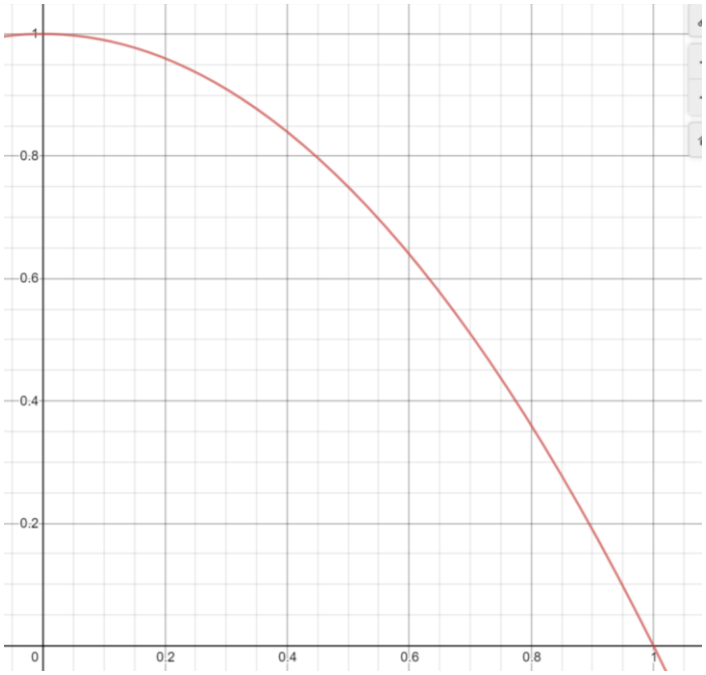


Sum Using the Midpoint Rule



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Now we need some vocabulary. Separate the graph in 4 parts and then label it with the correct terminology:



Define a Riemann Sum:

What is a left Riemann sum?

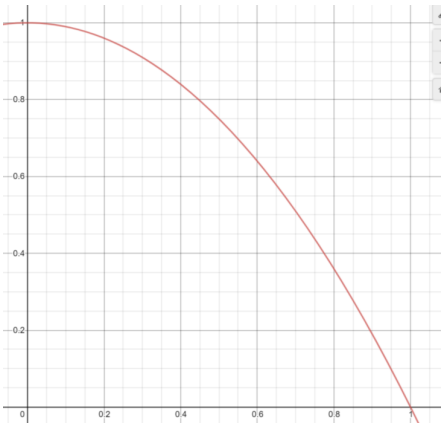
What is a right Riemann sum?

What is a midpoint Riemann sum?

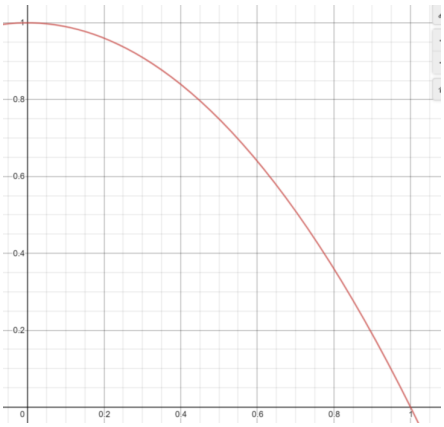
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Show how you can make a more precise sum:

8 Subintervals

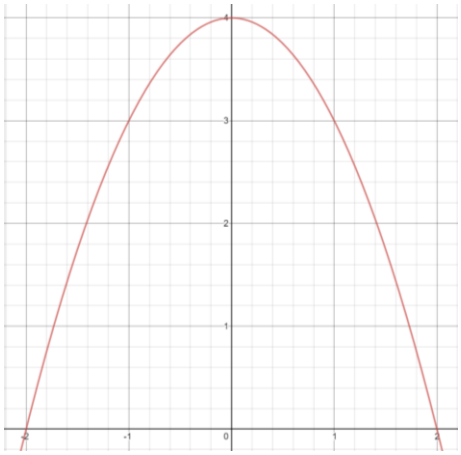


16 Subintervals

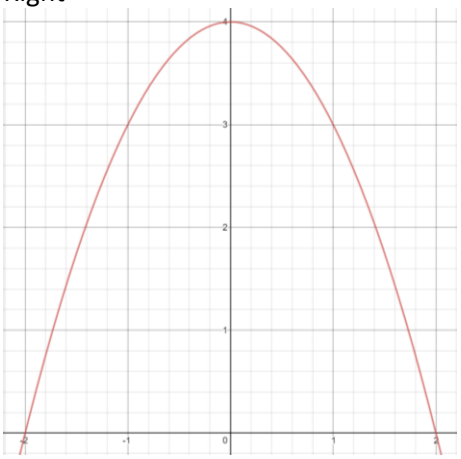


Example Compute the left, right, and midpoint Riemann sums for $f(x) = 4 - x^2$ between $[-2, 2]$ using 4 subintervals

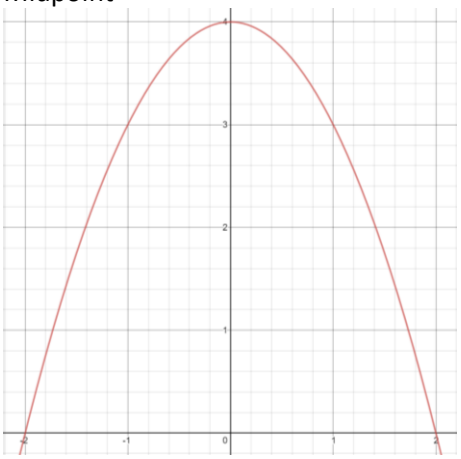
Left



Right



Midpoint



Graphs were created using Desmos.com. They have the greatest graphing software for free, check them out!

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Examples of Riemann Sums

Compute the left, right, and midpoint Riemann sum for $f(x) = 3 + x$ from $[1,3]$ using 4 subintervals.

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Examples of Riemann Sums – When It Goes Negative?

Compute the left, right, and midpoint Riemann sum for $f(x) = x^3$ from $[-2,2]$ using 4 subintervals.

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Examples: Different Ways to Compute Area Estimations

Using $f(x) = -x^2 + 2x + 3$, compute an area estimate using 4 subintervals from x=-1 to x=3.

Compute this estimate using an upper, lower, left-hand, and right-hand sum.

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Lesson: The Basics of Sigma Notation

Explain what sigma notation looks like and what it means:

Examples Compute the following sums:

a. $\Sigma_{k=1}^{4} k$

b. $\Sigma_{i=3}^{7} (2i - 1)$

c. $\Sigma_{k=1}^{4} (-1)^{k+1} k$

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We have a few handy rules for sigma notation:

1. Sum Rule:

2. Difference Rule:

3. Constant Multiple Rule:

4. Constant Value Rule:

Examples Use the rules above to help calculate the following sums:

a. $\Sigma_{k=1}^{3}(k+5)$

b. $\Sigma_{k=1}^{4}(-3k-2)$

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Lesson: Using Sigma Notation and Sum Formulas to Calculate Larger Sums

We've already discussed these basic algebraic rules for sums:

Sum Rule

$$\Sigma_{k=1}^{n}(a_k + b_k) = \Sigma_{k=1}^{n}a_k + \Sigma_{k=1}^{n}b_k$$

Difference Rule

$$\Sigma_{k=1}^{n}(a_k - b_k) = \Sigma_{k=1}^{n}a_k - \Sigma_{k=1}^{n}b_k$$

Constant Multiple Rule

$$\Sigma_{k=1}^{n}\, ca_k = c * \Sigma_{k=1}^{n}a_k$$

Constant Value Rule

$$\Sigma_{k=1}^{n}\, c = n * c$$

Now we will add on a few other handy formulas for sums:

The First n Terms

The First n Squares

The First n Cubes

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These formulas are particularly handy for larger sums.

Examples Sum the following using the sum formulas:

a. $\Sigma_{k=1}^{15} 3k^2$

b. $\Sigma_{i=1}^{52} (i^3 + 4i - 2)$

c. $\Sigma_{k=1}^{35} k(2k + 5)$

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Lesson: Sigma Notation with Riemann Sums

Our goal in this lesson is to figure out how to use Riemann Sums with Sigma Notation. To get you warmed up for this idea, we will practice figuring out how to write certain Riemann sums:

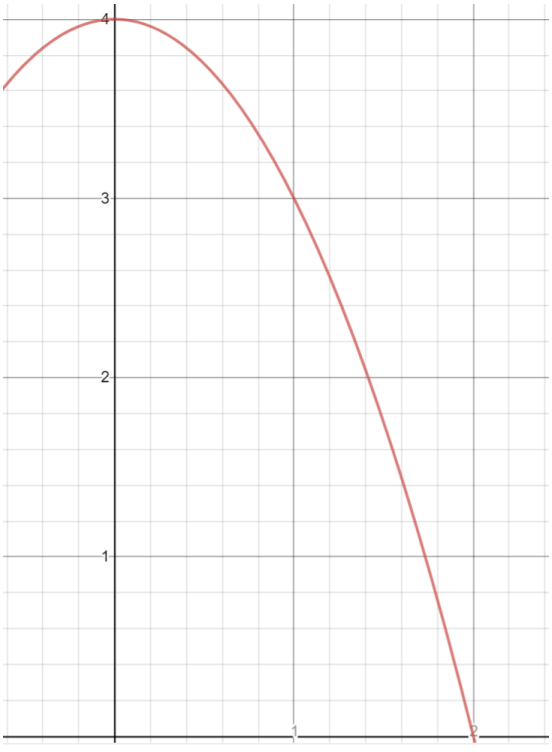
Examples Write the following sums using sigma notation.

a. $1 + 2 + 3 + 4 + 5$

b. $2 + 4 + 6 + 8 + 10 + 12$

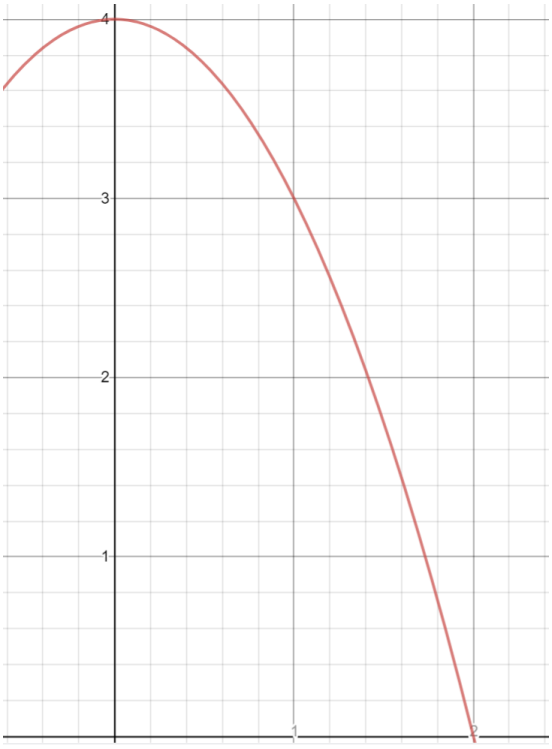
c. $1 - 3 + 5 - 7$

Previously we had discussed how to computer Riemann sums. Using $f(x) = 4 - x^2$ on $[0, 2]$, create 4 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?



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We will continue the pattern. Continuing to use $f(x) = 4 - x^2$ on $[0, 2]$, create 8 partitions and write out how to calculate the sum using right-hand endpoints. Can you figure out how to write this sum as a Riemann sum?



Can you determine how you might generalize this for n subintervals?

Write out the Riemann sum formula for using right-hand endpoints:

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How can we manipulate this formula so that we use left-hand endpoints instead? Write this down:

Example Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^2 + 1$ on $[0,2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

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Example: Evaluating Riemann Sums as $n \to \infty$

Example Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = 4 - x^2$ on $[0, 2]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

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Example: Evaluating Riemann Sums as $n \to \infty$

Example Use the right-endpoint Riemann sum formula to calculate a formula for determining the sum for n equal subintervals for $f(x) = x^3$ on $[0,1]$.

We have already determined that the best Riemann sum would have an infinite amount of subintervals. What happens if you evaluate your formula as $n \to \infty$? Draw an illustration of what this represents.

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Lesson: The Basics of Definite Integrals

Before we begin, we need to define some terminology. Draw and label the following:

Let $f(x)$ be a function defined on $[a, b]$. We say that a number J is the **definite integral of f over** $[\boldsymbol{a}, \boldsymbol{b}]$ and that J is the limit of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\Delta x_k$ if the following is true:

> Given any $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that for every partition P = $\{x_0, x_1, x_2, \dots, x_{n-1}, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$ we have:
>
> $$|\Sigma_{k=1}^{n} f(c_k)\Delta x_k - J| < \epsilon$$

This looks very similar to the definition of a limit in some ways, which I will state here:

> We have $\lim_{x \to c} f(x) = L$ if, for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that:
>
> $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

However, if the limit exists and using equal width partitions, we can instead state this (label what you write down):

Write down and label the definite integral notation:

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There are a few rules of definite integrals that we will want to note. When $f(x)$ and $g(x)$ are integrable functions over $[a, b]$, then we can use the following rules:

1. Order of Integration

2. Zero Width Interval

3. Constant Multiple

4. Sum and Difference

5. Additivity

6. Max-Min Inequality

7. Domination

Example if $\int_3^5 f(x)\,dx = 5$, $\int_0^3 f(x)\,dx = 2$, and $\int_0^5 g(x)\,dx = 9$, $\int_3^5 g(x)\,dx = 4$ find:

a. $\int_0^5 f(x)\,dx$

b. $\int_0^3 g(x)\,dx$

c. $\int_5^3 f(x)\,dx$

d. $\int_0^5 \big(3f(x) + g(x)\big)dx$

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Finally, we have one final theorem to state that helps us determine when a function is integrable.

Theorem: Integrability of Continuous Functions

A question that usually arises from this is, “What is a function that isn’t integrable then?” This is the classic example:

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Lesson: Finding Definite Integrals Using Area Formulas

If $f(x)$ is nonnegative and integrable over $[a, b]$, how do we define the area under the curve $y = f(x)$?

This means that we can calculate definite integrals using known area formulas when possible, as opposed to using Riemann sums.

Example Sketch the following curves and use known area formulas to evaluate the definite integrals.

a. $\int_{-1}^{1}(3 - |x|)\ dx$

b. $\int_{0}^{3}(2x)\ dx$

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c. $\int_0^2 (2x+1)\ dx$

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Lesson: The Average Value of a Function

If a function $f(x)$ is integrable on $[a, b]$, then its average value or mean is calculated by:

In this lesson, we will calculate integrals only using known area formulas.

Examples Find the average value of the following functions:

a. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$

b. $f(x) = |x| - 1$ on $[-1, 1]$

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Lesson: The Fundamental Theorem of Calculus, part 1 proof

Define the initial function we are working with and the corresponding graph:

Compute F’(x), including any relevant graphs. Interpret what this means:

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State the first part of the Fundamental Theorem of Calculus

Example Find $\frac{dy}{dx}$ given $y = \int_3^x (t^5 + 1)\, dt$

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Examples: The Fundamental Theorem of Calculus, part 1

Differentiate the following in 2 ways:

i. evaluate the integral then differentiate

ii. differentiate the integral directly

a. $\frac{d}{dx}\int_3^x \sin t \ dt$

b. $\frac{d}{dx}\int_1^{\cos x} 2t \ dt$

c. $\frac{d}{dx}\int_4^{\sqrt{x}} (5t^3 - \cos t) \ dt$

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Examples: The Fundamental Theorem of Calculus, part 1

Compute $\frac{dy}{dx}$ for each of the following:

a. $y = \int_4^x \sqrt{3t+2}\, dt$

b. $y = \int_x^5 \sin t \ \ dt$

c. $y = \int_{\sin x}^4 3t^2 \ \ dt$

d. $y = \int_{-3}^{e^{4x}} \frac{1}{\sqrt{3t+1}}\ dt$

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Step by Step Example: Fundamental Theorem of Calculus, part 1

Find $\frac{dy}{dx}$ for $y = x\int_5^{x^2} \sin t^3 \ dt$

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Lesson: The Fundamental Theorem of Calculus, part 2 proof

State the Mean Value Theorem for Definite Integrals

Example if $\int_a^b f(x)\,dx = 0$, must there be a point c in $[a, b]$ where $f(c) = 0$?

From the Fundamental Theorem of Calculus part 1, we know that the antiderivative of f exists in the form:

$$F(x) = \int_a^x f(t)\ dt$$

Now, we know that the most general form of an antiderivative for f would be $F(x) + c$. Now we want to make this more precise.

What is $F(b) - F(a)$?

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State the second part of the Fundamental Theorem of Calculus

Examples Compute the following:

a. $\int_1^3 2x\,dx$

b. $\int_0^{\frac{\pi}{2}} \cos x\,dx$

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Examples: The Fundamental Theorem of Calculus, part 2 – basic examples

Evaluate the following integrals:

a. $\int_1^4 (3x^2 - x)\ dx$

b. $\int_0^{\frac{\pi}{3}} 5\sec^2 x\ dx$

c. $\int_0^4 \left(4x^2 + \sqrt{x}\right) dx$

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Examples: The Fundamental Theorem of Calculus, part 2 – tricky trig examples

Evaluate the following integrals:

a. $\int_0^{\frac{\pi}{4}} \frac{1+\cos x}{2}\, dx$

b. $\int_0^{\frac{\pi}{3}} \frac{1+\cos 2x}{2}\, dx$

c. $\int_{\frac{\pi}{2}}^{\pi} \sin^2 x\ dx$

d. $\int_0^{\frac{\pi}{2}} (\sec x + \cos x)^2\ dx$

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Examples: The Fundamental Theorem of Calculus, part 2 – tricky non-trig examples

Evaluate the following integrals:

a. $\int_3^5 (x+1)^2 \ dx$

b. $\int_1^3 \left(\frac{x^6}{3} - \frac{1}{x}\right) dx$

c. $\int_0^4 \frac{x+\sqrt{x}}{x} \ dx$

d. $\int_{-2}^2 |x| \ dx$

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Examples: Which Formula Should I Use for Integrals?

In this video, we will compute the integral $\int_1^4 (2x + 1)\, dx$ 3 ways:

Way 1: Area Formulas

Way 2: Riemann Sum

Way 3: Using the fundamental theorem of calculus / evaluation formula

Use the space below to write out each way.

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Summary: Everything about Indefinite and Definite Integrals

Compare $\int 2x\,dx$, $\int_0^3 2x\ dx$, and $\int_3^0 2x\ dx$

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Lesson: Indefinite Integrals and the Substitution Method

Before we begin, let's just review how indefinite integrals work:

$\int (3x^4 - \sin 5x + 3e^{3x})\, dx$

Now we can explain substitution using the following example: $\int (4x^3 + 2x)(x^4 + x^2)^{18}\, dx$

Example Integrate:

a. $\int (4x^3 - 6x^2)(x^4 - 2x^3)^8\, dx$

b. $\int 3\sin(3x)\ dx$

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Sometimes we must manipulate the problem to get substitution to work.

Use the example $\int (6x + 9)^{12}\, dx$ to demonstrate this idea:

Examples Integrate:

a. $\int 4\sec^2(3x + 1)\ dx$

b. $\int \frac{1}{\sqrt{3x-5}}\, dx$

c. $\int \tan x\ dx$

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Sometimes we must go nuclear with our manipulations. Below are a few examples of creative ways to manipulate the problems:

a. $\int \sin^5 x \cos^3 x \ dx$

b. $\int (x-4)(x+3)^{15} \ dx$

c. $\int x^3 \sqrt[3]{x^2+4} \, dx$

d. $\int \frac{dx}{1-e^x} \ dx$

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Lesson: Indefinite Integrals and the Substitution Method to Solve Certain Trig Integrals

In this lesson, it is assumed that you already understand the basics of the substitution method. Here we will focus on how to use substitution to integrate the following integrals:

a. $\int \tan x \ dx$

b. $\int \cot x \ dx$

c. $\int \sec x \ dx$

d. $\int \csc x \ dx$

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Basic Examples of Indefinite Integrals and the Substitution Method

a. $\int 2x\sqrt{x^2+5}\ dx$

b. $\int \frac{1}{2\sqrt{x}\left(3+\sqrt{x}\right)^2}\ dx$

c. $\int 3\sin^3 3x\cos 3x\ dx$

d. $\int -\frac{1}{x^2}\sqrt{3+\frac{1}{x}}\ dx$

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Examples of Indefinite Integrals and the Substitution Method Where Some Manipulation is Required

a. $\int 5x\sqrt{3x^2+2}\,dx$

b. $\int 5x^{\frac{1}{2}}\cos\left(3x^{\frac{3}{2}}+1\right)\,dx$

c. $\int \frac{\cos(5x+2)}{\sin^2(5x+2)}\,dx$

d. $\int \sqrt{\frac{x^8}{x^5+4}}\,dx$

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Difficult Examples of Indefinite Integrals and the Substitution Method Where Creativity is Needed

a. $\int \sqrt{\frac{x^2-x}{x^6}}\ dx$

b. $\int 5x^5\sqrt{4x^3+1}\ dx$

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c. $\int \frac{7}{4+25x^2}\ dx$

d. $\int \frac{3}{x\sqrt{1-x^4}}\ dx$

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Lesson: Substitution with Definite Integrals

This lesson assumes you are already familiar with substitution for indefinite integrals. To get warmed up, evaluate the following:

a. $\int \frac{6x^2}{\sqrt{1-x^3}}\ dx$

b. $\int \frac{x}{\sqrt{1+x}}\ dx$

Using these examples, we can demonstrate 2 different ways to evaluate definite integrals with substitution.

$\int_1^3 \frac{6x^2}{\sqrt{1-x^3}}\ dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution

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Now you try using $\int_0^3 \frac{x}{\sqrt{1+x}}\, dx$

Way 1: Evaluate as an indefinite integral, then plug in the limits of integration

Way 2: Change the limits of integration based on your substitution

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Examples: Substitution with Definite Integrals

Evaluate the following:

a. $\int_1^4 \frac{1}{2x(1+\sqrt{x})^2}\ dx$

b. $\int_0^{\frac{\pi}{3}} \cos^{-2} x \sin x\ \ dx$

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c. $\int_1^4 x\sqrt{1+3x}\,dx$

d. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+e^{\tan x})\sec^2 x\;\;dx$

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Example: Definite Integrals and Substitution

$\int_2^4 \frac{dx}{x(\ln x)^6}$

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Step by Step Example: Substitution and Definite Integrals – Trig Problem!

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{x}\sin^2\left(x^{\frac{3}{2}}\right)\,dx$$

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Lesson: Definite Integrals of Symmetric Functions (Proof and Examples)

In this lesson, we will look at how substitution can be used to prove the following theorem:

Theorem Suppose that $f(x)$ is a continuous function on $[-a, a]$. Then:

a. if $f(x)$ is even, then $\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{2} f(x)\ dx$

b. if $f(x)$ is odd, then $\int_{-a}^{a} f(x)\ dx = 0$

Prove this below:

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Examples Evaluate the following integrals:

a. $\int_{-4}^{4}(x^3 + x)\, dx$

b. $\int_{-5}^{5}(|x| - x^4 + x^2)\, dx$

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Lesson: Calculating the Area Between 2 Curves With Integration

An integral is calculating what area?

Draw out an explanation of how we can understand using integration to find the area between 2 curves. Watch the explanation, and then rewrite it in terms that make sense to you.

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Example Find the area enclosed between $y = x^2 - 1$ and $y = 3$.

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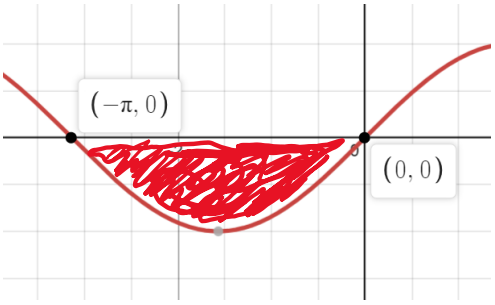
Example Find the area between $x = 2y^2$, $x = 0$, and $y = 3$

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Examples: Total Area vs Area

What's the difference between area and total area? Demonstrate this concept with a graph.

Example Find the total area for the curve $y = \sin x$ in the shaded region (graph from Desmos.com)



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Examples: The Area Between 2 Curves With Respect to x

Example Find the area between $y = x^2 - 2x$ and $y = -x^2 + 4x$

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Examples: The Area Between 2 Curves With Respect to y

Example Find the area between $x = 2y$ and $x = y^2 - 3$

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Example Find the area enclosed by $x + 4y^2 = 4$ and $x + y^4 = 1$ for $x \geq 0$

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Examples: Finding Area Between Hard to Visualize Curves

a. Find the area between the curves enclosed $y = x^4 - 9x^2 + 1$ and $y = x^2 - 8$

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b. Find the area enclosed by the curves $9x^2 + y = 9$ and $x^8 - y = 1$

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c. Find the area of the region enclosed by the curve $x = 3\sin y\ \sqrt{\cos y}$ and the line $x = 0, 0 \leq y \leq \frac{\pi}{2}$

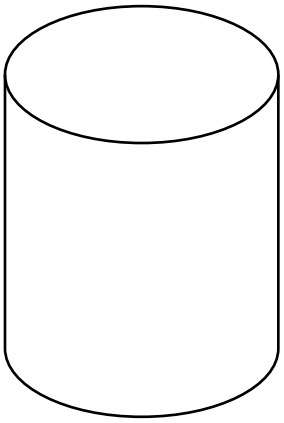
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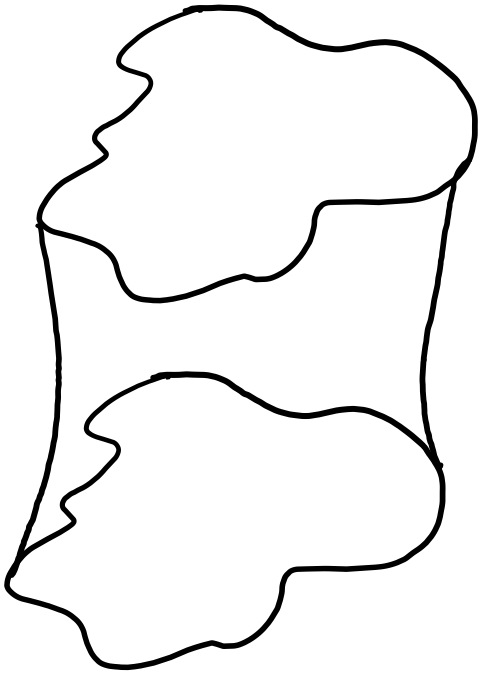
Lesson: Understanding Volume by Slicing with Parallel Planes

We're going to discuss how integration relates to volume. One way is using cross-sections.

What is a cross-section?

Below we have a few different kinds of cylinders. What would the cross-sections look like?

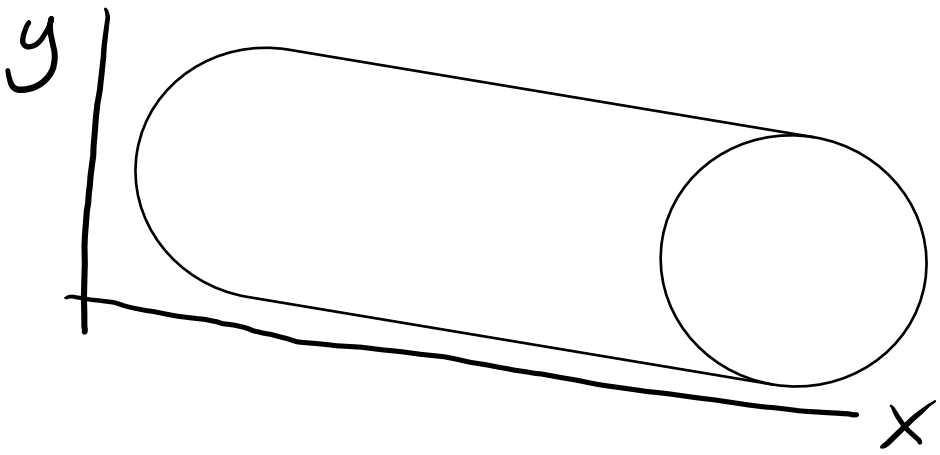




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Which way do you want these slices to be oriented?

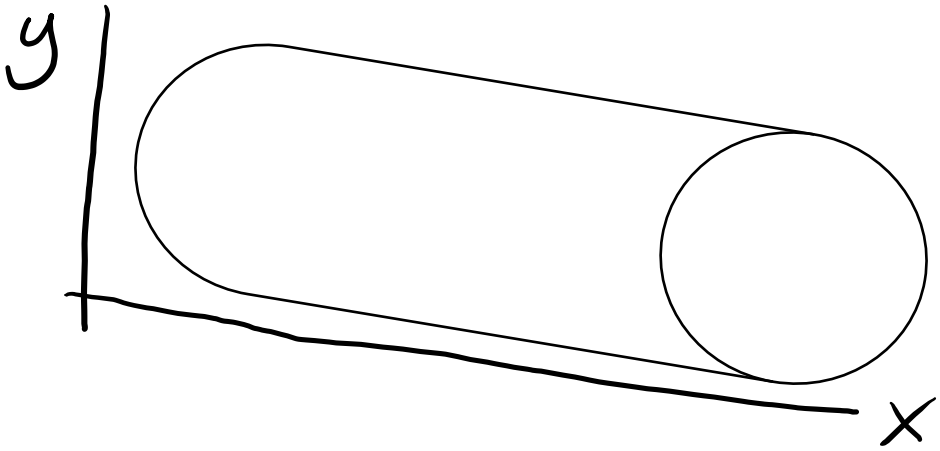
Below is how we would orient the circular cylinder.



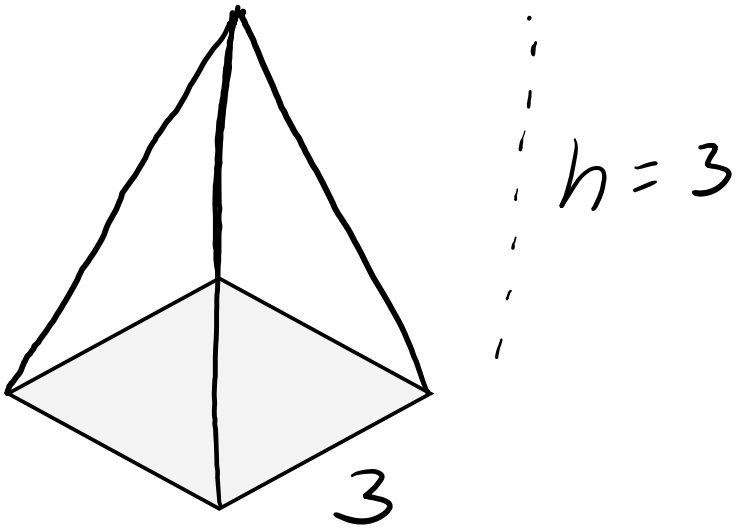
Draw a sketch showing how we would orient the second (odd looking) cylinder on the x- and y-axis.

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Explain how we derive volume using cross-sections for the cylinder.



Using the following pyramid with a square base, draw how we might opt to orient this object to make it easier to analyze the cross-sections.



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Summarize the explanation of how we would opt to take the volume of cross-sections in the case of the pyramid. In your explanation, explain how Riemann sums fit into this.

Write down both formulas for finding volume with cross-sections.

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What are the steps to calculating the volume of a solid with cross-sections?

Example Find the volume of a solid where the base is the region between the curve $y = 4\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis, where the cross-sections are equilateral triangles with the bases running from the x-axis to the curve y.

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Examples: Volumes Using Cross-Sections

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 4 - x^2$. Find the volume.

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The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x$. Find the volume when the cross-sections perpendicular to the x-axis are:

a. isosceles triangles of height 4

b. semicircles with diameters running across the base of the solid

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Examples: Volume by Slicing, Harder Examples

A square of side length 2x lies in a plane perpendicular to a line L. One vertex of the square lies on L. As the square moves a distance h along L, the square turns one revolution about L to generate a corkscrew shaped column with square cross sections. What is the volume?

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Find the volume of the tetrahedron. Please refer to the video for the reference picture.

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The base of the solid is the disk $x^2 + y^2 = 4$. The cross-sections by planes perpendicular to the y-axis between $y = -2$ and $y = 2$ are isosceles right triangles with one leg in the disk. Find the volume.

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Lesson: Solids of Revolution via the Disk Method

Before you do this lesson, you need to make sure you understand the lesson about volume by cross-sections!

The disk-method is basically an application of volume by cross-sections.

What is a solid of a revolution?

What are the cross-sections in this case?

How can we represent the area of these disk?

Formula for Finding Volume by Disks for Rotation About the x-axis

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Formula for Finding Volume by Disks for Rotation About the y-axis

Example Find the volume of the solid generated by revolving the bounded region about the x-axis:

$$y = x^3, y = 0, x = 1$$

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Example Find the volume of the solid generated by revolving the bounded region about the y-axis:

$$x = \sqrt{4 \sin y}\,, 0 \leq y \leq \pi, x = 0$$

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Example Find the volume of the solid generated by revolving the bounded region about the line $y = 1$.

$$y = \sqrt{2x}, y = 1, x = 2$$

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Examples: The Disk-Method, Revolved around the x-axis or y-axis

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = \sec x\,, y = 0, x = -\frac{\pi}{4}, x = \frac{\pi}{4}$ around the x-axis

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B. $x = \frac{2}{\sqrt{y+1}}, x = 0, y = 0, y = 3$

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Examples: The Disk-Method, Revolved Around a Line (not an axis)

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 4x, y = 0, x = 1$ about the line $x = 1$

B. $y = \sqrt{x}, y = 2, x = 0$ about the line $y = 2$

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Lesson: Solids of Revolution via the Washer Method

You should understand the idea of volume by cross-sections and volume by the disk method before you try this lesson.

The washer method is used with solids of revolution, but what characteristic do these solids have?

What do we need to know about the washers?

Formula for Volume by Washers for Rotation about the x-axis

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Example Find the volumes of the solid generated by revolving the enclosed region about the x-axis.

$$y = x, y = 2, x = 0$$

Example Find the volumes of the solid generated by revolving the enclosed region about the y-axis.

$$x = \tan y\,, y = 0, x = 1, y \leq 0 \leq \frac{\pi}{4}$$

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Examples: The Washer Method

Find the volumes of the solid generated by revolving the areas bounded by the lines and curves around the given axis.

A. $y = 2\sqrt{x}, y = 2, x = 0$ about the x-axis

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B. $y = \sec x,\ y = \sqrt{2}, -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

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Examples: Volume by Disks and Washers Around Different Axes

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, y = 0, x = 2$ about:

a. The x-axis

b. The line $y = -2$

c. The line $y = 6$

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d. The y-axis

e. The line $x = 3$

f. The line $x = -3$

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Examples: When to Use Subtraction with Finding the Radius

Find the volume of the solid generated by revolving the triangular region bounded by $y = 4x, x = 0, y = 4$ about:

a. The x-axis

b. The y-axis

c. The line $y = 4$

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Examples: Help! I Feel Like I'm Bad At Sketching

Sketch the region bounded by:

$y = 3x, x = 2, y = 0$

a. Show the solid of a revolution obtained by revolving this region around the x-axis

b. Show the solid of a revolution obtained by revolving this region around the y-axis

c. Show the solid of a revolution obtained by revolving this region around the line $x = 2$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

Sketch the region bounded by:

$y = \sqrt{x}, y = 2x - 1, x = 0$

a. Show the solid of a revolution obtained by revolving this region around the y-axis

b. Show the solid of a revolution obtained by revolving this region around the line $x = 4$

c. Show the solid of a revolution obtained by revolving this region around the line $y = 3$

d. Show the solid of a revolution obtained by revolving this region around the line $y = -2$

Lesson: Volume by Cylindrical Shells

Sometimes cross-sections can be awkward.

Demonstrate why cross-sections might be awkward using the example $y = 2x - x^2$ revolved around $x = -1$. Focus on the area enclosed by the x-axis.

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Another way to calculate volume is by using a cylindrical shell. Draw how a cylindrical shell is constructed. How do we compute the volume of a shell?

Now draw how a cylindrical shell might fit into your sketch. Your sketch does not have to be perfect – you just need to use it as a way to wrap your head around the idea. Draw multiple sketches if needed. Label the pieces of your sketch.

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One of the keys to using cylindrical shells is what?

Explain how the volume formula will relate to integration:

Shell Formula for A Revolution Around Any Vertical Line x=L

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What assumptions do we make about calculating volume in this manner?

Example Find the volume generated by revolving the bounded region about the y-axis:

$$y = x, y = -\frac{1}{2}x, x = 4$$

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Example Find the volume generated by revolving the bounded region about the line $y = -1$

$$x = \frac{1}{2}y, y = 0, x = 4$$

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Examples: Volume by Using Cylindrical Shells with a Piecewise Function

Let $g(x) = \begin{cases} \frac{\sin x}{x}, & 0 < x \leq \pi \\ 1, & x = 0 \end{cases}$

a. Show the $x * g(x) = \sin x,\ 0 \leq x \leq \pi$

b. Find the volume of the solid generated by revolving the region enclosed by the curve and the x-axis about the y-axis.

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Examples: The Shell Method Where The Integral is Split Up

Using the region enclosed by $y = x^2$ and $y = x + 6$, use the shell method to find the volume of solids generated by revolving the regions about the given lines:

a. $x = 3$

b. $x = -4$

c. the x-axis

d. $y = -2$

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Examples: Volume by Using Cylindrical Shells About Different Lines

For the region bounded by the curves:

$$y = 2x, y = 0, x = 3$$

Find the volume generated by revolving the region about:

a. The line $x = -1$

b. The line $y = -2$

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Lesson: Volume by the Washer Method vs Volume by Cylindrical Shells

Compute the volume of the solid bounded by the curves:

$$y = \frac{1}{2}x + 1, y = x, x = 0$$

In the following ways:

a. Revolved about the x-axis using the washer method

b. Revolved about the y-axis using the shell method

c. Revolved about the line x=3 using the shell method

d. Revolved about the line y=4 using the washer method

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Summary: How to Set Up Volume by Integration Problems

There are a few keys to setting up volume by integration problems:

1. Always draw the region **and** draw a sketch of the solid – you should always have 2 drawings
2. Decide what formula is appropriate for the situation and with respect to which variable
3. Label your solid to go with the formula
4. Fill in the formula and integrate

Let's review a few concepts.

Write out the formulas for the disk method.

Write out the formulas for the washer method.

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When do you use the washer method?

How can you determine if your revolution is a solid or has a hole in it?

With the disk/washer method, when do you integrate with respect to x?

With the disk/washer method, when do you integrate with respect to y?

In what circumstances do you have to “adjust” the radius? How does this adjustment look? What does this mean about rotating about the x-axis or y-axis?

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Write out the formulas for the shell method:

With the shell method, when do you integrate with respect to x?

With the shell method, when do you integrate with respect to y?

If you rotate around the follow axis, what is the radius in that situation?

a. x-axis

b. y-axis

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When do you need to subtraction with finding the shell radius?

If you need to use subtraction to find the shell radius, how do you determine the radius?

In the disk, washer, or shell methods: if you are integrating with respect to x, how can you determine the limits of integration?

In the disk, washer, or shell methods: if you are integrating with respect to y, how can you determine the limits of integration?

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If you struggle to sketch your solids, I have an entire video just focusing on this. *You absolutely need to be able to quickly sketch a solid, and it is worth your time to do this video if you have anxiety about doing this.*

Example Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

First, sketch the region:

a. Now we will find the volume generated by revolving this region about the x-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

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Example Use the region bounded by the following curves:

$$y = 3x, y = 0, x = 1$$

b. Now we will find the volume generated by revolving this region about the y-axis. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

c. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

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d. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

e. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

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f. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the disk/washer method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

g. Now we will find the volume generated by revolving this region about the x-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

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h. Now we will find the volume generated by revolving this region about the y-axis. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

i. Now we will find the volume generated by revolving this region about the line $x = -1$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

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j. Now we will find the volume generated by revolving this region about the line $x = 2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

k. Now we will find the volume generated by revolving this region about the line $y = 5$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

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l. Now we will find the volume generated by revolving this region about the line $y = -2$. We will use the shell method. Sketch the solid, label it, state what formula you will use, and then set up the formula.

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Lesson: Arc Length and How the Formula is Derived

Suppose we want to find the length of the curve $y = \sqrt{x-2}$ from $x = 2$ to $x = 6$. Draw a sketch of the curve.

To understand how we can use integration to find the length of a curve, we need to recall how we find the length of a line segment. Draw out a line segment from (x_1, y_1) to (x_2, y_2), and write out how we would find the formula for this line segment.

How can we apply this idea to a Riemann sum with a curve? Draw an illustration to accompany this explanation.

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Show how we can take the ideas from algebra and mold them to using integration to finding the length of the curve.

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State the arc length formula.

Example Find the length of the curve $y = 4x^{\frac{3}{2}}$ from $x = 0$ to $x = 1$

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Lesson: Arc Length with Discontinuous Derivative Values

There is an issue if we try to from the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$. Explain what the issue is.

How can we resolve this issue?

What is the formula for the length of the curve $x = g(y), c \leq y \leq d$

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Show how to find the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x = 0$ to $x = 2$.

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Examples: Finding Surface Areas of Revolution

Find the surface area of the revolution generated by revolving the given curve about the given axis.

a. $x = \sqrt{4y - 3}, 1 \leq y \leq 3$, y-axis

b. $y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}}, 1 \leq x \leq 2$, x-axis

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c. $x = \frac{1}{2}(e^{y} + e^{-y}), 0 \leq y \leq \ln 3$, y-axis

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Lesson: Finding the Area of a Surface of Revolution

How do you create a solid of revolution?

How to find the area of surface from revolving the graph of f(x) around the x-axis:

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How to find the area of surface from revolving around the y-axis:

Examples Find the areas of surface generated by revolving the curve around the given access.

a. $f(x) = \sqrt{x+1}, 1 \leq x \leq 5$, x-axis

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b. $x = \left(\frac{1}{3}\right) y^{\frac{3}{2}} - y^{\frac{1}{2}}, 1 \leq y \leq 3$, y-axis

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Lesson: A Guide to Approaching Work Problems

Constant force work formula:

Variable force work formula:

A note on calculating force:

The units are very important because they will likely guide you on how to do the problem.

Let's note units too for the following:

Measure	SI Units	US Customary
Force		
Work		
Mass		
Length		
Acceleration due to gravity		

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What are some important notes regarding how we think of force in different applications?

With problems involving objects:

With pumps/volume problems:

If you have completely forgotten how to do dimensional analysis, *please watch my review video on this topic.* This will really help you!

Going back to our integral, rewrite the integral to explain how we should be watching out for units.

A guideline for how to get to these units:

With objects

With pumps/volume

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Example You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

Why is it obnoxious to say “weighs” here? What’s a better way to think about this?

What elements go into how this formula is set up?

When we set up our integral, what ultimately needs to happen?

Show how to finish the problem and show how the units guide us that we are right.

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Example You want to haul a 25-m length of hanging rope. How much work will it take if the rope weighs 0.624N/m.

What if I wanted to use (25-x) instead of x to model the length left of the rope?

What does this mean?

Note the following fact about integration:

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Example You want to haul a 15-m length of hanging rope. How much work will it take if the rope weighs 0.08 kg/m.

Example A bag of sand weighing 160lbs was lifted at a constant rate. As it rose, sand leaked out at a constant rate. One-quarter of the sand was gone when the bag had been lifted to 20ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment).

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Example A vertical right-circular cylindrical tank measures 20 ft high and 10 ft in diameter. It is full of milk weighting 64 $\frac{lb}{ft^3}$. How much work does it take to pump the milk to the level of the top of the tank?

Example Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 5 ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to the rim of the tank? (See picture)

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Example Suppose a 10ft tall, 10ft diameter conical tank (tip down) is filled 8ft high with kerosene weighing 51.2 $\frac{lb}{ft^3}$. How much work does it take to pump the kerosene to 5ft above the top of the tank?

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Lesson: Hooke's Law for Springs

What is Hooke's law?

What are the relevant units to pay attention to?

Example It took 1500 J of work to stretch a spring from its natural length of 2m to a length of 4m. Find the spring's force constant.

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Example If a force of 80 N stretch a spring 2 m beyond its natural length, how much work does it take to stretch the spring 6 m beyond its natural length?

Example A spring has a natural length of 8 in. A 500-lb force stretches the spring to 13 in.

a. Find the force constant

b. How much work is done in stretching the spring from 8 in to 13 in?

c. How far beyond its natural length will a 800-lb force stretch the spring?

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Lesson: Fluid Forces

Fluid Force on a Constant-Depth Surface

Fluid Force Against a Vertical Flat Plate (Integration Formula)

Example Refer to the video for the sketch. Calculate the fluid force on one of the triangular plate using the given coordinates.

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Example In a pool fill with water to a depth of 12 ft, calculate the fluid force on one side of a 4ft x 6ft rectangular plate if the plate rests vertically at the bottom of the pool on its 4-ft edge.

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Examples: Fluid Forces

Calculate the fluid force on one side of a semicircular plate of radius 4 ft that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 7 feet.

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A metal cube that is 8ft on each side has a parabolic gate on one side at the bottom. See the picture in the video. It is designed to withstand a fluid force 500 lb without rupturing. The liquid you plan to store has a weight-density of 62.4 lb/ft^3.

a. What is the fluid force on the gate when the liquid is 6 feet deep?

b. What is the maximum height to which the container can be filled without exceeding the gate's design limitation?

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Examples: The Logarithm Defined as an Integral

Evaluate the following integrals.

a. $\int_{-4}^{-2} \frac{dx}{x}$

b. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

c. $\int 3x^2 e^{-x^3} dx$

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d. $\int_{\ln\left(\frac{\pi}{3}\right)}^{\ln\left(\frac{\pi}{2}\right)} 2e^x \sin e^x \, dx$

e. $\int_1^3 x^{3x}(1 + \ln x)dx$

f. $\int_1^9 \frac{\log_3 x}{x} dx$

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Examples: The Logarithm Defined as an Integral, Initial Value Problems

Solve the initial value problems.

a. $\frac{dy}{dx} = e^{-x}\csc^2(\pi e^{-x}),\ y(\ln 4) = \frac{2}{\pi}$

b. $\frac{d^2y}{dx^2} = 3e^{-x}, y(0) = 1, y'(0) = 0$

c. $\frac{d^2y}{dx^2} = \sec^2 x,\ y(0) = 0, y'(0) = 1$

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Examples: Separable Differential Equations, General Solutions Only

Examples Find general solutions to the following:

a. $\frac{dy}{dx} = 2x\sqrt{y}$

b. $\frac{dy}{dx} = (xy)^{\frac{3}{2}}$

c. $\frac{dy}{dx} = \frac{(x^5-1)y^3}{x^2(3y^2-2)}$

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Examples: Separable Differential Equations, Initial Value Problems

Examples Solve the initial value problems

a. $\frac{dy}{dx} = \sqrt{y}; y(0) = 4$

b. $\frac{dy}{dx} = \frac{x}{y}; y(2) = 6$

c. $\frac{dy}{dx} = 3x^2y^2 - y^2; y(0) = \frac{1}{2}$

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Examples: Population Growth

We can study how diseases die when properly treated by using the following logic let $\frac{dy}{dt}$ represent the rate at which the number of infected people changes is proportional to the number y. The number of people who are cured is proportional to y, the number who are infected with the disease. As of April 23, 2020, the number of coronavirus cases was 2,659,557. For simplicity, let's round that number to 2,660,000. Let's suppose that we can reduce the number of cases of the disease by 50% in a year. If there are 2,660,00 cases, then how many years will it take to the reduce the number to 2,000 cases?

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Examples: Radioactivity and Exponential Decay

Some atoms are unstable and spontaneously emit mass, which is called radioactive decay. An element that goes through this process is called radioactive. The decay of a radioactive element is described by the differential equation:

$$\frac{dy}{dy} = -ky, k > 0$$

If y_0 is the number of initial nuclei present, then the number at a later time is:

$$y = y_0 e^{-kt}, k > 0$$

An interesting consequence of this is that half-life can be calculated by the equation:

$$\frac{\ln 2}{k}$$

The half-life of the plutonium isotope is 24,360 years. If 15 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 75% of the isotope to decay?

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Examples: Newton’s Law of Cooling

A metal beam was brought from the outside cold into a machine shop where the temperature was held at $75°\text{F}$. After 10 min, the beam warmed to $45°\text{F}$ and after another 15 min it was $55°\text{F}$. Use Newton's Law of Cooling to estimate the beam's initial temperature.

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Lesson: Intro to Hyperbolic Functions

The hyperbolic sin and hyperbolic cosine functions are defined by the equations:

How do we say “sinh x” ?

What about “cosh x” ?

Using these definitions, we get some interesting identities. Determine the value of $2\sinh x\ \cosh x$:

Now determine the value of $\cosh^2 x - \sinh^2 x$

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List all of the relevant identities for hyperbolic functions:

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Lesson: Derivatives of Hyperbolic Functions – A Quick Overview

Find the derivative of $\sinh x$

Find the derivative of $\operatorname{csch} x$

Find the derivative of $\tanh x$

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Write the derivative rules for all of the hyperbolic functions below:

Write the integral formulas for all of the hyperbolic functions:

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Not included (no notes file):

112 (Lesson Introducing Where Calculus Starts)