**Lesson: Exponents and the Order of Operations**

## Exponents

Label the parts of this exponent:

$$2^3$$

What does $2^3$ literally mean?

## Example 1

Evaluate the following exponents

a. $3^3$

b. $5^3$

c. $2^4$

We need to understand exponents so that we can fully understand the order of operations. The order of operations is one of the handiest concepts in arithmetic that is often confused.

## Order of Operations

Write down what the order of operations is:



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**Classic Example of the Order of Operations**

Evaluate: $10 \div 5 \times 6$

**Example 2**

Evaluate the following. This page and the next page are both blank so that you have plenty of space to work out your calculations by hand.

a. $3 + 2^3 - 6 \div 3 \times 7$

b. $2 + 3^2 + 4 \times 2 - 5 \times 2$

c. $3(1 + 2^2 - 3)^2$



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**Examples: The Order of Operations**

a. $10 - 3(1 + 2 \times 3) - 2^2 \times 4$

b. $3(1 + 2^2 - 3)^2 + 5(4 - 2 \times 3)$

c. $\frac{1}{2}(5 + 3) - 2^2(5 - 6)$



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**Lesson: Advanced Notes on Exponents**

## Multiplicative Property of Negative One

Explain what is the difference between $-5^2$ and $(-5)^2$ from the standpoint of the order of operations:

## Example 1

Evaluate the following.

a. $(-3)^2$

b. $(-3)^3$

c. $-3^2$

d. $-3^3$

e. $-7^2$

f. $(-2)^5$

## Example 2

What is the sign on the following problems? Do not worry about the final answer, you only need to circle if the final answer is positive or negative.

a. $(-4)^7$ Positive Negative

b. $(-58)^{12}$ Positive Negative

c. $-72^{57}$ Positive Negative

d. $(-13)^{72}$ Positive Negative

e. $-64^{14}$ Positive Negative

f. $(-23)^{91}$ Positive Negative



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There is one final rule we need to discuss.

**Zero Exponents**

For any real number a,

**Example 3**

Evaluate the following:

a. $5^0$

b. $x^0$

c. $35^0$

d. $-6^0$

e. $(-7)^0$

f. $3^0 + 7^0$

g. $-5^0 + 2^0$

h. $\left(-3 + \sqrt{52}\right)^0$



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**Lesson: Algebraic Expressions**

First we will define the terminology, which is already written out for you.

- **Constant:** any real number in an expression that is not attached to a variable
- **Variable:** a letter than can stand for any set of real numbers, and we can substitute values in for it if desired
- **Coefficient:** a number that is being multiplied by a variable
- **Term:** this is a single number, variable, or a number and a variable multiplied together, often separated by a plus + or minus – sign.
- **Factor:** a number or variable that is being multiplied with another number or variable
- **Algebraic Expression:** a mathematical string of constants and terms that can be added, subtracted, multiplied or divided together

This terminology will be used often, so it's important to understand what it means.

**Example 1**

Classify each part of the expressions

| Expression | Constants | Variables | Coefficients | Terms | Factors |
|---|---|---|---|---|---|
| a. $5x$ | | | | | |
| b. $2x + y$ | | | | | |
| c. $8$ | | | | | |
| d. $x$ | | | | | |
| e. $2x^2 - 4x + 7$ | | | | | |

Algebraic expressions are often used in everyday situations; we just don't realize it.

**Example 2**

You want to go to the movies, and on Tuesdays a theatre runs a deal so that all tickets cost $8. How much would it cost for each of the following situations?

a. A ticket for only you?

b. A ticket for you and 2 of your friends?

c. Tickets for 5 people, a giant tub of popcorn that costs $20.

d. Tickets for z people?



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In the last part of this example, we wrote an algebraic expression where z represented some quantity. Notice that if you input any of the previous number of tickets from examples a, b, and c into z, you would get the same answers.

To evaluate expression, you can simply plug in the desired values for a variable.

**Example 3**

Evaluate each expression for $x = -2$, $y = 4$, and $z = \frac{1}{3}$ as applicable:

a. $3x - 7y$

b. $6z - x^2 + 2y$

c. $x + z^2$



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**Lesson: The Commutative, Associative, and Distributive Properties**

Your math homework and English homework are due on the same day. Does it matter which one you do first?

This situation illustrates the idea of the Commutative Law**,** which we can now state below:

**Commutative Law**

Let $a$ and $b$ represent any two real numbers. Now we can define the following properties:

- The Commutative Law for Addition:

- The Commutative Law for Multiplication:

**<u>Example 1</u>**

Rewrite the following using the commutative law.

a. $3 + y$

b. $3c$

c. $\frac{2}{5}k$

d. $2 - z$



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You are buying chips, a drink, and a notebook. Does it matter what order the cashier rings up this items in?

These situations illustrate the idea of the **Associative Law**, which we can now state below:

Let $a$, $b$, and $c$ represent any three real numbers. Now we can define the following properties:

- The Associative Law for Addition:

- The Associative Law for Multiplication:

**<u>Example 2</u>**

Rewrite the following at least 2 different ways using the associative law.

a. $4xz$

b. $8 + 2 + 7 + 3$



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These rules are helpful when we are computing longer expressions.

**How to Simplify Calculations Using the Commutative and Associative Laws**

a. Suppose we want to calculate: $2 + 3 + 8 + 7$
   Using the commutative law, we more cleverly group these numbers together like this:
   $$2 + 8 + 3 + 7$$
   Usually we recognize 10s faster than some other quantities, so from here we can rearrange and group like this to find the final answer:

b. How can you rearrange $\left(\frac{1}{5} * \frac{1}{3}\right) \times (10 * 6)$ to make the calculations simpler to work with?

**The Distributive Law**

Let $a$, $b$, and $c$ represent any three real numbers. Then the following property holds:



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The distributive property effectively gives us permission to either add or multiply certain expressions in the order that simplifies the calculation.

**Example 3**

Use the distributive property as needed to simplify the following calculations.

a. $10\left(\frac{2}{5}-\frac{1}{2}\right)$

b. $\frac{2}{3}(6+9)$

c. $20\left(\frac{1}{4}+\frac{2}{5}\right)$

d. $\frac{2}{5}(14-4)$

The distributive law works in the opposite direction as well. If we start with $ab+ac$, according to the distributive law what would this equal?

Another name for this process is called **factoring.**

**Example 4**

Use the distributive law to factor the following:

a. $km+kn$

b. $5x+10y$



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There are a few other properties we need to mention:

**<u>The Additive Identity</u>**
For any number a, $0 + a = a + 0 = a$

**<u>The Multiplicative Identity</u>**
For any number a, $1 \times a = a \times 1 = a$

**<u>The Additive Inverse (aka the opposite of a number)</u>**
For any number a, the addition of its opposite will result in zero, ie: $a + (-a) = 0$.

**<u>The Multiplicative Inverse (aka the reciprocal)</u>**
For any number a, it's inverse can be written in the form $\frac{1}{a}$ and has the property: $a \times \frac{1}{a} = 1$

It is interesting to note that there exist number systems (not discussed in this class) where these properties do **not** exist!



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**Lesson: Like Terms and Combining Like Terms**

In algebra, we often want to simplify expressions if possible to make our lives easier. This is where the discussion of like terms comes into play.

**<u>Like Terms</u>**

Like terms are terms that share the exact same variables with the exact same exponents. Only like terms may be added.

Do like terms work in real life? Ie, can you come up with an example of like terms that could still be added in the real world?

Building off of your example, what wouldn't be like terms and could NOT be added in that scenario?

**<u>Example 1</u>**

In the following list below, classify which terms are like terms.

Discuss the two examples with the class to ensure they understand how to identify like terms. Students may color code, underline, or draw shapes around anything in the lists to indicate they are like terms.

a. $2x, 7y, 6x^2, 8y, x, -2x^2$

b. $3x^2y^2, 7x^2y, 2xy^2, 8x^2y, x^2y^2, -3x^2, 5xy^2$

The idea of like terms only comes up when we are talking about addition or subtraction. We do not worry about this when we want to multiply two things together. You may only add or subtract terms that are like terms.



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**Example 2**

Combine like terms. The next page is also left blank if you need more space.

a. $3x + 2y - 5x + 9y$

b. $4x^2 + 2x + 6y^2 + 2x^2 + 8x$

c. $\frac{1}{2}x + \frac{2}{3}y - \frac{3}{2}x - 2\frac{1}{3}y$

d. $3x^2y^3 + 7x^2y^2 - 2x^2y^3 + x^2y^2$

e. $\frac{1}{4}x + \frac{2}{5}y - \frac{1}{2} + \frac{1}{5}y$



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**Lesson: Translating English into Math**

As we move through this course, one major goal is to learn how to recognize and interpret certain situations as math problems. There are particular words in English that denote certain operations in mathematics, and we will classify those here.

| **Addition** | **Subtraction** | **Multiplication** | **Division** |
|---|---|---|---|
| | | | |

**Example 1**

Translate the following sentences into an algebraic expression. There is nothing to solve for you, you are merely setting up expressions.

a. 3 more than Lucy's height

b. Two less than a number

c. Three more than twice his salary.

d. Five less than three times a number.

e. Two-thirds of the cost.

f. Three more than twice his and her salary combined.



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**Lesson: Basic Exponent Rules – Product, Power, and Quotient Rules**

When working with rules of exponents, it is very important to be able to identify the base of each exponent. To begin, we will practice this.

**Example 1**

State the base for each exponent in the problems below.

| Expression | Exponent | Base for this Exponent |
|---|---|---|
| $5^3$ | | |
| $-5^4$ | | |
| $-5x^6$ | | |
| $3y^7$ | | |

**Product Rule**

First we want to remember what the actual definition of an exponent means.
We want to find: $2^3 * 2^2$. We will calculate this a few ways to understand the rules.

First we will work towards finding the exact answer. Evaluate each exponent below as you normally would with the **order of operations**, and then multiply the numbers together:

Now we will re-examine this result solely from the viewpoint of exponents. Write out each exponent as a multiplication problem, and notice the results you get from this exercise.

What do you notice about the exponents?

**Product Rule (formal mathematical definition)**

Let x be a real number, and let a and b be integers. Then:



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**Example 2**

Use the product rule to evaluate the following.

a. $x^2 * x^4$

c. $x^3 + x^7$

b. $3x^3 * -1x^4 * 2x$

d. $\frac{2}{3}x^4y^2z^5 * 6x^3y^2 * \frac{1}{2}y^3z^4$

**Power Rule**

Using the order of operations, first rewrite the expression inside the parentheses without exponents:

$(2^3)^2$ =

Now rewrite this entire result just utilizing what we know about the definition of exponents:

What is the relationship between the exponents in this case?

**The Power Rule (formal mathematical definition)**

Let x be a real number, and let a and b be integers. Then:



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**Example 3**

Use the power rule to evaluate the following:

a. $(x^3)^5$

b. $(x^2y^7z)^4$

c. $(3x^4y^5)^3$

d. $\left(\frac{2}{3}x^5y^2\right)^3$

Is there a difference between $(x+y)^n$ and $(xy)^n$? Explain.

**The Quotient Rule**

Using the order of operations, first rewrite the expression inside the parentheses without exponents:

$$\frac{2^5}{2^3}$$

First we will evaluate and find the final answer by using the order of operations below.

Notice the final answer of this question. How can we rewrite this number in its exponent form?

What relationship do you see between the exponents in the original question and the final answer?

**The Quotient Rule (formal mathematical definition)**

Let x be a real number, and let a and b be integers. Then:



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**Example 4**

Use the quotient rule to simplify the following:

a. $\frac{x^7}{x^3}$

b. $\frac{x^5y^8z^{10}}{xyz}$

c. $\frac{15x^4y^3z^{10}}{2x^2y^2z^5}$

d. $\frac{12x^7y^3}{8x^3z^4}$



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**Examples: Exponent Rules, Basic Examples**

Simplify the following using only the product, power, and quotient rules. Your final answer should not contain negative exponents.

a. $x^2y^7 * 5x^3y^5z$

b. $(4x^7y^3)^2$

c. $\frac{15x^7y^3}{10x^2y}$

d. $\frac{2}{5}x^2y^5 * \frac{1}{3}x^7yz^3$

e. $\left(\frac{5}{3}x^3y^2z^3\right)^3$

f. $(-3x^5)^2$

g. $(-3x^5)^3$

h. $(2x^4y^2z^6)^2 * 3x^2z^3$

i. $\frac{(x^3y^7)^2}{x^4y}$

j. $\frac{(x^4y^2)^3}{(x^2y)^2}$



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**Examples: Exponent Rules, With Numbers Only**

Simplify the following using only the product, power, and quotient rules. Your final answer should be a number.

a. $3^2 * 3$

b. $(-4)^2 * 2$

c. $\frac{2^7}{2^4}$

d. $\left(\frac{2}{3}\right)^3 * 3^2$

e. $\frac{6^2}{2^3}$

f. $\left(\frac{1}{2}\right)^4 * 2^2$



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**Examples: Exponent Rules, Trickier and Harder Examples**

Simplify the following using only the product, power, and quotient rules. Your final answer should not include negative exponents.

a. $(3x^2y^4)^2 * (x^4y^5)^3$

b. $\frac{(2x^5y^2)^3}{(x^2yz^5)^2}$

c. $\left(\frac{2}{3}x^5y^7z^3\right)^2 * 9x^3y^3$

d. $\left(\frac{3}{5}x^5y^2z^7\right)^2 * \frac{xy^3}{z^5}$

e. $\frac{(3x^5y^2z^4)^3}{(6x^2yz^3)^2}$

f. $\frac{(4x^3y^2z^5)^2}{(2xz^2)^4}$



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**Lesson: Negative Exponents**

Simplify: $\frac{25}{125}$

If we wrote these numbers as exponents, this calculation would look like this:

$$\frac{5^2}{5^3} = \frac{1}{5}$$

Consider what would happen if we used the quotient rule on this problem:

$$\frac{5^2}{5^3} = 5^{2-3} - 5^{-1} = \frac{1}{5}$$

This problem shows us that exponents can also be negative.

**<u>Three Rules for Negative Exponents</u>**

Let x and y be a real numbers, and let n be an integer. Then:

$$x^{-n} = \frac{1}{x^n}$$

$$\frac{1}{x^{-n}} = x^n$$

$$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n = \frac{y^n}{x^n}$$

In general your final answer cannot have a negative exponent, so it is essential that you understand what to do with it.



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**Example 1**

Rewrite the following expressions without negative exponents.

a. $4^{-2}$

b. $\frac{1}{3^{-3}}$

c. $\left(\frac{2}{5}\right)^{-2}$

d. $\left(\frac{1}{3}\right)^{-2}$

One of the biggest challenges with negative exponents is identifying how to apply them to the proper base. As a quick review, remember how we identified the bases in the last section:

**Example 2**

Identify the base of the following exponents:

| Expression | First Base | First Exponent | Second Base | Second Exponent |
|---|---|---|---|---|
| $3^5$ | | | | |
| $-2x^5$ | | | | |
| $5x^3y^7$ | | | | |

This is extremely important to remember when working with negative exponents. In general, our final answer can't have negative exponents so you need to be used to reformatting your answers properly if negative exponents come up.

**Example 3**

Simplify the following expressions, and write your final answer without negative exponents.

a. $\frac{a^3b^{-2}}{c^{-4}d^6}$

b. $\frac{3a^{-3}b^4c^{-5}}{d^2e^{-2}f^5}$

c. $\frac{a^5b^3c^4}{4d^{-2}e^{-4}f^{-2}}$

d. $\frac{a^{-3}b^{-2}c^{-6}}{d^4e^7f}$



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With this in mind, we should now be able to work on combining all of the rules of exponents we've learned.

**Example 4**

Simplify the following. Your final answer cannot contain negative exponents.

a. $(3x^{-2}y^{4})^{3}$

b. $\left(\frac{2x^{-4}y^{6}}{x^{5}y^{-3}}\right)^{4}$

c. $(-2x^{3}y^{-2}z)^{-3}$

d. $\left(\frac{2x^{3}y^{-4}}{3x^{-2}y^{-5}}\right)^{-3}$



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**Examples: Negative Exponents, Basic Examples**

Your final answer should not contain negative exponents.

a. $\left(\frac{4}{5}\right)^{-2}$

b. $3x^{-6}y^{4}z^{-2}$

c. $\frac{4x^{-4}y^{7}}{z^{-5}}$

d. $\frac{6x^{3}y^{-5}}{x^{7}y^{2}}$

e. $\frac{14x^{-3}y^{5}z^{3}}{21x^{-2}y^{-3}z^{9}}$

f. $(3x^{-4}y^{-2}z^{4})^{2}$

g. $\left(\frac{5x^{2}y^{-4}}{z^{2}}\right)^{5}$

h. $(x^{-3}y^{2}z^{-4})^{-3}$



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**Examples: Negative Exponents, Tricky Examples**

Your final answer should not contain negative exponents.

a. $(3x^{-4}y^{5}z^{-2})^{-3}$

b. $(-5x^{2}y^{-3}z^{4})^{-2}$

c. $\left(\frac{4x^{-5}y^{2}}{x^{-3}y^{-5}}\right)^{-2}$

d. $\left(\frac{3x^{3}y^{-3}z^{4}}{2x^{-2}y^{4}z^{-2}}\right)^{-3}$

e. $\frac{(2x^{-4}y^{5})^{4}}{x^{-3}y^{7}}$

f. $\frac{(3x^{-3}y^{2}z^{-3})^{-3}}{(2x^{4}y^{-5}z^{3})^{-4}}$



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**Examples: Negative Exponents versus Negative Numbers**

**<u>Examples</u>** Simplify:

a. $6^{-2}$

b. $-6^2$

c. $(-6)^2$

d. $(-6)^{-2}$

**<u>Examples</u>** Simplify:

a. $5x^{-2}$

b. $(5x)^{-2}$

c. $-5x^2$

d. $(-5x)^{-2}$



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**Examples** Simplify

a. $-7^2$

b. $(-7)^{-2}$

c. $7^{-2}$

d. $-7^{-2}$

**Examples** Simplify

a. $3x^{-2}$

b. $(-3x)^{-2}$

c. $(-3x)^{-3}$



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**Lesson: Scientific Notation**

**Scientific Notation**

A number in scientific notation is of the form: $A \times 10^n$ where $1 \leq |A| < 10$ and n is an integer

Before we can write numbers in this form, it is best if we can identify examples that are or are not in this format.

**Example 1**

Identify if the following examples are in scientific notation. If not, circle and explain why.

a. $5.34 \times 10^7$ Yes, it's in scientific notation No, it's not in scientific notation

b. $-2.7832 \times 10^{\frac{1}{2}}$ Yes, it's in scientific notation No, it's not in scientific notation

c. $-4.5382 \times 10^{-8}$ Yes, it's in scientific notation No, it's not in scientific notation

d. $0.8 \times 10^{23}$ Yes, it's in scientific notation No, it's not in scientific notation

We will now examine how to rewrite the following example in scientific notation.

**Example 1**

Rewrite the following in scientific notation.

**Example 2**

Rewrite the following examples in their expanded form.



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**Example 3**

Use the rules of exponents to multiply the following results together.

a. $(2 \times 10^{6}) \times (3 \times 10^{8})$

b. $(2 \times 10^{17}) \times (2.1 \times 10^{23})$

c. $(8 \times 10^{7}) \times (2 \times 10^{-3})$

**Example 4**

Use the rules of exponents to divide the following results.

a. $\frac{8 \times 10^{12}}{4 \times 10^{3}}$

b. $\frac{6 \times 10^{7}}{2 \times 10^{19}}$

c. $\frac{3 \times 10^{-12}}{6 \times 10^{-5}}$



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**Example 5**

Add the following:

a. $(5 \times 10^{78}) + (2 \times 10^{78})$

b. $(8 \times 10^{6}) + (2 \times 10^{6})$

c. $(3 \times 10^{4}) + (2 \times 10^{3})$



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**Rational Exponents**

**Video 1: How to work with rules like $x^{\frac{1}{n}}$ or $x^{\frac{m}{n}}$ or $x^{-\frac{m}{n}}$**

In this video, we will learn how to evaluate expressions that look like this:

$8^{\frac{1}{3}}$ or $25^{\frac{3}{2}}$

Define $x^{\frac{1}{n}}$ and show how to use this rule to evaluate $25^{\frac{1}{2}}$

**<u>Examples</u>** Evaluate the following:

a. $125^{\frac{1}{3}}$

b. $32^{\frac{1}{5}}$

c. $9^{\frac{1}{2}}$

d. $(-9)^{\frac{1}{2}}$

e. $-9^{\frac{1}{2}}$

f. $\left(\frac{27}{125}\right)^{\frac{1}{3}}$



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Define $x^{\frac{m}{n}}$ and show how to use this rule to evaluate $25^{\frac{3}{2}}$

**<u>Examples</u>** Evaluate the following:

a. $27^{\frac{2}{3}}$

b. $4^{\frac{3}{2}}$

c. $16^{\frac{5}{4}}$

d. $-8^{\frac{2}{3}}$

e. $(-8)^{\frac{2}{3}}$

f. $(-16)^{\frac{3}{4}}$

g. $-81^{\frac{3}{4}}$

h. $\left(\frac{125}{27}\right)^{\frac{2}{3}}$



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Define $x^{-\frac{m}{n}}$ and show how to use this rule to evaluate $25^{-\frac{3}{2}}$

**<u>Examples</u>** Evaluate the following:

a. $25^{-\frac{1}{2}}$

b. $27^{-\frac{1}{3}}$

c. $16^{-\frac{3}{4}}$

d. $4^{-\frac{5}{2}}$

e. $\left(\frac{25}{4}\right)^{-\frac{3}{2}}$

f. $(-25)^{-\frac{3}{2}}$

g. $(-27)^{-\frac{2}{3}}$

h. $-16^{-\frac{5}{4}}$



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**Lesson: Using Exponent Rules with Rational Exponents**

First, let's review the main exponent rules:

Next, let's do a quick review on operations with fractions. Complete the following operations:

a. $\frac{2}{3} \times \frac{5}{6}$

b. $\frac{2}{5} \times 10$

c. $\frac{2}{3} + \frac{2}{5}$



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Now we will combine all of these ideas into the following examples!

**<u>Examples</u>** Simplify the following. Your final answer should not include any negative exponents.

a. $x^{\frac{2}{3}} * x^{\frac{1}{5}}$

b. $\left(4x^{\frac{2}{3}}y^{6}\right)^{\frac{3}{2}}$

c. $\frac{x^{\frac{2}{3}}y^{-\frac{1}{5}}}{x^{\frac{1}{6}}y^{\frac{2}{5}}}$



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d. $\left(\frac{125x^4y^{-\frac{1}{3}}}{x^{-\frac{1}{2}}y^{\frac{2}{5}}}\right)^{-\frac{2}{3}}$

e. $\left(\frac{4x^6y^{-\frac{1}{3}}}{x^{-3}y^{\frac{1}{2}}}\right)^{-\frac{3}{2}}$



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**Lesson: Simplifying Radicals with Rational Exponents**

Convert the following radicals to rational exponents, simplify, and then rewrite in radical form.

a. $\sqrt{7^2}$

b. $\sqrt{9^2}$

c. $\sqrt[3]{x^{15}}$

d. $\sqrt[4]{x^2}$



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**Rational Exponents**
**Video 3: Convert a radical expression to exponential form**

**<u>Examples</u>** Convert the following expressions to exponential form, and then simplify if needed.

a. $\sqrt{7}$

b. $\sqrt[3]{8}$

c. $\sqrt{5^2}$

d. $\sqrt[3]{7^6}$

e. $\sqrt[4]{x^{16}}$

f. $\sqrt[6]{x^2}$



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**Lesson: A Review of Exponent Rules for Algebra Students**

Power Rule: $x^m * x^n =$

Product Rule: $(x^m)^n =$

Quotient Rule: $\frac{x^m}{x^n} =$

Negative Exponents

$x^{-n} =$

$\frac{1}{x^{-n}} =$

$\left(\frac{x}{y}\right)^{-n} =$

Simplify the following using exponent rules. Your final answer should not contain negative exponents.

a. $3x^7y^{-4} * 5x^{-2}y^9$

b. $\left(\frac{2x^4y^5}{3z^5}\right)^3$

c. $\frac{10x^4y^7}{2x^2y^3z^{-5}}$

d. $\frac{4x^3y^{-2}}{6x^{-4}y^{-5}}$

e. $\frac{8x^{-3}y^4z^{-3}}{10x^5y^{-2}z^{-7}}$

f. $\left(\frac{3x^{-3}y^{-2}}{6x^2y^4}\right)^{-2}$

g. $\left(\frac{2x^{-7}z^3}{3x^{-2}y^{-3}z^5}\right)^{-3}$

h. $\left(\frac{5x^{-2}y^{-3}z^3}{x^3y^4z^{-7}}\right)^{-2}$



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**Lesson: An Intro to Polynomials**

We can extend our knowledge of exponents to another algebraic topic – polynomials

Before we define what a polynomial is, it is usually easier to see a few examples of what is or is not a polynomial.

The following ARE polynomials:

- $x^2 - 3x + 8$
- $x^7 - 5x^4 + 2x^3 + x$
- $9$
- $7x + \pi$
- $x^3 + \frac{2}{3}$

The following ARE NOT polynomials:

- $\sqrt{x} - 4$
- $4x^{-4} - 7x^{-3}$
- $\frac{4}{x} - \frac{2}{x+3}$

Now we can more formally define a polynomial.

## Polynomial

An expression with real numbers, variables, and whole number exponents.

## Descending Order

Ordering a polynomial from the largest exponent to the smallest exponent, and any freestanding number without a variable is last

## Example 1

Put the polynomials in descending order.

a. $x^3 + 4x^5 - 2x^7 + 3x^2 + 5x^8$

b. $x^2 - x^3 + x^4 - x^5 - x^6 + 3$



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**Term**
Any part in a polynomial that is separated by a "+" or "-" sign is considered to be its own term

**Variables**
The letters in a polynomial that represent numbers

**Coefficients**
The numbers attached to the variables

**Constant Term**
The freestanding number within a polynomial that is not attached to any variables

**Example 1**
Examine the polynomials, then determine the variables, coefficients, and constant terms

| Polynomial | Terms | Variable(s) | Coefficient(s) | Constant |
|---|---|---|---|---|
| $2x + 3$ | | | | |
| $3x^2 - x + 5$ | | | | |
| $5$ | | | | |
| $2x^2y + 3y - x$ | | | | |



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## Degree of a Term

The degree of a term is the sum of the exponents of that term.

## Example 2

Identify the degree of each term

| Term | Degree | Term | Degree |
|---|---|---|---|
| $x^4$ | | $x^4y^3$ | |
| $3y^6$ | | $x^7y^9$ | |
| $x$ | | $x^4y^2z^5$ | |
| $6$ | | $xy^4$ | |

## Degree of a Polynomial

The degree of a polynomial is the same as the degree of the term of highest degree within that polynomial

## Leading Coefficient

The leading coefficient is the coefficient attached to the term of highest degree in a polynomial

## Trinomial

A polynomial with 3 terms

## Binomial

A polynomial with 2 terms

## Monomial

A polynomial with 1 term

## Example 3

Identify the degree of each polynomial and the type of polynomial it is.

| Polynomial | Degree of Each Term, Respectively | Deg of Polynom | Leading Coefficient | Type of Polynomial |
|---|---|---|---|---|
| $x^5 + 4x^3 - 2x$ | | | | |
| $-5x^2 + 18x$ | | | | |
| $6x + 2x^4 - 3x^2$ | | | | |
| $x^5 + 2x^7 - 4$ | | | | |
| $8$ | | | | |



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**Evaluating Polynomials**

Evaluating polynomials is not unlike evaluating other expressions we've seen in this course.

**Example 4**

Evaluate the polynomials for $x = -2$.

a. $x^2 + 3x + 1$

b. $x^3 - x^2 - 3x + 4$

c. $\frac{1}{4}x^3 - \frac{1}{2}x^2 + 5x$



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**Lesson: How to Add and Subtract Polynomials**

To add or subtract polynomials, combine like terms.

**<u>Examples</u>**

Perform the following operations.

a. $(2x + 4) + (5x^2 - 3x + 1)$

b. $(5x^2 - 3x + 1) - (x^3 - 2x^2 - x + 1)$

c. $(3x^3 + 2x^2 - 5x) + (4x^3 + 5x^2 + 2x)$

d. $(2x^2 + 5x - 4) - (x^3 - 2x^2 + 3x - 5)$



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**Lesson: How to Multiply Polynomials**

To begin, let's first focus on the process of distributing a monomial with a polynomial.

**<u>Examples</u>** Multiply the following:

a. $3x(4x^2 + 5x - 7)$

b. $x^2y^4(3x^2 + 2xy - 8y^4)$

c. $\frac{1}{2}x^3y^4(4xy^3 - 2x^3y + 8xy)$

Now write down the process for how to multiply $(x + 2)(x^2 + 3x)$



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**Examples** Multiply the following:

a. $(x+3)(x+2)$

b. $(3x+1)(x-6)$

c. $(x^2+4)(x-2y)$

d. $(x+4)(x-4)$

e. $(x+3)(x^2+4x+5)$

f. $(2x^2+3x+1)(3x^2-4x+2)$



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How do you multiply $(x+5)^2$?

**<u>Examples</u>** Multiply the following:

a. $(x+2)^2$

b. $(2x^2+1)^2$

c. $(3x^2+x+5)$

How do you multiply 3 polynomials together, such as $(x+2)(x+1)(x-3)$?



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**Examples** Multiply the following together:

a. $(x-3)(x+4)(x-1)$

b. $(2x+1)(x-2)(x-1)$



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**Lesson: The Basics of Dividing Polynomials**

In this section, we will discuss how to divide polynomials that are in these forms:

$\frac{x^2+3x}{x}$ or $(4x^7y^3 + 8x^4y^2 - 12x^5y^5) \div 2x^2y$

To divide in these cases, you can just use rules of exponents. Sometimes you will want to separate the problem into separate fractions.

**<u>Examples</u>**

a. $\frac{x^2+3x}{x}$

b. $\frac{5x^4-10x^3+15x^2}{5x}$

c. $(27x^4y^2 + 18x^3y) \div 9x^2y$

d. $(4x^7y^3 + 8x^4y^2 - 12x^5y^5) \div 2x^2y$



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**Lesson: Long Division for Polynomials**

In this section, we will discuss how to divide polynomials that are in these forms:

$(x^2 + 5x + 6) \div (x + 3)$ or $(2x^2 + 5x + 3) \div (2x + 3)$

In these types of problems, we need to use long division. In another video, we talked about how to divide polynomial problems like these:

$\frac{4x^3+2x^2}{2x}$

But we did not need long division. What is different about these types of problems that require long division?

Show how to use long division to divide $(x^2 + 5x + 6) \div (x + 3)$



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**Examples** Divide the following.

a. $(x^2 + 7x + 10) \div (x + 5)$

b. $(2x^2 + 5x + 3) \div (2x + 3)$

c. $\frac{x^2-11x+30}{x-5}$



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Now we will look at what to do when there is a remainder.

Divide the following: $(x^2 + 3x + 1) \div (x + 2)$

Now show how to divide when there are "missing" terms, such as with $(x^4 - 81) \div (x^2 - 9)$



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**Examples** Divide.

a. $(5x^2 - 10x^3 + 8 - 9x) \div (x + 2)$

b. $(x^3 + 1) \div (x + 1)$

c. $(5x^3 - 4x^2 + 1) \div (x + 2)$



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**Lesson: The Greatest Common Factor**

## The Greatest Common Factor (GCF)

The Greatest Common Factor is the biggest common factor between a set of terms.

## Examples

Find the greatest common factor in the following polynomials, then factor the polynomial.

a. Polynomial: $3x + 3y$

   GCF:

   Factor out the GCF:

b. Polynomial: $3xy + 6xz$

   GCF:

   Factor out the GCF:

c. Polynomial: $10x^2 - 25xy + 15xz$

   GCF:

   Factor out the GCF:

d. Polynomial: $3x^7y + 8x^3z^5 - 5y^4z^3$

   GCF:

   Factor out the GCF:

## Examples



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Factor the following polynomials.

a. $12xy^2 + 10x^2y^4 - 14xy^5$

b. $5xy - 10yz + 12xz$

c. $3x + 6x^2 - 12x^3$

d. $5x^4y^2z^5 + 10x^2y^4z^6 - 25x^5yz^3$



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**Lesson: Factoring by Grouping**

Grouping can best be understood with the following example.

Factor $ax + ay + bx + by$ by grouping:

## Examples

Finish the following examples where the grouping had been started.

a. $3(x - y) + 2z(x - y)$

b. $2a(3b + 2c) - 5(2c + 3b)$

c. $4x(y - z) - 3(z + y)$

d. $4z(4x - 7) - (4x - 7)$



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**Examples**

Factor the following.

a. $5x + 10y + xz + 2yz$

b. $10x^2 + 15x^3y - 2 - 3xy$

c. $3xy + 4y - 27x - 36$

d. $3xy + 6x + 21y + 42$

e. $3x^2 - 48y + 8x - 18xy$



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**Lesson: The Basics of Factoring Trinomials of the Form $x^2 + bx + c$**

Factoring trinomials will require us to multiply polynomials, so first we will review this skill.

## Examples

Multiply.

a. $(x + 3)(x - 5)$

b. $(2x - y)(5x + 2y)$

## Definition of a Trinomial

A trinomial is a polynomial with three terms.

## How to Factor a Trinomial Using Trial and Error

Show how to factor $x^2 + 3x + 2$

Show how to factor $x^2 + 7x + 12$



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**Examples**

Factor the following.

a. $x^2 + 6x + 5$

b. $x^2 + 12x + 11$

c. $x^2 - 8x + 15$

d. $x^2 - 13x + 30$

e. $x^2 + 10x - 24$

f. $x^2 - 5x - 14$

g. $2x^2 + 16x + 30$

h. $5x^4 + 55x^3 - 45x^2$



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**Lesson: The Basics of Factoring Trinomials of the Form $-x^2 + bx + c$**

How can you factor $-x^2 - 5x - 6$?

What about $-x^2 - 15x - 36$?

**<u>Examples</u>**

Factor the following.

a. $-x^2 + 13x - 30$

b. $-x^2 + 16x - 55$

c. $-x^2 - 2x + 3$



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**Lesson: The Basics of Factoring Trinomials of the Form $x^2 + bxy + cy^2$**

How do you factor $x^2 + 5x + 6$?

How do you factor $x^2 + 5xy + 6y^2$?

**<u>Examples</u>** Factor the following.

a. $x^2 + xy - 12y^2$

b. $x^2 - 2xy + y^2$

c. $3x^2 + 21xy + 18y^2$



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**Lesson: The Basics of Factoring Trinomials of the Form $\boldsymbol{ax^2 + bx + c}$**

In this lesson we will focus on trinomials that do not have a leading coefficient.

To begin, first let's get warmed up by factoring the following using trial and error:

a. $x^2 + 7x + 12$

b. $x^2 - 3xy + 10y^2$

How do you factor $2x^2 + 5x + 3$ using trial and error?

How do you factor $3x^2 + 16x + 5$ using trial and error?

Another way to factor $3x^2 + 16x + 5$ is by grouping. Show how to use this technique below:



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**Examples** Factor the following:

a. $7x^2 - 11x + 4$

b. $3x^2 + 10x + 8$

c. $5x^2 - 11x - 36$

d. $3x^2 - 17xy - 6y^2$

e. $-10x^2 + 19x - 6$



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**Lesson: Factoring Special Trinomials**

How do you factor $x^2 + 6x + 9$?

What's another way to write this factorization?

What do you call this type of trinomials?

**<u>Examples</u>** Factor the following:

a. $x^2 + 12x + 36$

b. $9x^2 - 12x + 16$

c. $x^2 - x + \frac{1}{4}$

d. $36x^2 - 60xy + 25y^2$



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How do you factor $x^2 - 4$?

What is this type of factoring called?

**<u>Examples</u>** Factor the following:

a. $x^2 - 25$

b. $x^2 - 16$

c. $x^2 - 1$

d. $25x^2 - 36$

e. $16x^2 - 81y^2$

f. $16 - x^2$

g. $25y^2 - 64x^2$

h. $\frac{49}{64} - x^2$



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**Lesson: How to Factor The Sum and Difference of Cubes**

Write down the formula for the sum of cubes

$a^3 + b^3 =$

Write down the formula for the difference of cubes

$a^3 - b^3 =$

Show how to factor $x^3 + 8$

Show how to factor $8x^3 - 27$

**<u>Examples</u>** Factor:

a. $x^3 - 1$

b. $x^3 - \frac{1}{8}$

c. $x^3 + 64$



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**Review: An Overview of All the Types of Factoring**

We will review all of the different types of factoring by presenting an example of each type.

**Greatest Common Factor**

1. $4x^2 + 8xy$

2. $6x^3y^2 - 8x^2y^5 + 10x^5y$

**Grouping**

3. $3x + 6y - xz^2 - 2yz^2$

4. $12x^2 - 2y + 3x - 8xy$

**Basic Trinomials of the Form $\boldsymbol{x^2 + bx + c}$**

5. $x^2 + 7x + 10$

6. $x^2 - 2x - 15$



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**Trickier or Special Forms of Trinomials of the Form $x^2 + bx + c$**

7. $-x^2 - 3x + 28$

8. $6x^2 + 54x + 120$

9. $x^2 - 6xy + 8y^2$

10. $x^2 - 8x + 16$

11. $x^2 - 25$

12. $x^2 - \frac{9}{4}$



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**Trinomials of the Form $\boldsymbol{ax^2 + bx + c}$**

13. $2x^2 + 11x + 5$

14. $7x^2 - 15x + 8$

15. $-3x^2 - 15x^2 + 198x$

16. $16x^2 - 24xy + 9y^2$

**More Practice with the Difference of Squares**

17. $x^2 - 81$

18. $25x^2 - 4$

19. $\frac{1}{4}x^2 - 9y^2$



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**Sum and Difference of Cubes**

20. $x^3 + 8$

21. $x^3 - 27$

22. $27x^3 - 1$

23. $8x^3 + 64y^3$



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**Lesson: Factoring by GCF and Trinomials (the more memorable types of factoring)**

How do you factor $3x + 3y$?

What type of factoring is this called?

**<u>Examples</u>** Show how to use the GCF to factor the following:

a. $4x + 12x^2 - 8x^3$

b. $5x^2y + 10x^7y^3 - 15x^5y^2$



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**Examples** Show how to use the GCF to factor the following:

a. $4x^2 + 12xy - 3y^2$

b. $15x^3y^4 - 25x^6y^2 + 20x^7y^8$

What is a trinomial?

Show how to multiply: $(x + 2)(x + 7)$



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Show how to factor $x^2 + 3x + 2$ using the *trial and error* method. Write down some reasoning explaining how this method works.

**<u>Example</u>** Factor the following using trial and error.

a. $x^2 + 5x + 6$

b. $x^2 - 7x + 12$

c. $x^2 - 25$



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Show how we can factor $2x^2 + 5x + 3$ using *trial and error.*

**<u>Examples</u>** Factor the following.

a. $5x^2 + 7x + 2$

b. $3x^2 + 13x - 10$



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**Lesson: Factoring by Grouping and the Sum and Difference of Cubes**

Show how to use grouping to factor: $ax + ay + bx + by$

What are some key hints that you probably need to use the grouping technique?

Write out the steps of how to use grouping for this problem:

**<u>Example</u>** Show how to finish the factoring for the following:

a. $5(x - 2y) + z^2(x - 2y)$

b. $6x^2(3y - z) - (3y - z)$



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**Examples** Use grouping to factor the following:

a. $4x^3 - 12xy + x^2y - 3y^2$

b. $xy + 4x - y - 4$

For the next example we will use $12x^2 - 15y + 18x - 10xy$

What is the issue when we try to use grouping?

How can we resolve this issue and still use grouping?



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Write the sum and difference of cubes formulas below:

**<u>Examples</u>** Factor the following:

a. $x^3 + 8$

b. $y^3 - 1$

c. $8x^3 + 27y^3$



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**Lesson: A Few More Tricks for Factoring**

List all the types of factoring that you might have to use:

It is always possible that you will have to combine these methods. There are also a few other tricks you can try, which we will discuss in this lesson.

Using **<u>Grouping</u>** with Trinomials

Write out *instructions* on how to use grouping for $4x^2 + 16x + 15$



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Below are a few challenging examples of factoring. Write out any notes of how to work these out that might be helpful to you. Remember, you are expected to use these notes as a resource to help you finish your homework – take the time to write down reminders that will be meaningful and helpful to you!

a. $-x^2 - 15x - 36$

What trick can you use here? Show how to use it.

b. $4y^3 - 28y^2 + 48y$

What trick can you use here? Show how to use it.

c. $-6x^2 + 10x - 3$



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d. $x^2 + 7xy + 12y^2$

How should you deal with having the multiple variables?

Finally, let's work with a few trickier examples of the difference of squares. Follow the pattern:

a. $x^2 - 16$

b. $x^2 - 25$

c. $x^2 - \frac{16}{25}$



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d. $16 - x^2$

e. $25x^2 - 16y^2$

f. $\frac{16}{25}x^2 - \frac{4}{9}y^2$



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**Lesson: Synthetic Division and the Remainder Theorem**

The outcome of division can be restated using the division algorithm.

State the division algorithm:

Show how we use polynomial long division with $(x^2 - 3x - 1) \div (x + 2)$, and then restate the result using the format of the division algorithm.



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Now show how to do the operation $(x^2 - 3x - 1) \div (x + 2)$ with synthetic division:

Restate this result using the division algorithm:

**<u>Example</u>** Use synthetic division to perform: $\frac{4x^3-3x^2-7}{x+1}$. Rewrite the result using the division algorithm.



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So what are 3 keys about using synthetic division to remember?

What is the remainder theorem?



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**<u>Example</u>** Use the remainder theorem to find the desired functional values: $f(x) = -3x^4 + 5x^3 - 30x + 20, f(2)$. What is the equivalent division problem associated with this? Write the result of your division using the division algorithm.



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**Example** Use the remainder theorem to find the desired functional values: $f(x) = 3x^2 + 5, k = 3 - i$



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**Lesson: Synthetic Division and Potential Zeros of Polynomials**

What is a “zero” of a polynomial?

How does the remainder theorem help us to determine if we have a zero?

What are some other terms that we use for “zero” of a polynomial?



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**<u>Example</u>** Determine if the given k is a zero of the polynomial. If it is not, then what is the value of $f(k)$?

a. $f(x) = x^2 - 4x - 5; k = 5$

b. $f(x) = 2x^3 - 3x^2 + 5; k = 1$

c. $f(x) = x^2 - 2x + 2; k = 1 + i$



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**Rational Expressions**
**Video 1: The Basics**

What are rational expressions?

What do rational expressions look like? Write down some examples:

**<u>Simplifying Rational Expressions</u>**
Take down some notes on this:



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**Examples** together:

a. $\frac{25x^7y^9}{15x^3y^2z^7}$

**Examples** on your own:

b. $\frac{21x^7y^9}{14y^8z^3}$

c. $\frac{16x^4y^7z^3}{15x^3y^4z^8}$



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**Examples** together:

a. $\frac{x^2+5x+6}{x^2-9}$

**Examples** on your own:

b. $\frac{x^2+7x+10}{x^2+10x+25}$

c. $\frac{x^2-x-6}{x^2-9x+14}$

d. $\frac{x^2+10x+16}{x^3+2x^2+x+2}$

Play the video and check your work.



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**Examples** on your own:

e. $\frac{x^3-8}{x^2-9x+14}$

f. $\frac{10x^3+15x^7}{10x^3+40x^2+40x}$



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**Rational Expressions**
**Video 2: Other Things to Know**

When it comes to fractions, where are you allowed to have a zero? In the numerator (top) or denominator (bottom)?

Therefore, what do we have to worry about with rational expressions?

**Examples** on your own:

Evaluate the following rational expressions for $x = 2$.

a. $\frac{x^2+6x+8}{x^2+8x+15}$

b. $\frac{x^2+9x+20}{x^2+3x-10}$



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**Example** on your own: (evaluate for x=2)

c. $\frac{x^2-4}{x^2+9x+14}$

**To summarize…**

When a rational expression has a zero in the numerator, what does this mean?

When a rational expression has a zero in the denominator, what does this mean?

Define the term **domain:**

What are we concerned with for the domain of rational expressions?



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**A review of interval notation:**

**Examples**:

Write the following in interval notation:

a.

b.

c.



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**Example** together:

Determine i) where the expression equals zero ii) where the expression is undefined iii) the domain of the expression

a. $\frac{x+3}{x-5}$

b. $\frac{2x+1}{x^2+7x+10}$

**Examples** on your own:

Determine i) where the expression equals zero ii) where the expression is undefined iii) the domain of the expression

c. $\frac{x-3}{x-1}$

d. $\frac{x^2-81}{x^2+5x+6}$



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**Rational Expressions**
**Video 3: The Negative One Trick with Simplifying**

In its current form, can you simplify: $\frac{x-3}{3-x}$? Why or why not?

Show how you can manipulate this expression to simplify it.

There is no formal name for this trick. Some might refer to it as a *multiplicative property of -1,* but I refer to as "that -1 trick." I know, it's not exactly formal sounding!

Let's do one more example of this. Simplify $\frac{5-x}{3x-15}$



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The first thing to get used to it how to rewrite expressions using this trick.

**Examples**

Rewrite the order of each expression by factoring out -1 first.

a. $4 - x$

b. $3x - 1$

c. $x - 6$

Now let's use this to simplify rational expressions.

**Examples** Simplify the following.

a. $\frac{6x-12}{2-x}$



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**Examples** Simplify the following.

b. $\frac{4x-20}{30x-6x^2}$

c. $\frac{x^2-4}{2-x}$

d. $\frac{x-6}{36-x^2}$

e. $\frac{5-x}{x^2-7x+10}$



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**Examples** Simplify the following.

f. $\frac{x^2-10x+25}{25-x^2}$

g. $\frac{27-x^3}{x^2-5x+6}$

h. $\frac{x^3-3x^2+5x-15}{6-3x}$



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**Examples** Simplify the following.

i. $\frac{54-9x}{x^2-36}$

j. $\frac{8-x^3}{x^2-9x+14}$

k. $\frac{y^3-x^3}{x^2-y^2}$



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**Example/Question: Troubleshooting the -1 Trick**

Explain the actual DETAIL behind WHY $\frac{4}{5-x} = -\frac{4}{x-5}$

**<u>Examples</u>** Write an equivalent fraction for the following:

a. $\frac{5}{x-8}$

b. $\frac{x-3}{x+2}$



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**Lesson: Multiplying Rational Expressions**

How do you multiply the fractions $\frac{a}{b} \times \frac{c}{d}$? Include a one-sentence explanation.

**<u>Examples</u>** Multiply the following.

a. $\frac{4}{5} \times \frac{7}{9}$

b. $\frac{2x^3}{7y^2} \times \frac{5x^5}{9y^7}$



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Show how to make cancellations before multiplying $\frac{12}{25} \times \frac{15}{16}$

We can apply this same logic to rational expressions.

**<u>Examples</u>** Multiply the following and simplify when possible.

a. $\frac{21x^7}{15y^3} \times \frac{5y^8}{42x^4}$

b. $\frac{2x^2+5x+3}{x^2-25} \times \frac{x^3+125}{4x^2+6x}$



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**Lesson: Dividing Rational Expressions**

How do you divide $\frac{a}{b} \div \frac{c}{d}$? Write a one sentence explanation of how to do this too.

**<u>Example</u>** Divide the following: $\frac{5x}{9} \div \frac{2}{7x^4}$

When can you make cancellations in the process?



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**<u>Examples</u>** Divide.

a. $\frac{12x^7}{45} \div \frac{42x^3}{15}$

b. $\frac{x^2+4x+4}{x+5} \div (x+2)^5$

c. $\frac{x^3+1}{6x^3-6x^2+6x} \div \frac{x^2-1}{8}$



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Explain why $\frac{\frac{5x+5y}{24}}{\frac{15x^2+15xy}{12}}$ is both a fraction and a division problem.

**<u>Example</u>** Divide: $\frac{\frac{16x^2-25}{x^5}}{\frac{36x-45}{9x^7}}$



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**Lesson: Finding the Least Common Denominator for Addition and Subtraction – A Quick Review for Fractions (with only numbers)**

We are going to work on adding and subtracting rational expressions with different denominators. Before we do that, we are going to remind ourselves how to find common denominators for fractions that only have numbers.

The simplest case: what is the least common denominator between the fractions $\frac{1}{2}$ and $\frac{1}{3}$?

Why is that the common denominator?

Now let's look at something slightly more complicated. What is the least common denominator between $\frac{1}{6}$ and $\frac{1}{10}$?

First, take a guess. It's totally fine if this guess is wrong, but just make a guess.

*Why* is this your guess? What did you do?



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There is a specific procedure that you want to use when finding least common denominators in this class. Write down instructions that were given in the video, but write them our in a way that makes sense to you.

Repeat this procedure to find the least common denominator $\frac{1}{15}$ and $\frac{1}{10}$.



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Repeat this procedure to find the least common denominator $\frac{1}{12}$ and $\frac{1}{8}$.

Now we will focus on how to write fractions with this desired denominator.

The simplest case: using the least common denominator you found for $\frac{1}{2}$ and $\frac{1}{3}$, show how to write 2 equivalent fractions with the desired denominator. Write out instructions on how you do this. Summarize what was discussed in the explanation in the video.



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Using the least common denominator you found for $\frac{1}{6}$ and $\frac{1}{10}$, show how to write 2 equivalent fractions with the desired denominator. Write out instructions on how you do this. Summarize what was discussed in the explanation in the video.

Using the least common denominator you found for $\frac{1}{12}$ and $\frac{1}{8}$, show how to write 2 equivalent fractions with the desired denominator. Write out instructions on how you do this. Summarize what was discussed in the explanation in the video.



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**Lesson: Finding the Least Common Denominator for Rational Expressions**

The simplest case: what is the least common denominator between the fractions $\frac{1}{2x}$ and $\frac{1}{3y^2}$?

**<u>Examples</u>** Find the least common denominator between the following fractions.

a. $\frac{1}{2x^2}, \frac{1}{3xy^2}$

b. $\frac{1}{2x^2y^5}, \frac{1}{3x^7y^2}$

c. $\frac{1}{2x^3y^5z^2}, \frac{1}{3x^5y^7z^8}$



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**Examples** Find the least common denominator for the following.

a. $\frac{1}{14x^3y^2}, \frac{1}{21x^2y^7}$

b. $\frac{1}{24x^3y}, \frac{1}{12x^2y^5}$

What is the least common denominator between $\frac{1}{x+2}$ and $\frac{1}{x}$?



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What is the least common denominator between $\frac{1}{x^2-4}$ and $\frac{1}{x^2+5x+6}$?

**<u>Examples</u>** Find the least common denominator between the following fractions.

a. $\frac{3}{5x-10}, \frac{1}{7x-14}$

b. $\frac{1}{x^2-12x+35}, \frac{1}{x^2+x-20}$



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**Examples** Find the least common denominator between the following fractions.

a. $\frac{1}{x-3}, \frac{1}{3-x}$

b. $\frac{1}{x^2-4}, \frac{1}{2-x}$

c. $\frac{1}{x^2+9x+20}, \frac{1}{x^2-25}, \frac{1}{x^2-x-20}$



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**Lesson: Writing Equivalent Rational Expressions with the Same Denominator**

**<u>Examples</u>** Find the least common denominator between the following rational expressions, and then write each rational expression with it's equivalent and the LCD.

a. $\frac{2}{15x^2y^5}, \frac{4}{9x^3y^2}$

b. $\frac{5}{x+3}, \frac{4}{x}$



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**<u>Examples</u>** Find the least common denominator between the following rational expressions, and then write each rational expression with it's equivalent and the LCD.

a. $\frac{x}{x^2+9x+18}, \frac{2}{x^2+5x+6}$

b. $\frac{x}{5x^2-30x}, \frac{x-2}{2x^2-17x+30}, \frac{2}{2x^2-5x}$



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**Lesson: Adding and Subtracting Rational Expressions**

Show how to add or subtract the following:

a. $\frac{5}{x+2} - \frac{3}{x+2}$

b. $\frac{5}{x-4} + \frac{7}{x}$

c. $\frac{5}{12xy^2} - \frac{7}{8x^3y}$



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d. $\frac{x}{x+3} - \frac{2}{x^2-9}$

e. $\frac{x}{x^2-25} - \frac{5}{x^2-3x-40}$



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**Example/Question: When Are You Allowed to Cancel with Rational Expressions?**

What is the wrong way to cancel $\frac{(x+1)-3(x-3)}{2(x-3)+4(x+1)}$ and explain why you can't cancel like that? How can you tell it's wrong?

What is the difference between $\frac{(x+1)-3(x-3)}{2(x-3)+4(x+1)}$ and $\frac{3(x+1)(x-3)}{2(x-3)(x+1)}$?

How do you simplify $\frac{(x+1)-3(x-3)}{2(x-3)+4(x+1)}$?



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**Lesson: Complex Fractions**

First let's review how to simplify a complex fraction we've already discussed. Show how to simplify:

$$\frac{\frac{2x}{x+5}}{\frac{2}{x^2-25}}$$

Explain what makes the problem below different from the one we just reviewed above.

$$\frac{\frac{1}{2}-\frac{1}{3}}{\frac{5}{6}+\frac{1}{3}}$$



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Outline the steps required to simplify. Write out the instructions as they were explained in the video.

$$\frac{\frac{1}{2}-\frac{1}{3}}{\frac{5}{6}+\frac{1}{3}}$$

**<u>Example</u>** Simplify the following

$$\frac{\frac{7}{9}-\frac{1}{3}}{1+\frac{5}{9}}$$



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**Examples** Simplify the following.

a. $\dfrac{\frac{5}{x}+\frac{2}{y}}{\frac{1}{y}-\frac{3}{x}}$

b. $\dfrac{\frac{6}{x+3}-\frac{4}{x-1}}{\frac{2}{x-1}+\frac{5}{x+3}}$



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**<u>Example</u>** Simplify the following.

$$\frac{1-\dfrac{4}{t+5}}{\dfrac{4}{t^2-25}+\dfrac{t}{t-5}}$$



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**Complex Fractions**
**Video 2: More Complicated Examples**

Complex fractions look like this:

$$\frac{\frac{4x}{15y^2}}{\frac{6x^4}{25y^8}} \quad \text{or} \quad \frac{\frac{1}{3}-\frac{1}{6}}{\frac{3}{2}+\frac{5}{6}} \quad \text{or} \quad \frac{\frac{4}{x^2}-\frac{4}{y^2}}{\frac{1}{xy}+\frac{5}{y}}$$

First we will discuss the basics.

**<u>Examples</u>**

a. $\frac{\frac{1}{x^2}-\frac{1}{y}}{\frac{3}{x}-\frac{4}{y^2}}$

b. $\frac{\frac{3}{x}-x}{\frac{2}{x}+x}$



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**Examples**

a. $\dfrac{\frac{3}{x-1}-\frac{2}{x+3}}{\frac{5}{x+3}-\frac{2}{x-1}}$

b. $\dfrac{\frac{5}{x-2}-\frac{3}{x+5}}{\frac{1}{x+5}+\frac{2}{x-2}}$



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**Examples**

a. $\dfrac{1-\frac{1}{x+5}}{\frac{x}{x-5}+\frac{4}{x^2-25}}$

b. $\dfrac{1+\frac{12}{x^2-16}}{\frac{x}{x+4}-\frac{6}{x-4}}$

c. $\dfrac{\frac{1}{x+2}+\frac{4}{x^2-4}}{\frac{3}{x-2}+1}$



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**Lesson: Solving Linear Equations, A Review of the Basics**

To begin, let's review some of what we've learned about working with equations and using graphs to solve them.

**Linear Equation in One Variable**
An equation that can be written in the form ax+b=0 where a and b are real numbers

**Solution**
A value that when substituted into the variable of an equation makes the equation true

**Example 1**
Determine if $x = -1$ is a solution to the following equations.
a. $5 - 4x = 9$

b. $2x + 3 = 1 - 3x$

We will now work with some basic ideas of solving linear equations. There are two key concepts we must understand to do this.

**The Addition Principle**
Let a, b, and c be real numbers. If a=b then:



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**Example 1**

Use the addition principle to solve the following:

Choose at least one from each category. Label the examples you choose as a, b, c, etc to keep the class organized.

a. $x - 4 = 2$

b. $x + 3 = 3$

c. $x - \frac{1}{3} = \frac{1}{2}$

**The Multiplication Principle**

Let a, b, c be real numbers. If a=b then:

**Example 2**

Use the multiplication principle to solve the following:

a. $3x = 15$

b. $5x = 0$

c. $-\frac{2}{5}\mathrm{x} = 10$



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**Example 3**

Solve the following examples:

a. $5 - 4x = 33$

b. $3x + 2 = 2 + 4x - 8$

c. $3(4 - 2x) = 3x + 6$



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**Lesson: Solving Linear Equations, No Solution and Infinite Solutions**

There are 2 special outcomes we have to watch out for:

**<u>An Example Where There Is No Solution</u>**

$3(2x - 1) = 2(2x + 1) - 2(3 - x)$

**<u>An Example Where There Are Infinite Solutions</u>**

$4x + 5 = 3(2x + 1) - 2(x - 1)$

**<u>A Summary of the 3 Possible Solution Types for Linear Equations</u>**



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**Examples: Solving Linear Equations, Basic Examples with No Fractions or Decimals**

a. $\mathbf{3x - 5 = 7}$

b. $\mathbf{6 - 2x = 6}$

c. $\mathbf{4x + 7 = 2x - 9}$

d. $\mathbf{3 - 4x - 7 + 2x = 4 - 6x - 8 + 4x}$

e. $\mathbf{4x + 7 + 6x = 7(1 - 2x)}$

f. $\mathbf{3(5 - 2x) - 6(3 - 2x) = 5(1 + 2x)}$

g. $\mathbf{4(2x - 1) - 2(6 - 3x) = 2(3x - 4)}$

h. $\mathbf{3(2x - 5) - 4x + 1 = 4(x - 3) - 2(1 - 3x)}$



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**Lesson: Solving Linear Equations with Fractions**

To begin, we will look at 3 examples to demonstrate how to clear fractions without parentheses. Take notes as shown in the video.

a. $\frac{1}{2}x + \frac{1}{5} = \frac{6}{5}$

b. $\frac{1}{2}x + \frac{1}{5} - \frac{1}{10}x = 1 + \frac{3}{10}x - \frac{2}{5}$

c. $\frac{1}{2}x + 1 + \frac{1}{4}x = \frac{3}{4} + \frac{2}{3} + \frac{1}{3}x$



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Now we will look at how to clear fractions in equations with parentheses.

a. $\frac{3}{4}(3+2x)-\frac{1}{2}(5-3x)=\frac{5}{4}$

b. $\frac{1}{2}(x+2)+\frac{1}{3}(2x-5)=\frac{1}{3}(4+2x)$



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**Examples: Solving Linear Equations with Fractions (No Parentheses)**

a. $\frac{1}{5}x - \frac{1}{3} = \frac{2}{3}$

b. $\frac{1}{3}x - 1 + \frac{1}{5}x + \frac{2}{5} = \frac{2}{5}x + \frac{1}{5}$

c. $\frac{1}{4}x - \frac{1}{10} + \frac{1}{5}x - \frac{4}{5} = \frac{3}{20}x + \frac{1}{10} + \frac{3}{10}$

d. $\frac{2}{3}x + \frac{1}{3} - \frac{1}{2}x = \frac{5}{6}x - \frac{1}{6} - \frac{2}{3}x + \frac{1}{2}$



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**Examples: Solving Linear Equations with Fractions (With Parentheses)**

a. $\frac{1}{2}(3x-2)-\frac{2}{5}(2x-3)=\frac{1}{5}(x+1)$

b. $\frac{3}{4}(4x+5)+\frac{1}{2}(2-3x)=\frac{3}{4}(2x+1)$

c. $\frac{1}{2}(2x+3)-\frac{1}{3}(2x-1)=\frac{1}{4}(x+7)$

d. $\frac{3}{4}(2x-1)+\frac{5}{8}(2-4x)=\frac{1}{4}(2-4x)$



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**Lesson: Solving Linear Equations with Decimals**

To begin, we will look at 3 examples to demonstrate how to clear decimals without parentheses. Take notes as shown in the video.

a. $0.03x + 0.04 = 0.19$

b. $0.05x + 0.1 - 0.03x = 0.08$



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Now we will look at 3 examples to demonstrate how to clear decimals with parentheses. Take notes as shown in the video.

a. $0.4(x+2)-0.3(2x+1)=0.09$

b. $0.1x+0.03(4-2x)=0.5(x+3)$



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**Examples: Solving Linear Equations with Decimals (no parentheses)**

a. $0.1x - 0.06 - 0.2x - 0.04 = 0.3$

b. $0.3x - 0.6 + x - 0.2 = 0.6x - 0.8 + 0.7x$

c. $1 + 0.2x - 0.7 + 2x = 0.5 - 0.4x + 1.8 + 0.6x$



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**Examples: Solving Linear Equations with Decimals (with parentheses)**

a. $0.4(3x+1)-1.5(2x+1)=0.6(3-3x)$

b. $0.2(3x+1)-0.05(2x+8)=0.1(2x+4)$

c. $0.09(2-4x)-0.1(2x-1)=0.2(2-x)-0.06(2+6x)$



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**Lesson: A Quick Review of How to Solve Quadratic Equations**

What is the form of a quadratic equation?

How is this different from a quadratic function?

What are the 4 ways you can solve a quadratic equation?

Which technique should you use when solve a quadratic equation?

We will review these 4 techniques on the next page!

What type of answers are possible when you solve a quadratic equation?



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**Example** Show how to use each of the 4 different techniques for solving quadratic equations by using each one at most once for the following examples.

a. $5x^2 + 6x + 1 = 0$

b. $(x + 4)^2 = 12$

c. $x^2 - 4x + 8 = 0$

d. $x^2 - 5x + 1 = 0$



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Optional extra help: If you need a more thorough (and more complete) review of the methods we discussed today, watch these:

- A review of ALL the different types of factoring (just factoring, nothing about solving quadratic equations): https://youtu.be/nrI7dJRXe5I and https://youtu.be/yGY5eqZkJcw
- How to solve a quadratic equations by factoring: https://youtu.be/5uof6OEyod0
- How to solve quadratic equations by the square root property: https://youtu.be/a1-smI3Xx5w
- How to solve quadratic equations by completing the square: https://youtu.be/6jXpMTozmhI
- How to solve quadratic equations by the quadratic formula: https://youtu.be/ryp2rdkhks8
- A review that compares all the techniques on the same problem: https://youtu.be/2ny9HbQ8doY



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**A Quick Refresher of the 4 Ways to Solve Quadratics**

For all of these examples, we will solve $x^2 + 6x + 5 = 0$.

**Solving by Factoring**

**Solving by Completing the Square**

**Solving by Using the Quadratic Formula**

The quadratic formula is: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$



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One final way we can solve quadratics is by using the square root property.
Use the square root property to solve $(x+3)^2 = 12$



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**Choosing the Best Method to Solve a Quadratic**

There are 4 ways we can solve quadratics:

1.

2.

3.

4.

Using each of these methods only once, determine the best way to solve these quadratics:

a. $x^2 + 6x = 2$
b. $x^2 + 3 = 2x$
c. $(2x + 1)^2 = 49$
d. $3x^2 + 2 = 3x$



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**<u>Examples</u>** Using each of these methods only once, determine the best way to solve these quadratics:

a. $(3x-2)^2=-24$

b. $2x^2+5x-1=0$

c. $x^2-12=x$

d. $x^2-2x+5=0$



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**Lesson: A Quick Review of The Discriminant**

What is the discriminant?

Fill in the table below:

| If the discriminant is … | How many solutions are there? | What type of solution? |
|---|---|---|
| **Positive and a perfect square** | | |
| **Positive (not perfect square)** | | |
| **Zero** | | |
| **Negative** | | |

**Example** Use the discriminant to determine how many solutions there are to: $x^2 - 4 = 0$.

**Example** Use the discriminant to determine how many solutions there are to: $x^2 + 6x + 9 = 0$.



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**Lesson: Solving for Quadratic Variables**

**<u>Example</u>** Solve for t: $s = \frac{1}{2}gt^2$

**<u>Examples</u>** Solve for x: $kx^2 - tx = m$



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**Solving Quadratic Equations with Factoring**

In this video, we will look at how to solve equations by factoring, which looks like this:

$$x^2 + 5x + 6 = 0 \rightarrow (x + 2)(x + 3) = 0$$

Quadratic equations are equations of the form:

Where a, b, and c are real numbers and $a \neq 0$.

How would you solve the quadratic equation $x^2 + 7x + 6 = 0$ by factoring?

**<u>Examples</u>** Solve the following by factoring:

a. $x^2 - x - 12 = 0$

b. $x^2 + 9 = -6x$

c. $x^2 = 25$



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**<u>Examples</u>** Solve the following by factoring:

a. $x^2 + 7x = 0$

b. $\frac{1}{10}x^2 + \frac{1}{2}x + \frac{3}{5} = 0$

c. $2x^3 + 28x^2 + 90x = 0$



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**Solving Quadratic Equations using the Square Root Property**

In this video, we will look at how to solve equations by using the square root property, which looks like this:

$$(x+2)^2 = 9 \quad \rightarrow \quad x+2 = \pm 3$$

Quadratic equations are equations of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$.

What are the solutions to $x^2 = 25$?

In general, how do you solve equations of the form $Q^2 = k$, where Q is an expression and k is a constant? In other words, what is the **square root property**?

**<u>Examples</u>** Solve the following equations using the square root property:

a. $(x+3)^2 = 25$

b. $(x+3)^2 = 12$



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**Examples** Solve the following equations using the square root property:

a. $(5x-2)^2+5=29$

b. $(6x+7)^2+4=-41$



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**Solving Quadratic Equations by Completing the Square**

In this video, we will look at how to solve equations by factoring, which looks like this:

$x^2 + 6x = 7 \qquad b = 6, \quad \frac{1}{2}b = 3, \quad \left(\frac{1}{2}b\right)^2 = 9$

Quadratic equations are equations of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$.

Before we begin, you should be familiar with the square root property. Solve $(2x + 1)^2 = 49$ using the square root property.

First let's talk about how to complete the square. Show how to complete the square on $x^2 + 6x$ below:



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**Examples** Complete the square on the following:

a. $x^2 + 8x$

b. $x^2 - 10x$

c. $x^2 + 3x$

d. $x^2 - \frac{1}{2}x$

e. $x^2 + \frac{2}{3}x$



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Show how to solve $2x^2 + 12x - 16 = 0$ by completing the square:



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**Examples** Solve the following by completing the square:

a. $x^2 + 4x - 6 = 0$

b. $x^2 + 5 = 2x$



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**Examples** Solve the following by completing the square:

a. $3x^2 + 15x = -6$

b. $\frac{1}{2}x^2 + 3x - 2 = 0$



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**Examples: The Distance Formula**

Find the distance between the following sets of points.

a. (6, 2) and (2, 11)

b. (3, -1) and (-4, -2)



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**Solving Quadratic Equations by the Quadratic Formula**

Complete the square on: $ax^2 + bx + c = 0$

State the quadratic formula:



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**Examples** Use the quadratic formula to solve the following:

a. $x^2 + 3x + 2 = 0$

b. $2x^2 + 3x = 5$

c. $\frac{1}{2}x^2 + \frac{1}{3}x + \frac{5}{6} = 0$



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**Lesson: The Discriminant**

Restate the Quadratic Formula, and then label the discriminant.

| Value of the Discriminant | Interpretation |
|---|---|
| | |
| | |
| | |
| | |



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**Examples** Find the values of the discriminant for the following equations. Then solve the equations (using the most efficient method) to determine if this makes sense.

a. $x^2 - 4x + 4 = 0$

b. $x^2 + 6x + 5 = 0$

c. $3x^2 + 2x + 5 = 0$



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**Solving Equations that End Up Having Quadratics**

In this lesson, we will review some techniques from previous lessons.

Solve $\frac{2}{x} - \frac{1}{x+1} = \frac{2}{3}$

Solve $\sqrt{2x+8} = x + 4$



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**<u>Examples</u>** Solve the following:

a. $\frac{4}{x-2}+\frac{x}{x-3}=\frac{2}{x-3}$

b. $\sqrt{3x+5}=x-3$



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**Solving Equations that Are Similar to Quadratics**

In this lesson, we will look at how to solve equations that look like this:

$$x^{\frac{2}{3}} - x^{\frac{1}{3}} - 12 = 0 \qquad \text{or} \qquad (2x+3)^2 + 5(2x+3) + 6 = 0$$

Before we begin, what do the following equations have in common if you **look at their exponents**?

a. $x^2 + 5x + 6 = 0$

b. $x^{\frac{2}{3}} + 5x^{\frac{1}{3}} + 6 = 0$

c. $x + 5x^{\frac{1}{2}} + 6 = 0$

d. $x^4 + 5x^2 + 6 = 0$

Explain how to solve $x^4 - 13x^2 + 36 = 0$



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**<u>Examples</u>** Solve the following:

a. $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 15 = 0$

b. $x - 2x^{\frac{1}{2}} - 8 = 0$

d. $x^{-2} + 8x^{-1} - 9 = 0$



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**Examples** Solve the following:

a. $(x+3)^2 - 4(x+3) - 5 = 0$

b. $(2x+1)^2 - 4(2x+1) - 21 = 0$



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**Lesson: A Review of Imaginary Numbers**

*Please note:* Knowing imaginary numbers if one of the "prerequisite" topics for this course. Ie – you should already know this (though I also realize you probably need a review!). If you have *never* seen imaginary numbers before, then I strongly suggest that you watch my series on imaginary numbers (because, yes, I have one). So if you felt like this went too fast and you want more details, please take responsibility for your education and go back to review this!

How do we represent an imaginary number?

What is $i^2$?

What is a complex number?

When are 2 complex numbers equal?

What does $\sqrt{-a}$ mean?

**Examples** Write using the definition of $\sqrt{-a}$

a. $\sqrt{-12}$

b. $\sqrt{-25}$



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**Examples** Complete the following operations.

a. $\sqrt{-5} * \sqrt{-3}$

b. $\frac{\sqrt{-12}}{\sqrt{-18}}$

c. $\frac{\sqrt{-50}}{\sqrt{9}}$

To add or subtract complex numbers, we can add/subtract the real parts and the imaginary parts similar to working with like terms.

**Examples** Find the sum or difference.

a. $(2 - 6i) + (8 + 2i)$

b. $(5 - 7i) - (-4 + 2i)$



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When multiplying complex numbers, recall that $i * i = i^2 = -1$!

**<u>Examples</u>** Multiply.

a. $(2 + 4i)(3 - 5i)$

b. $(4 - 5i)(3 - 2i)$

c. $(3 + 2i)(3 - 2i)$

Recall, the conjugate is found by using the opposite side for the second term!

**<u>Examples</u>** What is the conjugate of the following?

a. $4 + 2i$

b. $-3 - 4i$

c. $i$



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To find the quotient of 2 complex numbers, we multiply by the conjugate of the denominator.

**Examples** Find each quotient. Write your answer in *standard form,* ie $a + bi$ form

a. $\frac{3-4i}{2+i}$

b. $\frac{5}{i}$

How do the powers of $i$ cycle through?

**Examples** Show how to simplify the following powers of i. Write your work in great detail so that you remember how to do this!

| | |
|---|---|
| a. $i^{17}$ | c. $i^{-14}$ |
| b. $i^{28}$ | d. $i^{-5}$ |



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**<u>Examples</u>** The following quadratic equations have imaginary solutions. Show how to solve them.

a. $x^2 = -25$ using the **square root property**

b. $x^2 + 75 = 0$ using the **square root property**

c. $2x^2 - 3x = -6$ using the **quadratic formula**

d. $x^4 - 16$ using **factoring** and also the **square root property**



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**How to Evaluate Square Roots that Have Imaginary Number Answers and Dealing with Negatives in Square Roots**

In this video, we will look at how to evaluate things like this:

$\sqrt{-4}$ or $\sqrt{-12}$ or $\sqrt{-3} * \sqrt{-6}$

What is an imaginary number?

What is a complex number?

How do you evaluate something like $\sqrt{-9}$?

**<u>Examples</u>** Compute the following:

a. $\sqrt{-25}$

b. $\sqrt{-81}$

c. $\sqrt{-12}$

d. $\sqrt{-7}$

e. $\sqrt{-\frac{25}{16}}$

f. $\sqrt[3]{-8}$

g. $\sqrt[4]{-16}$

h. $\sqrt{-32}$



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How do you evaluate problems like $\sqrt{-3} * \sqrt{-2}$?

**<u>Examples</u>** Evaluate the following:

a. $\sqrt{-6} * \sqrt{-5}$

b. $\sqrt{-6} * \sqrt{-2}$

c. $\sqrt{-10} * \sqrt{-15}$

d. $\frac{\sqrt{-6}}{\sqrt{-2}}$

e. $\frac{\sqrt{-50}}{\sqrt{-2}}$

f. $\frac{\sqrt{-7}}{\sqrt{-2}}$



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**Operations with Imaginary Numbers**

In this video, we will look at how to evaluate things like this:

$2i(3i-5)$ or $(4i+1)-(5i+3)$ or $\frac{4}{3+2i}$

**Adding and Subtracting with Imaginary Numbers:**
Use the example $(2i+3)+(5i-6)$ to show how to add or subtract with imaginary numbers.

**<u>Examples</u>** Simplify:
a. $(3i-2)-(4i-9)$

b. $(5i+3)+(2i-1)$

What can we say about $i^2$?



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With that in mind, explain what is unique about multiplying: $2i(3i+1)$

**<u>Examples</u>** Simplify:

a. $4(2-3i)$

b. $6i(3i+2)$

c. $5i(2-8i)$

d. $(5i+1)(2i-3)$

e. $(4i-3)(6i+7)$



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**Examples** Simplify:

a. $(5i + 2)^2$

b. $(3i - 5)^2$

Why do you need to rationalize $\frac{3}{2i+1}$? Explain how to do this.

**Examples** Rationalize the following:

a. $\frac{4}{2i+6}$

b. $\frac{7}{i+3}$



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**Examples** Rationalize the following:

a. $\frac{3i+1}{5i-2}$

b. $\frac{7i+3}{2i+1}$

c. $\frac{i-3}{5i+3}$



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**Evaluating Higher Powers of Imaginary Numbers**

In this video, we will look at how to evaluate things like this:

$i^{21}$ or $i^{54}$

First we need to investigate the powers of $i$ and the patterns with it. Write this down below:

Now that we can see a pattern, explain how to evaluate the following:

$i^{15}$

$i^{22}$



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**<u>Examples</u>** Evaluate the following:

a. $i^{36}$

b. $i^{43}$

c. $i^{52}$

d. $i^{33}$

e. $i^{25}$



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**Lesson: Applications and Modeling with Linear Equations**

We will revolve a few different types of applications for linear equations.

**Example** Two sides of a triangle have the same length. The third side measures 8 inches less than twice the common length. The perimeter of the triangle of 20 inches. What are the lengths of the three sides?

**Example** Dan ran the 100-m dash with a time of 9.92 seconds. If this pace could be maintained for an entire 26-mile marathon, what would his time be? (Note: 1 m = 3.281 ft, 1 mi = 5280 ft)



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**Example** Sam earned $25,000 from royalties on his music. He set aside 20% of this for a down payment on a new home. The balance will be used for a new car (eventually). He invested a portion of the money in a CD account (bank certificate of deposit) that earns 2.25%, and the remainder in another CD account that earns 1.75%. If the total interest earned after 1 year is $405, how much money was invested in each account?



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**Lesson: The Basics of Applications and Word Problems**

As a reminder, here is a review of some of the key words that imply certain operations in math:

| **Words That Imply Addition** | **Words That Imply Subtraction** | **Word That Imply Multiplication** | **Words That Imply Division** |
|---|---|---|---|
| Add | Subtract | Multiplied by | Divided by |
| Increased by | Difference | Times | Quotient of |
| Altogether | Less than | Of | Out of |
| Sum | Fewer | Product of | Ratio |
| Longer than | Decrease | Twice | |
| Plus | Minus | | |
| Above | Take away | | |
| More than | Shorter than | | |

**Example** Juan is memorizing a list of names for a history class quiz. So far, he has memorized 8 of the names, which is one-fourth of the list. How many names are on this list?

**Example** Corey wants to rent a truck for the day. It costs $30 for the base fee, plus $0.23 for each mile driven. When he returned the truck, he owed $41.19. How many miles did he drive?



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**Example** A café serves both tea and coffee. On Sunday, the café sold 95 teas and coffees combined. The café sold 9 more coffees than teas. How many teas were sold on Sunday?

**Example** Latoya bought a new sweater and a pair of jeans for $64. The jeans cost $8 less than the sweater. How much was the sweater? How much were the jeans?

**Example** There are red, orange, and green balls in a ball pit. There are 12 more orange balls than red balls, and there are 9 more green balls than red balls. How many are there of each color?



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**Lesson: Consecutive Integer Word Problems**

How do you represent consecutive integers with variables?

**Example** The sum of 2 consecutive integers is 73. Find the integers.

**Example** The sum of 3 consecutive integers is -33. Find the integers.

How do you represent even consecutive integers with variables?

How do you represent odd consecutive integers with variables?

What is noteworthy about how you represent even or odd consecutive integers?



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**Example** The sum of 2 consecutive even integers is 110. Find the integers.

**Example** The sum of 3 consecutive odd integers is -87. Find the integers.

**Example** Find 3 consecutive odd integers such that their sum is seven more than four times the largest integer.



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**Lesson: Solving a Linear Equation with Multiple Variables**

**<u>Examples</u>**

a. Solve for x: $x + y = 7$

b. Solve for y: $4y = z$

c. Solve for a: $3a - 6b = 2c$

d. Solve for b: $4a + 8b = 16c$

e. Solve for k: $\frac{1}{3}k = m$

f. Solve for a: $3(a - 2b) = 6c$

g. Solve for b: $\frac{2}{3}(3a + 2b) = c$

h. Solve for a: $-\frac{2}{5}(2a - 4b) = 4$



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**Lesson: Ratios and Proportions**

What is a ratio?

**Example** The unit price of an item is the ratio of the price of an item to the amount of the item. Find the unit price if 5oz of peanuts costs $1.

**Example** Express the ratio of 12 boys to 14 girls.

What is a proportion?

**Examples** Solve the following proportions:

a. $\frac{x}{5} = \frac{6}{10}$

b. $\frac{x+2}{12} = \frac{x-4}{8}$



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**Examples: Proportions**

a. $\frac{3}{x} = \frac{9}{21}$

b. $\frac{3x+10}{15} = \frac{2}{5}$

c. $\frac{2x-1}{6} = \frac{4x}{10}$



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**Lesson: Proportion Word Problems**

It 3 cookies cost $1.36, how much will 7 cookies cost?

At a motorcycle race, the ratio of parents to children was 5 to 3. If 500 children attended the motorcycle race, how many parents were at the race?

One euro is equivalent to $1.18 US dollars. How many euro would you get for $25?



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**Lesson: Distance, Rate, and Time Problems**

Two cars leave Glen Ellyn, IL, with one driving north and the other driving south. The northbound car is traveling 8mph faster than the southbound car, and after 2 hours they are 420 miles apart. Find the speed of each car.

Marshall and Lilly live together. Marshall leaves the house at 2pm, and then Lilly leaves the house at 2:30pm. If Marshall is driving 30 mph and Lilly is driving 35 mph in the same direction, how long will it be until their cars meet?

A Lamborghini and a Ferrari pass one another at noon on a highway in California. The cars are going opposite directions. If the Lamborghini is traveling 75 mph, and the Ferrari is traveling 80 mph, how long will it take for the cars to be 310 miles away from one another?



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**Lesson: Money Word Problems**

Sarah has five more nickels than quarters. When combined, the coins are worth $4.75. How many nickels and quarters does she have?

Maria has $75 worth of $1 and $5 bills. In total, she has 27 bills. How many $5 bills does she have?

Tickets at the movie theater course $15 for an adult and $10 for a child. In the last hour, sales were $550. How many adult tickets versus child tickets were sold?



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**Lesson: Applications of Quadratic Equations**

**Example** Find the problem of 2 pairs of 2 consecutive integers whose product is 156.

**Example** You want to buy a rug for a room that is 18ft x 26ft. You want to have a uniform strip of floor around the rug, and you can afford 384 square feet of carpeting. What dimensions should the rug have?



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**Example** The area of a square is 32 more than its perimeter. Find the length of a side.



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**Solving Formulas with Squares and Square Roots**

In this lesson, we will look at how to solve for formulas that look like this:

$$y = 3x^2 \qquad \text{or} \qquad M = \sqrt{\frac{AB}{C}} \qquad \text{or} \qquad kx^2 + cx - n = 0$$

**Solving Formulas with Squares**

Solve the following for the indicated letters:

a. for x: $y = \frac{4x^2}{z}$

b. for k: $b = \frac{3}{k^2}$



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**Solving Formulas with Square Roots**

Solve the following for the indicated letters:

a. for b: $a = \sqrt{\frac{b}{c}}$

b. for z: $x = \sqrt{\frac{3y}{z}}$



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**Solving Formulas in other forms**

Solve the following for the indicated letters:

a. for x: $kx^2 + cx - d = 0$

b. for b: $ab^2 - 5b + m = 0$

c. for k: $\frac{1}{2}k^2 - 5nk + m = 0$



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**Solving Applications of Quadratic Equations**

An object is thrown upwards off a cliff that is 48 feet above the ground. The height of the object t seconds after the object is released is given by:

$$h = -16t^2 + 32t + 48$$

How long does it take the object to hit the ground?

How long does it take the object to reach a height of 20 feet?



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A 12 ft by 16 ft pond is going to be surrounded by a border of grass of uniform width all the way around. If the area of the enclosure must be 480 feet squared, what is the width of the grass border?



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**Lesson: Applications of Quadratic Equations**

The area of a rectangle is 15ft squared, and the length of the rectangle is 7ft less than twice the width. Find the dimensions of the rectangle.

An 12ft ladder leans against the side of a house. The bottom of the ladder is 6ft from the side of the house. How high is the top of the ladder from the ground? If necessary, round your answer to the nearest tenth.

A kite flying in the air has a 14ft line attached to it. Its line is pulled taut and casts a 9ft shadow. Find the height of the kite. If necessary, round your answer to the nearest tenth.



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The longer leg of a right triangle is 2in longer than the shorter leg. The hypotenuse is 4in longer than the shorter leg. Find the side lengths of the triangle.



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**Lesson: A Quick Review of Some Things About Graphing**

Remind yourself how to graph the following points. Draw your own graph.

Point 1: $(-3, 4)$ Point 2: $(0, -2)$

Draw another rectangular coordinate system below and label the quadrants and origin.

Often when graphing, we use equations in two variables.

**<u>Example</u>** Find the ordered pair associated with the equation $3x + y^2 = 7$ when $x = 1$



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**Example** Find at least 3 ordered pairs for the following equations.

a. $y = \sqrt{x+2}$

b. $2x + 3y = 6$

Finding 3 points can be a strategy for graphing.

**Example** Sketch a graph for the previous 2 examples.

What is the x-intercept? And how do you find it?

What is the y-intercept? And how do you find it?



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**Examples** Find the x- and y-intercepts for the following equations.

a. $y = 3x - 1$

b. $x = \sqrt{y - 1}$

c. $x^2 - 3x + 2 = y$



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**Lesson: The Distance Formula**

Graph the points $(2, 4)$ and $(-4, -6)$. Explain how you can find the distance between these points by summarizing the explanation given in the video (make a decent effort to explain this – you will have an easier time remembering the formula if you understand WHY it works).

State the distance formula:



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**Example** Find the distance between the following points: $(4, -1)$ and $(7, 3)$

There are 2 ways we can use the distance formula.

**Example** Show how to use the distance formula to determine whether the following points are the vertices of a right triangle.

a. $(0, -2), (-4, 3)$ and $(5, 1)$

b. $(-2, -8), (0, -4)$ and $(-4, -7)$



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What does it mean for 3 points to be collinear?

**<u>Example</u>** Show how to use the distance formula to determine if the 3 points are collinear.

a. $(-1, 4), (-2, -1), (1,14)$

b. $(-2, -4), (-6, 11), (0, -12)$



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**Lesson: The Midpoint Formula**

What is the midpoint formula?

So what is the midpoint formula effectively doing?

**<u>Example</u>** Find the midpoint between the points $(-5, -6)$ and $(-3, 2)$

**<u>Example</u>** The midpoint of a line segment is $(5, 8)$ and the endpoint of that line segment is $(11, 10)$. What is the OTHER endpoint on this line segment?



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**Example** The midpoint of a line segment is $(3, 1)$ and the endpoint of that line segment is $(6, 8)$. What is the OTHER endpoint on this line segment?

**Example** The midpoint of a line segment is $(2a, 5b)$ and the endpoint of that line segment is $(-1a, 9b)$. What is the OTHER endpoint on this line segment?



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**Lesson: Graphs of Basic Functions**

In this section, we will review some basic graphs that are vital to know for this class and calculus. Knowing the "basic shapes" of certain graphs will make your life much easier and give you greater intuition when graphing.

Show hot to graph the following and highlight the **key points** and **key patterns.** This will come in handy for the next lesson. Take your time to sketch your own NEAT CLEAN CLEAR graphs.

$f(x) = x$

$f(x) = x^2$



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$f(x) = x^3$

$f(x) = \sqrt{x}$



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$f(x) = \sqrt[3]{x}$

$f(x) = |x|$



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**Lesson: Piecewise Functions**

What is a piecewise function? Draw an example of what they look like.

**<u>Example</u>** Given the function: $f(x) = \begin{cases} 3x - 1, & x \geq 2 \\ \frac{1}{2}x - 4, & x < 2 \end{cases}$, evaluate the following:

a. $f(3) =$

b. $f(-1) =$

c. $f(2) =$

**<u>Example</u>** Given the function: $f(x) = \begin{cases} 4, & x > 3 \\ 4x - 2, & -2 < x \leq 3 \\ x^2 - 5, & x \leq -2 \end{cases}$ evaluate the following:

a. $f(4) =$

b. $f(-5) =$

c. $f(-2) =$

d. $f(3) =$



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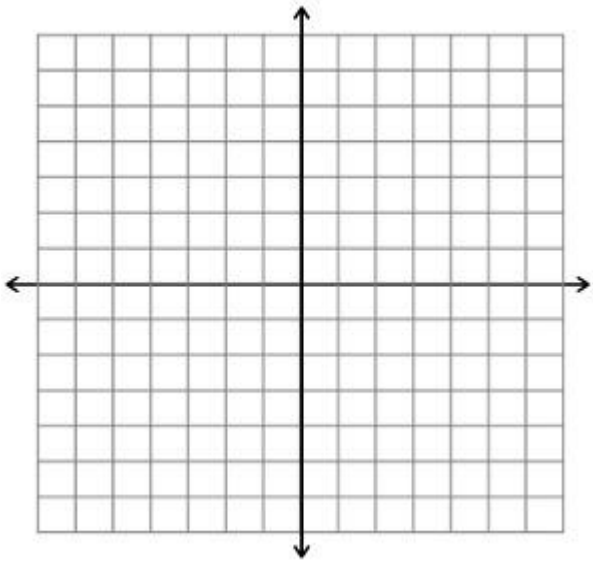
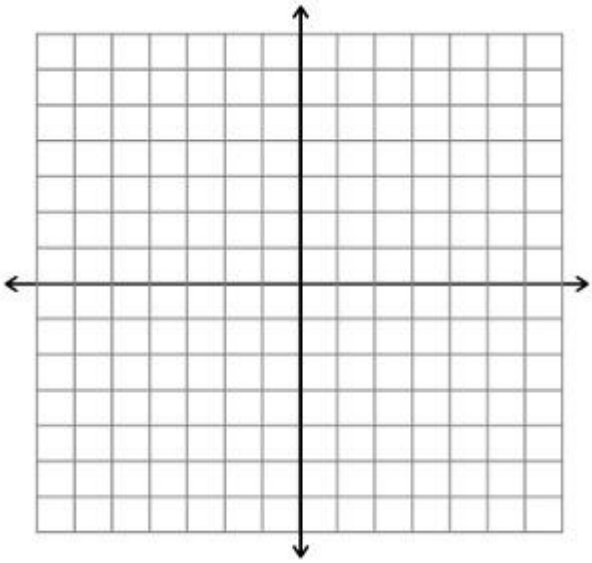
**Example** Show how to graph the following piecewise functions.

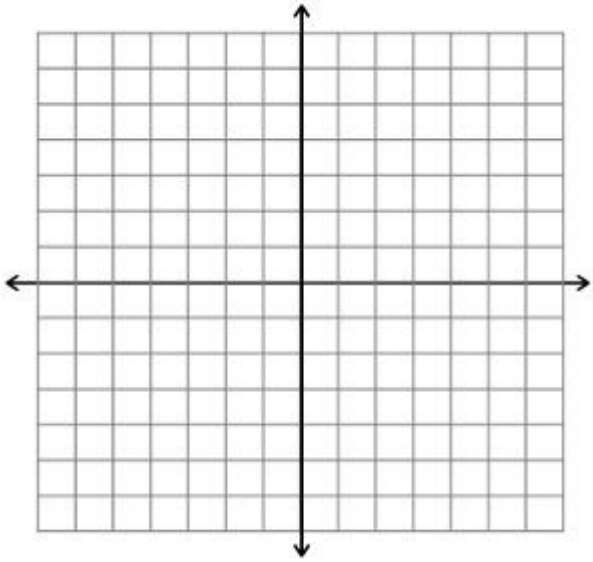
a. $f(x) = \begin{cases} 4, & x > 5 \\ -1, & x \leq 5 \end{cases}$



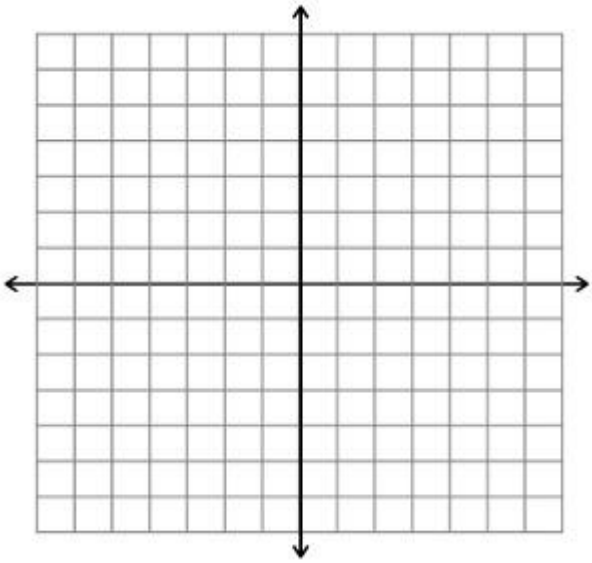
b. $f(x) = \begin{cases} 3x + 1, & x > -3 \\ -2x, & x \leq -3 \end{cases}$



c. $f(x) = \begin{cases} 3, & x > 2 \\ 2x, & -1 < x \leq 2 \\ -x + 3, & x \leq -1 \end{cases}$



d. $f(x) = \begin{cases} 3x + 1, & x \geq 1 \\ 6, & -1 \leq x < 1 \\ x^2, & x < -1 \end{cases}$





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**Lesson: The Greatest Integer Function**

Show how to graph the greatest integer function $f(x) = [\![x]\!]$

**<u>Example</u>** Graph $f(x) = \left[\!\left[\frac{1}{2}x - 2\right]\!\right]$



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**Lesson: A Few Basic Graphs**

There are a few graphs that are helpful to immediately know as we move forward. Let's break these down.

The Parabola: $f(x) = x^2$

Create table of relevant points and then sketch your own graph.

Absolute Value: $f(x) = |x|$

Create table of relevant points and then sketch your own graph.



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X cubed: $f(x) = x^3$

Create table of relevant points and then sketch your own graph.

Square Root: $f(x) = \sqrt{x}$

Create table of relevant points and then sketch your own graph.



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Cube Root: $f(x) = \sqrt[3]{x}$

Create table of relevant points and then sketch your own graph.



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**Lesson: Review of Lines and Linear Functions**

What makes a function **linear?** What are 2 forms of a linear function? How do you interpret this equation?

**Examples** What is the domain and range of $f(x) = 4x + 1$. Show how to graph it.

**Examples** What is the domain and range of $f(x) = 5$. Show how to graph it.

**Examples** What is the domain and range of $x = -4$. Show how to graph it.



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What is the **standard form** of a line?

**<u>Examples</u>** Give the domain and range of $3x - y = 6$. Show how to graph it.

How do you find the **slope** of a line?

What is the slope of a vertical line?

What is the slope of a horizontal line?



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**Examples** Find the slopes between the following sets of points.

a. $(5, 6)$ and $(-3, 2)$

b. $(-4, 7)$ and $(2, 7)$

c. $(6, 2)$ and $(6, -1)$

**Example** Find the slope of the following lines:

a. $4x - 2y = 8$

b. $-5x + 3y = -15$



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**Lesson: The Slope of a Line**

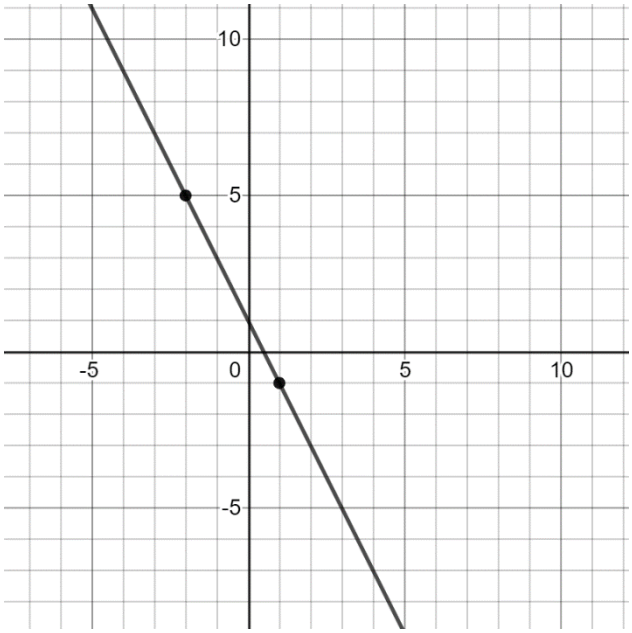
What does slope represent?

Draw the various possible slopes of lines:

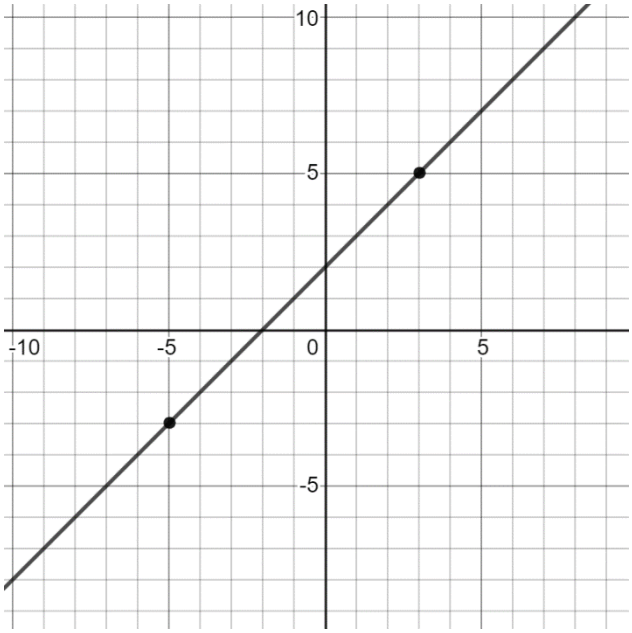
How do find the slope of a line?

**<u>Examples</u>** Find the slope of the following lines:

a.




b.



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What is the slope formula?

**<u>Examples</u>** Use the slope formula to find the slope between the pairs of points:

a. $(3, 5)$ and $(4, 8)$

b. $(6, -2)$ and $(-2, 4)$

c. $(7, 2)$ and $(7, 4)$

d. $(2, 4)$ and $(-1, 4)$



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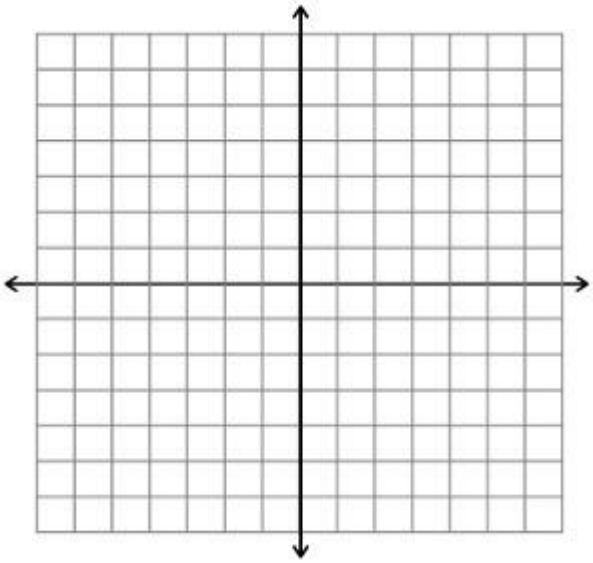
**Lesson: Using Slope to Graph Lines; Intro to Slope-Intercept Form**

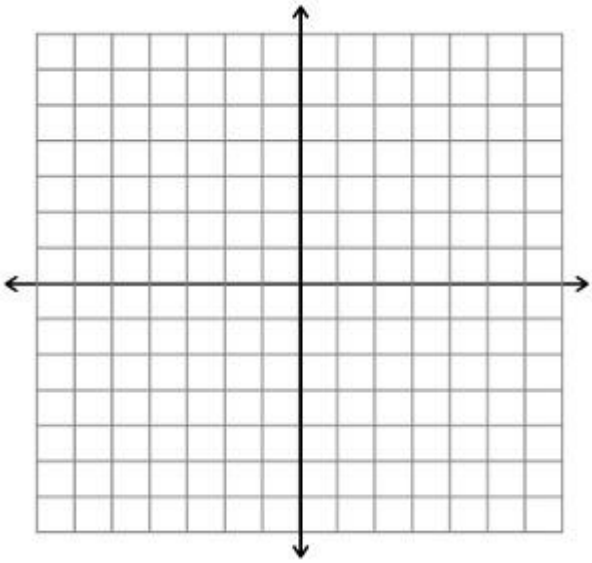
When the slope of a line and a point are known, a line can be easily graphed.

**<u>Examples</u>** Given the slope and the point, graph the following lines:

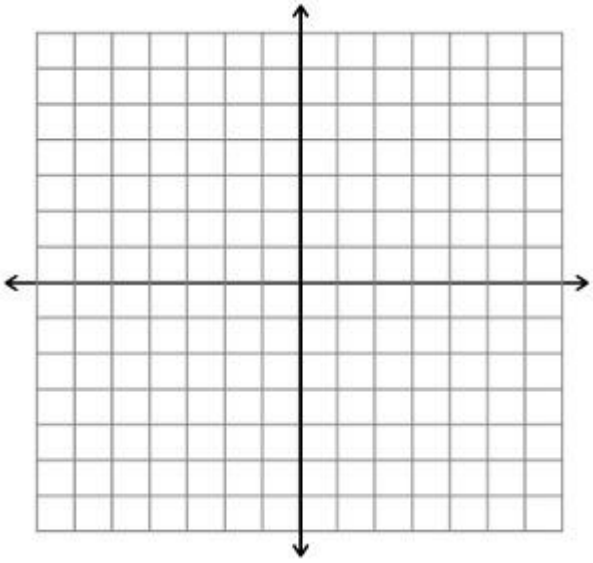
a. $m = 2$ and $(-1, 2)$



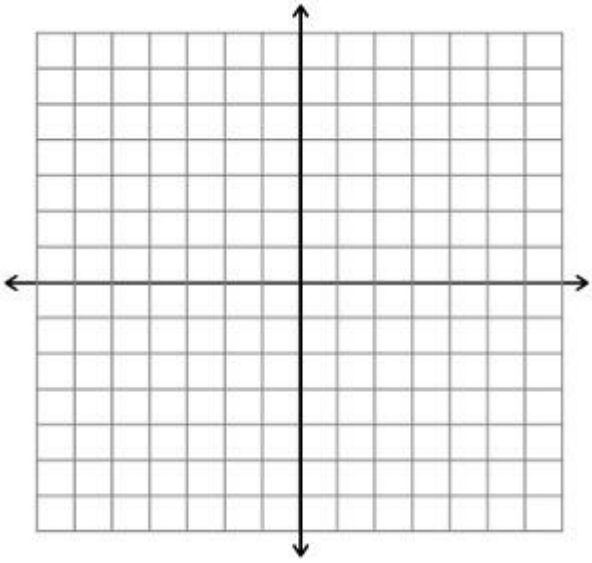
c. $m = 1$ and $(2, -1)$



b. $m = -\frac{1}{3}$ and $(-3, 4)$



d. $m = \frac{2}{3}$ and $(0, -3)$



What is the slope-intercept form of a line?



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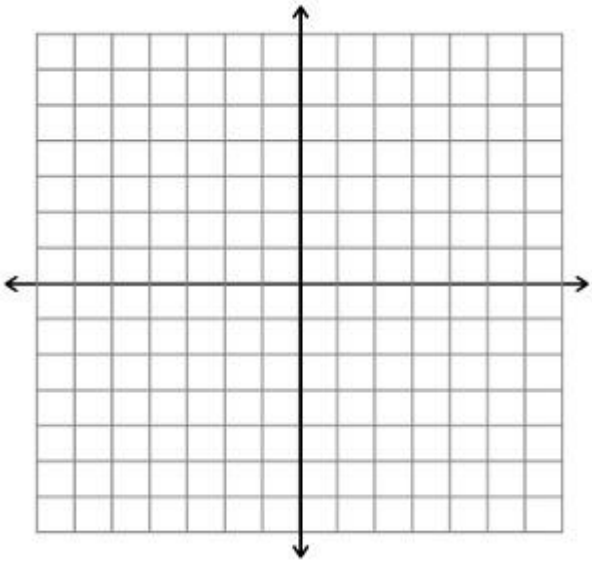
**Examples** Identify the slope $m$ and the y-intercept $(0, b)$ in each of the following equations:

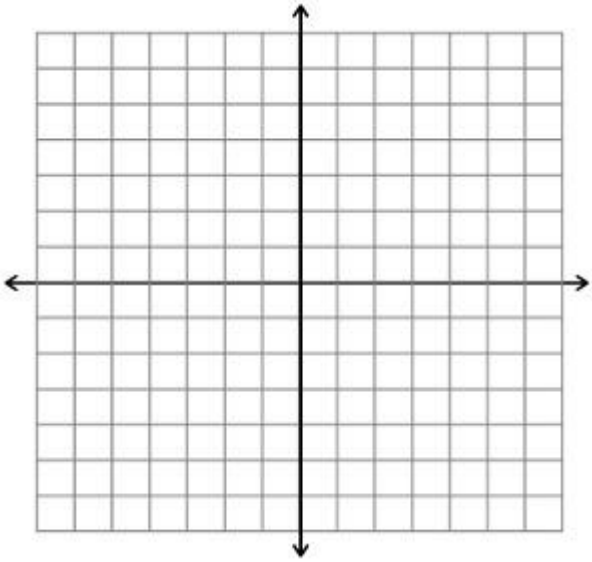
| Equation | Slope, m | What stands for b | State the y-intercept (as a point) |
|---|---|---|---|
| a. $y = -2x + 3$ | | | |
| b. $y = \frac{1}{4}x - 7$ | | | |
| c. $y = 3x + 9$ | | | |
| d. $y = x$ | | | |

**Examples** Graph the following lines:

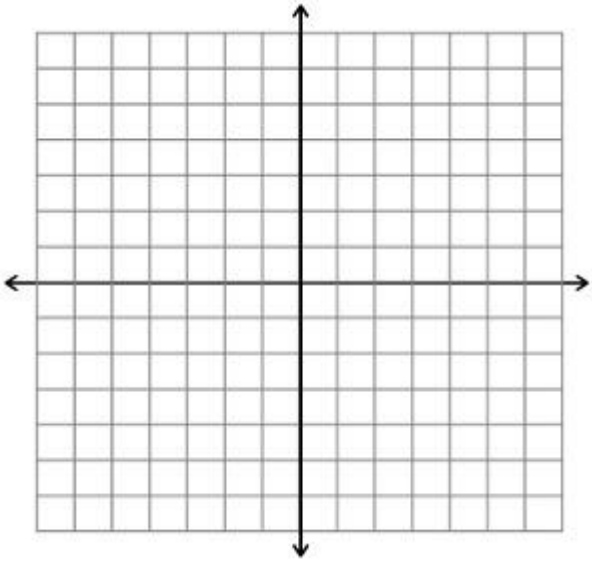
a. $y = 2x + 3$



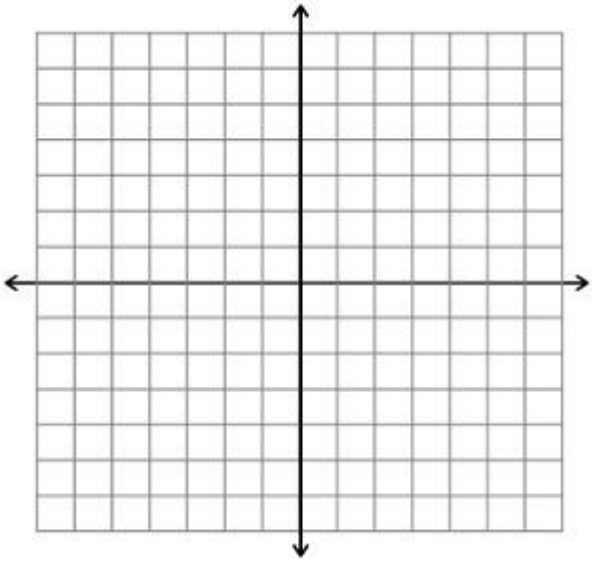
c. $y = \frac{1}{3}x - 4$



b. $y = -x + 4$



d. $y = 3x$





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**Lesson: Review of Writing Equations of Lines**

What is the point-slope form of a line?

**<u>Example</u>** Write the equation of a line through the point $(4, 3)$ with slope $m = -2$.

**<u>Example</u>** Write the equation of a line that goes through the points $(2, 4)$ and $(-4, 10)$. Put your answer in standard form.

What is the slope-intercept form of a line?



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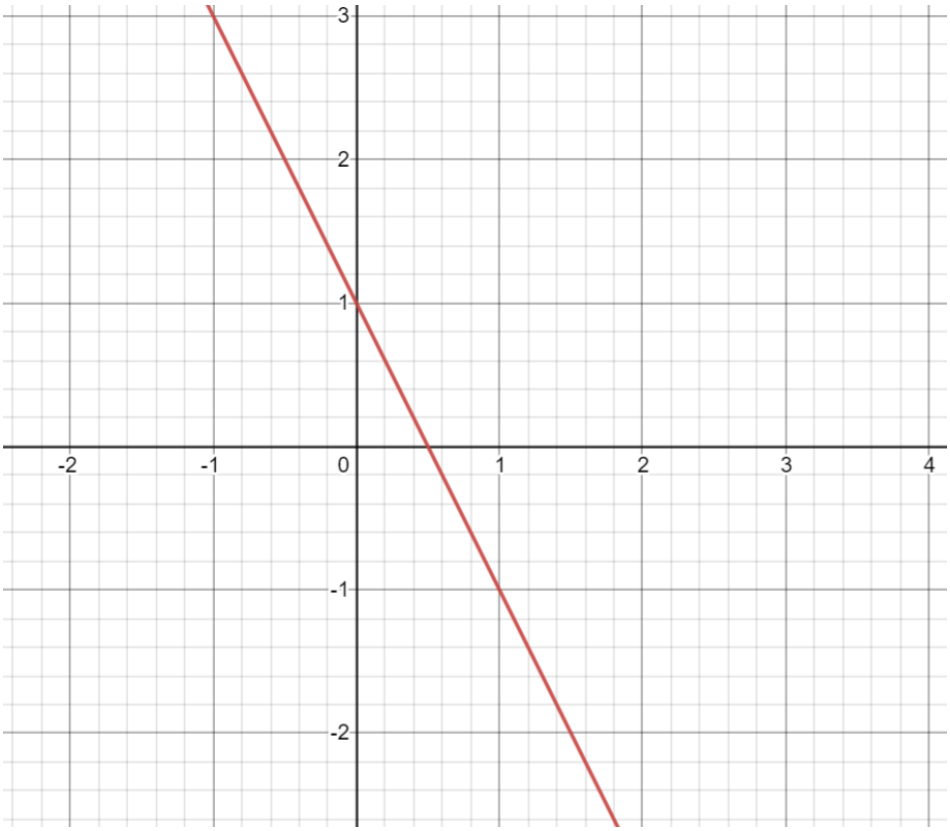
**Example** Show how to write $4x - 6y = 12$ in slope-intercept form.

**Example** Find the slope, x- and y-intercept, and equation of the line:



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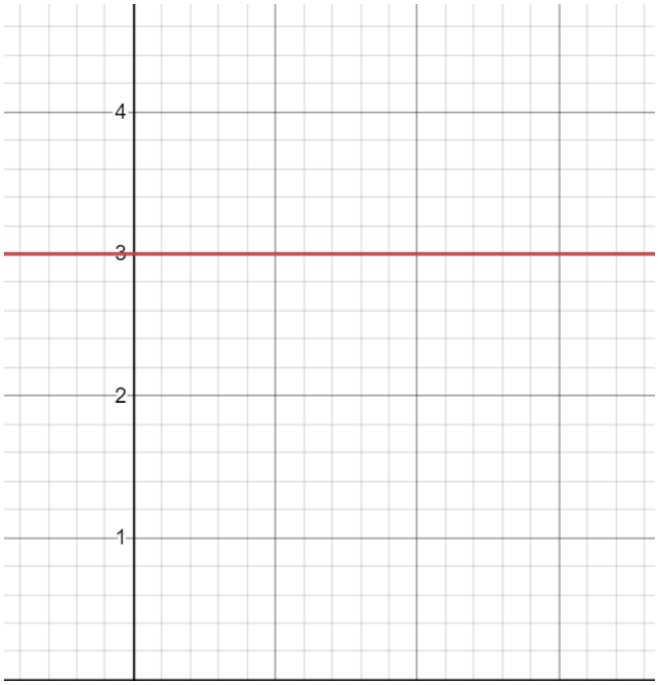
What is the form of a vertical line?

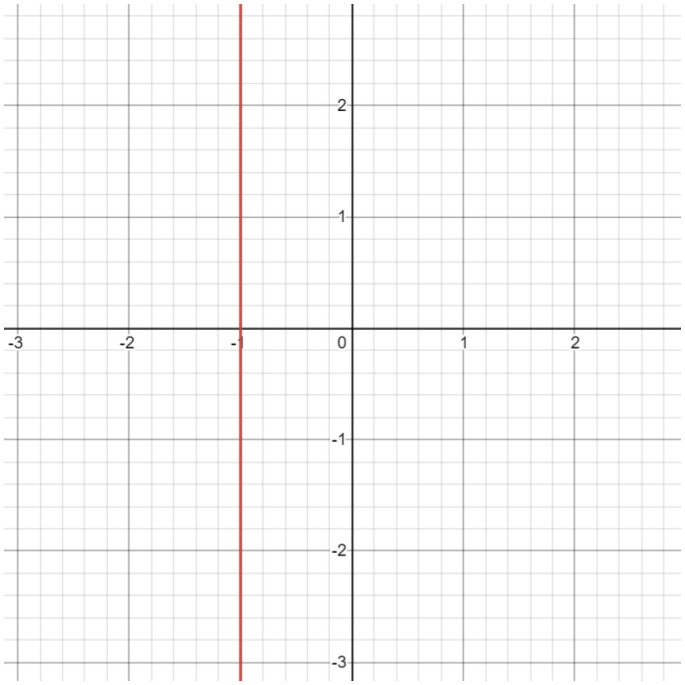
What is the form of a horizontal line?

**<u>Example</u>** Determine the equation of the following lines.

a.




b.




What makes 2 lines parallel?

What makes 2 lines perpendicular?



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**Examples** Write the equation of a line that goes through $(-2, -5)$ that satisfies the conditions:

a. parallel to the line: $3x - 2y = -6$

b. perpendicular to the line: $x - 2y = -4$



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**Quick Reference of Line Formulas**

Slope-Intercept Form

Standard Form

Point-Slope Form

Slope Formula

Form of a Horizontal Line

Form of a Vertical Line

What makes 2 lines parallel?

What makes 2 lines perpendicular?



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**Examples: Writing the Equation of a Line Through 2 Points**

**<u>Example</u>** Write the equation of a line through the given 2 points. Put your answer in standard form, and your standard form should not include any fractions or decimals.

a. $(4, 5)$ and $(2, 9)$

b. $(2, 7)$ and $(-4, 7)$

c. $(-4, 2)$ and $(2, 5)$

d. $(3, 5)$ and $(3, 8)$



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**Examples: Writing the Equation of a Line Through 2 Points**

**<u>Example</u>** Write the equation of a line through the point $(2, 3)$ that satisfies the given conditions. Your final answer should be in standard form with no fractions or decimals.

a. Parallel to $5x - 2y = 10$

b. Perpendicular to $4x - 2y = 1$

c. Parallel to $y = 5$

d. Perpendicular to $x = 7$



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**Lesson: Rewriting Linear Equations into Slope-Intercept Form of a Line**

What is the slope-intercept form of a line?

Explain how to put the equation $2x - 4y = 8$ into slope-intercept form.

**<u>Examples</u>** Put the following equations in slope-intercept form:

a. $6y - 3x = 12$

b. $-5x + 2y = -4$



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**Lesson: Determining Whether Two Lines are Parallel or Perpendicular**

What makes 2 lines parallel?

**<u>Examples</u>** Are the equations $y = \frac{2}{3}x + 5$ and $2x - 3y = 6$ parallel?

What makes 2 lines perpendicular?

**<u>Examples</u>** Determine the negative reciprocal of each of the following:

a. $\frac{2}{3}$

b. $-\frac{1}{5}$

c. $-7$

d. $1$



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**Example** Are the equations $y = \frac{3}{5}x + 2$ and $-3x + 5y = -15$ perpendicular?

**Examples** Are the following pairs of lines parallel, perpendicular, or neither?

a. $3x - 2y = 6$ and $4y = -6x - 8$

b. $y - 3x = 0$ and $12x - 4y = 24$

c. $y = 3$ and $x = -2$



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**Examples** Lines A and B contain the given points. Determine whether the lines are parallel, perpendicular, or neither.

a. Line A: $(2, 11), (5, 10)$ Line B: $(1, 3), (-6, -18)$

b. Line A: $(-5, 5), (-7, -2)$ Line B: $(0, 11), (-4, 9)$



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**Lesson: Putting Equations in Standard Form**

What is the standard form of a line?

**<u>Examples</u>** Write the following equations in standard form:

a. $y = 2x - 4$

b. $y = -5x + 3$

c. $y = -\frac{1}{2}x + 1$

d. $y = \frac{2}{3}x - 2$



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**Lesson: Writing Equations of Lines with The Point-Slope Form**

What is the point-slope form of a line?

**<u>Examples</u>** Given the point and the slope, write the point-slope form of a line.

a. $m = -2, (1, 3)$

b. $m = -\frac{2}{3}, (5, -2)$

What is the slope-intercept form of a line?

What is the standard form of a line?



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**Examples** Given the point and the slope, write the equation of the line in the desired format.

a. $m = -1, (2, -5)$ in slope-intercept form

b. $m = 3, (5, -4)$ in standard form

c. $m = -\frac{1}{2}, (6, 3)$

d. $m = \frac{2}{3}, (-5, -1)$



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**Lesson: Writing the Equation of a Line from Two Points**

Write the equation of the line that contains the following 2 points in the desired format.

a. $(4, 2)$ and $(5, 5)$ in slope-intercept form

b. $(-2, 4)$ and $(-4, -2)$ in standard form

c. $(5, -3)$ and $(3, 5)$ in slope-intercept form

d. $(-1, -3)$ and $(1, 5)$ in standard form



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**Lesson: Writing the Equation of a Line Parallel/Perpendicular to Another Line**

What makes 2 lines parallel?

What makes 2 lines perpendicular?

**<u>Examples</u>** Write the equation of a line with the given conditions.

a. Parallel to $y = 3x + 1$ that goes through $(-2, -5)$ in slope-intercept form

b. Perpendicular to $y = -\frac{2}{3}x - 5$ that goes through $(6, -2)$ in standard form

c. Parallel to $2x - 3y = 6$ that goes through $(1, -2)$ in slope-intercept form



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**Examples** Write the equation of a line with the given conditions.

a. Perpendicular to $2x - 4y = 8$ that goes through $(-4, -2)$ in standard form

b. Parallel to $3x - y = 12$ that goes through $(-1, -5)$ in slope-intercept form

c. Perpendicular to $-2x + 3y = 12$ that goes through $(2, -5)$ in standard form



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**Lesson: Equations of Circles and How to Graph Circles**

Draw the circle in the video and label the relevant parts of it.

Using your sketch above, explain how the distance formula can be used to help us determine the equation of a circle (you will need to summarize the explanation in your own words).

What is the **Center-Radius Form** of the equation of a circle?



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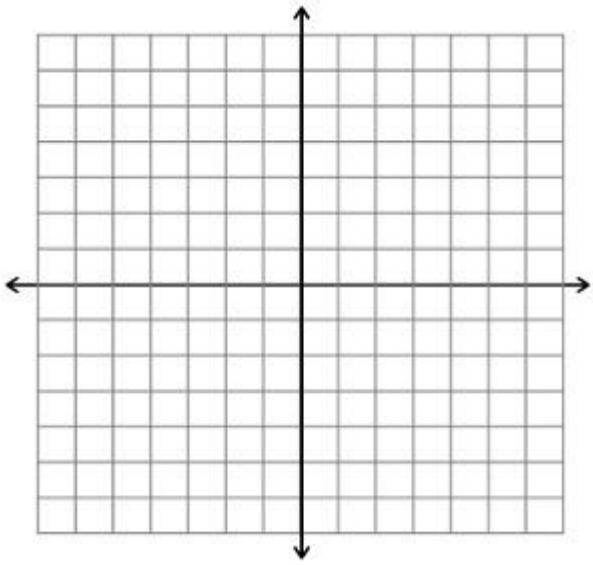
**Example** Find the center radius form of a circle using the given information.

a. center $(-2, 3)$ and radius 4

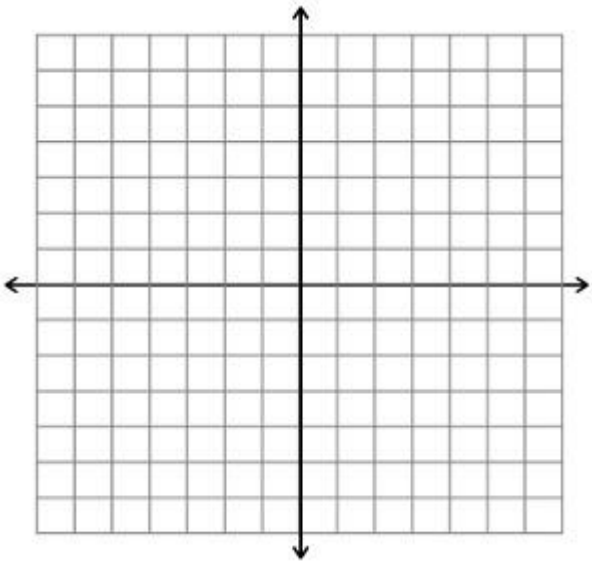
b. center $(0, -1)$ and radius 2

**Example** Graph the following circles.

a. $(x-1)^2 + (y+3)^2 = 4$



b. $x^2 + y^2 = 9$





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Expand $(x-1)^2+(y+3)^2=4$ and show how this creates the general form.

What is the **General Form** of the equation of a circle?

**<u>Example</u>** Review how to use completing the square with the equation: $x^2-6x+4=0$.



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**<u>Example</u>** Show how to use completing the square to write each in the center-radius form, and then sketch a graph of the circle.

a. $x^2 - 4x + y^2 + 8y - 44 = 0$

b. $x^2 + y^2 + 2x - 6y + 14 = 0$



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**Examples: Working with the General Form Equation of a Circle**

**<u>Example</u>** Show how to use completing the square to write in the center-radius form, and then sketch a graph of the circle if possible.

a. $4x^2 + 4y^2 + 4x + 16y - 19 = 0$

b. $x^2 + y^2 - 4x - 4y + 8 = 0$



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c. $x^2 + y^2 + 2x - 4y + 18 = 0$

d. $9x^2 + 9y^2 + 12x - 18y - 23 = 0$



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**Examples: Graphing Circles Using Technology**

You can use this page to take notes for how to solve the following 2 examples:

a. Find all coordinates of points whose distance from (-3, 2) and (-2, 1) is $\sqrt{5}$.

b. There's an earthquake! There are 2 receiving stations with the following coordinates:

$$A = (-1, 1), B = (-2, 1), C = (-3, -1)$$

The epicenter of the earthquake is 2 units from stations A and C, and $\sqrt{5}$ units from B. Where is the epicenter?



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**Lesson: Review of Functions**

How do you represent a function?

What is a function often interchangeable with? How can you rewrite $f(x) = 3x + 8$?

So what IS a function? What does it actually represent?

What's another handy feature of functions?



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What are the domain and range?

Does domain and range only work for functions?

**<u>Examples</u>** Are the following functions, and what is the domain and range?

a. $\{(2,3), (5, -1), (-3, 7)\}$

b. $\{(-1, 5), (-8, 3), (-1, 6)\}$



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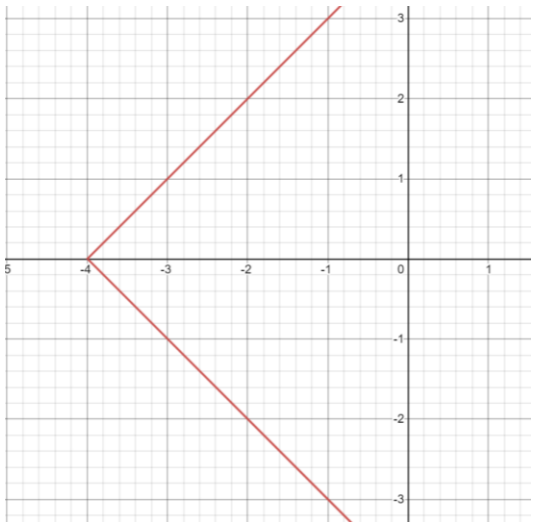
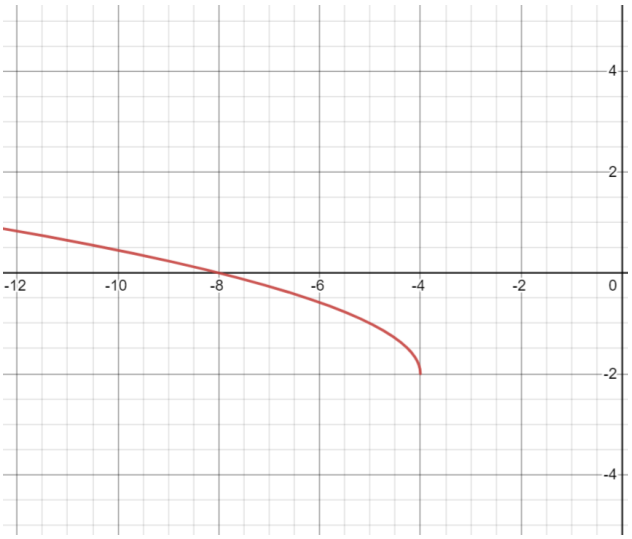
What is the Vertical Line Test, and how does it help you determine if you have a function?

**<u>Example</u>** State the domain and range of the following, and use the vertical line test to determine if the graph represents a function.

a.




b.



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**Examples** Determine if the following represent functions.

a. $y = 3x^2$

b. $x = y^2$

c. $y = \frac{5}{x^2}$



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**Example** Write the following in function notation: $3x + 2y = 6$

**Example** Suppose $f(x) = 3x - 5$ and $g(x) = x^2 + 4x - 1$ and $h(x) = \{(2,5),(6,-3),(-1,-4)\}$. Find the following:

a. $f(7) =$

b. $g(3) =$

c. $f(k) =$

d. $g(x + h) =$

e. $h(6) =$



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**Lesson: Increasing, Decreasing, and Constant Functions**

Below are the "formal math" definitions of whether a function is increasing, decreasing, or constant. Let I be an open interval and let $x_1$ and $x_2$ be two points in that interval.

- A function is **increasing** over I if when $x_1 < x_2, f(x)_1 < f(x_2)$
- A function is **decreasing** over I if when $x_1 < x_2, f(x)_1 > f(x_2)$
- A function is **constant** over I if when $x_1 < x_2, f(x)_1 = f(x_2)$

While these are the formal definitions, what we really want to know are the intuitive meanings of these definition.

Sketch a graph that is increasing:

Decreasing:

And constant:



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**Example** Sketch the graph in the video, and then identify where the graph is increasing, decreasing, or constant.

a.

b.



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**Lesson: An Intro to Functions**

Define a relation:

Define a domain:

Define a range:

**<u>Example</u>** In the set of ordered pairs $\{(2,3), (4, -1), (7, 2)\}$, define the domain and range of the relation.



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Define a function:

What is the visual explanation that explains what a function is or is not?

**<u>Example</u>** In the set of ordered pairs $\{(2,3), (4, -1), (7, 2)\}$, is this a function?



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**Example** Are the following functions?

a. $\{(4,-1),(3,2),(7,-1)\}$

b. $\{(2,4),(3,5),(2,-1)\}$

What is the notation for a function? How do we describe/use this notation?

**Example** In the set of ordered pairs $\{(2,3),(4,-1),(7,2)\}$, what is $f(4)$?



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**<u>Example</u>** Suppose $f(x) = 2x + 1$ and $g(x) = x^2 - 3$. Find the value of the following:

a. $f(2)$

b. $g(1)$

c. $f(a)$

d. $g(h + k)$



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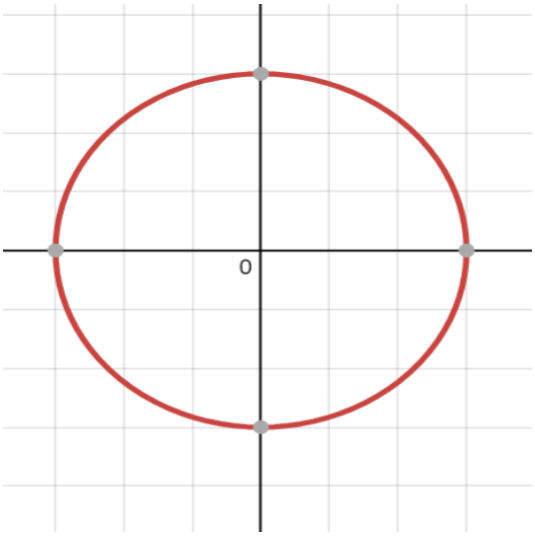
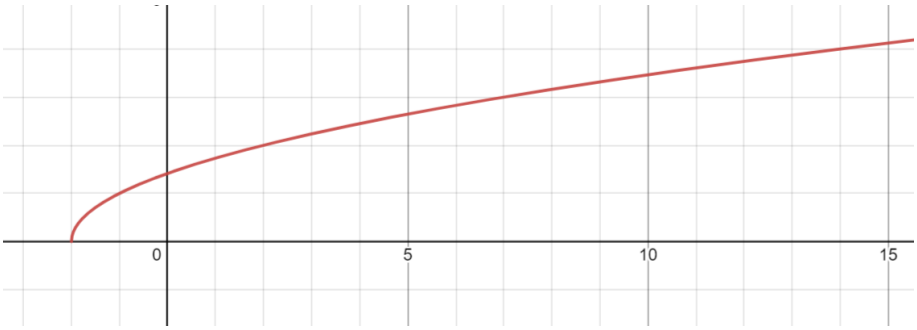
What is the vertical line test?

**<u>Example</u>** Do the following graphs represent a function?

a.




b.



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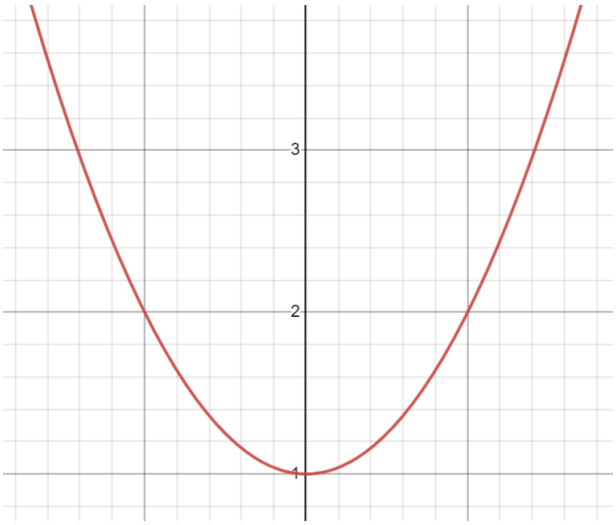
**Lesson: Determining the Domain and Range of Functions In General**

How do you determine the domain and range of a set of points like $\{(2,4), (5,8), (4, 6)\}$?

What is the domain and range of that set of points?

In general, it is easiest to determine the domain and range when we have the graph in front of us.

For the following graph, write out how you should determine the domain and range.




Write a sentence on how you should find the domain:

State the domain:

Write a sentence on how you should find the range:

State the range:



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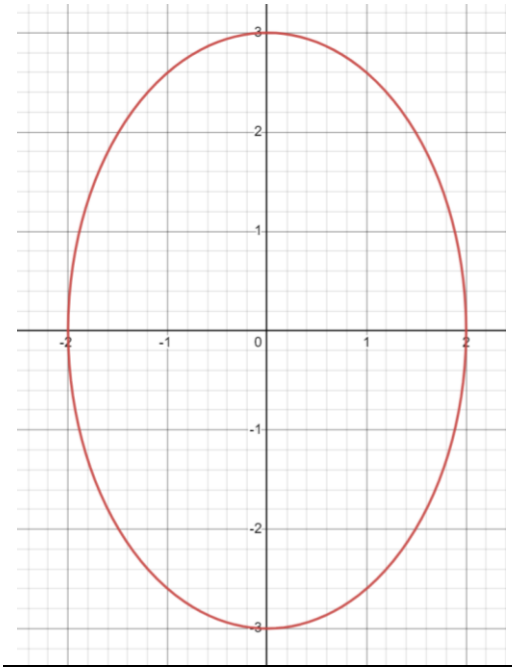
**Examples** Determine if the following graphs represent functions, and then state the domain and range.

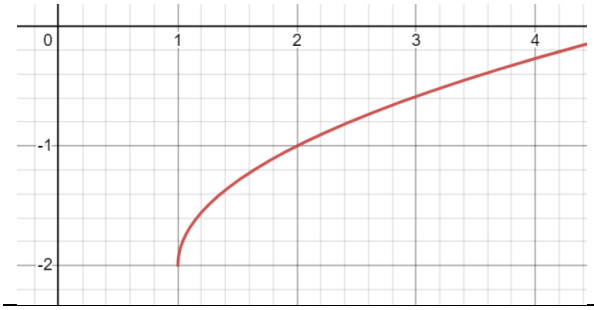
a.




Domain:

Range:

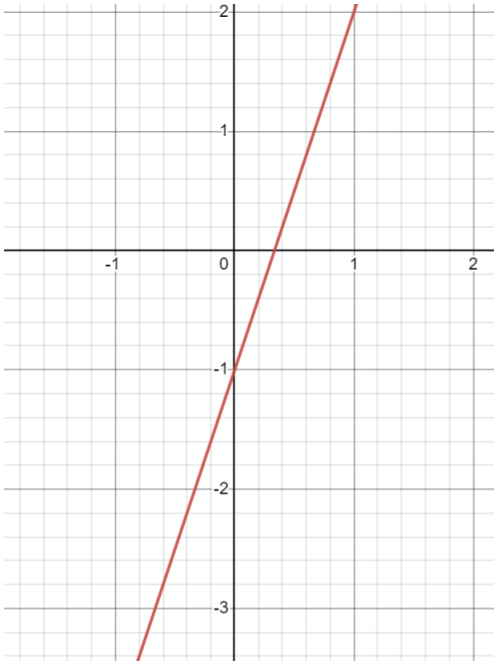
b.




Domain:

Range:

c.




Domain:

Range



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**Lesson: The Domain of Some Specific Types of Functions**

What is a polynomial? Write some examples.

What is the domain of any polynomial function?

What is a rational function?

What is the domain of any rational function?



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**Examples** Determine the domain of the following.

a. $f(x) = 3x - 1$

b. $g(x) = \frac{4}{x+2}$

c. $h(x) = 3x^3 - 2x^2 + 4x - 1$

d. $f(x) = \frac{2x+1}{x^2-9}$



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What is a square root function?

What is the domain of a square root function?

**<u>Examples</u>** Find the domain of the following:

a. $f(x) = \sqrt{x - 2}$

b. $g(x) = \sqrt{4x - 1}$

c. $h(x) = \sqrt{4 - x}$



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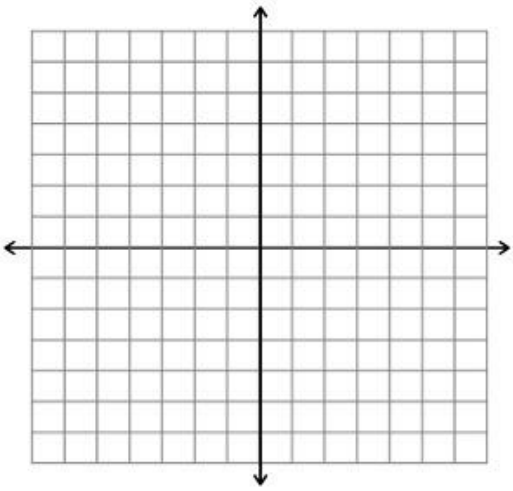
**Lesson: How to Graph a Function**

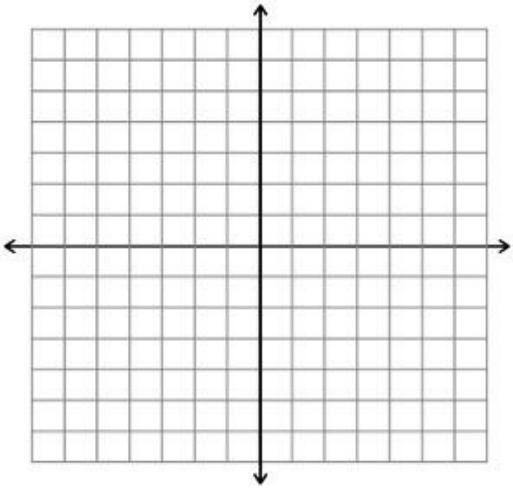
How do you graph a function?

**<u>Examples</u>** Graph the following functions.

a. $f(x) = 3x + 1$



b. $g(x) = 4$





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**Lesson: Translations and Reflections of Functions**

Using the graph $f(x) = |x|$ as the "base function," sketch how vertical translations work (summarize what was discussed in the video).

How do you describe vertical translations in function notation?

Using the graph $f(x) = x^2$ as the "base function," sketch how horizontal translations work (summarize what was discussed in the video).

How do you describe horizontal translations in function notation?



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Use the graph $f(x) = \sqrt{x}$ to show how reflections over the x- and y-axis work (summarize what was discussed in the video).

How do you describe reflections over the x-axis and y-axis using function notation.

Use the graph $f(x) = |x|$ to show how stretching and shrinking works (summarize what was discussed in the video).



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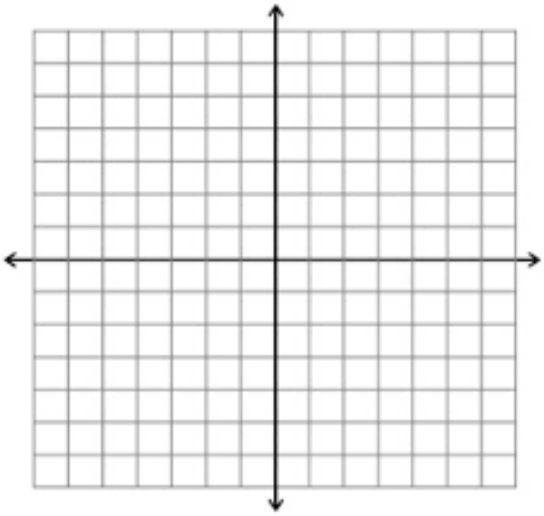
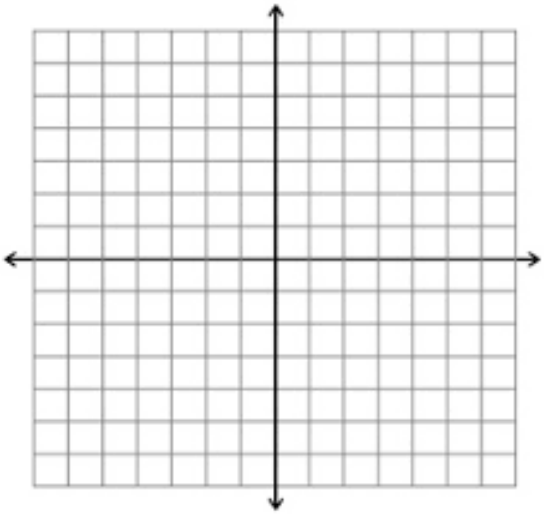
Explain how stretching and shrinking works with function notation.

**<u>Example</u>** Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $f(x) = -x^2 + 3$



b. $g(x) = (x + 2)^3 - 3$





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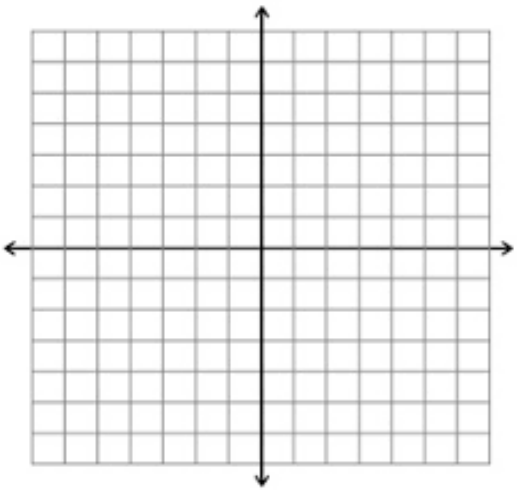
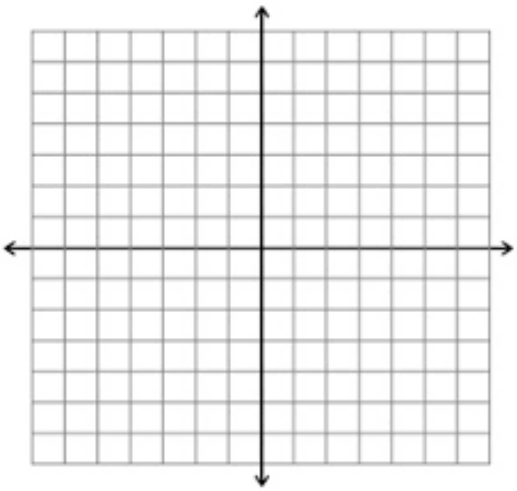
**Example** Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

a. $g(x) = -2(x+1)^2 - 5$



b. $h(x) = 2\sqrt{-x} + 1$



**Examples** Write the equation of the given functions with the following characteristics.

a. The function $f(x) = x^2$ moved 3 units left and 5 units down.

b. The function $g(x) = \sqrt[3]{x}$ reflected over the y-axis, translated 2.5 units right and 1.75 units up.



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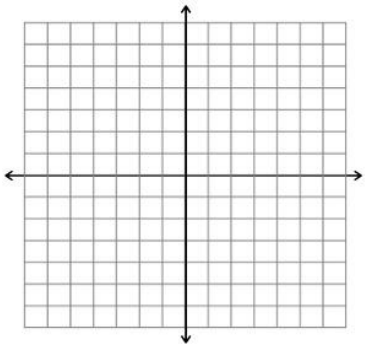
**Transformations of Functions**

These are looking at graphs of things like:

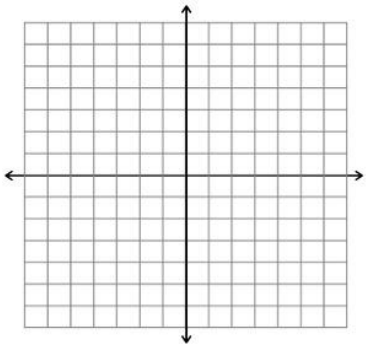
$$f(x) = x^2 - 5 \quad \text{or} \quad f(x) = |x+3| - 2 \quad \text{or} \quad f(x) = \sqrt{x-3}$$

First we need to discuss the shape of a few basic functions. Write these down and note the important points.

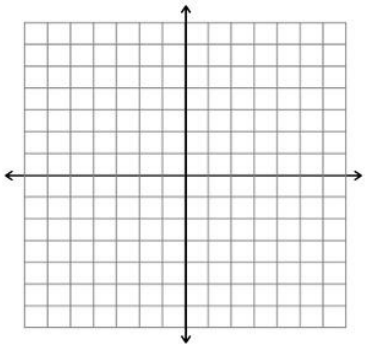
a. $f(x) = x^2$



b. $f(x) = |x|$



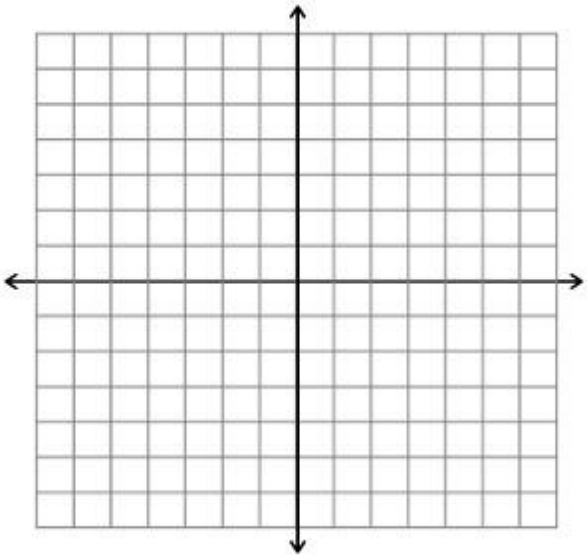
c. $f(x) = \sqrt{x}$



Using a few different colors, sketch out what the graphs of

a. $f(x) = x^2$ versus b. $g(x) = x^2 + 3$ versus c. $h(x) = x^2 - 1$

look like all on the same graph below:





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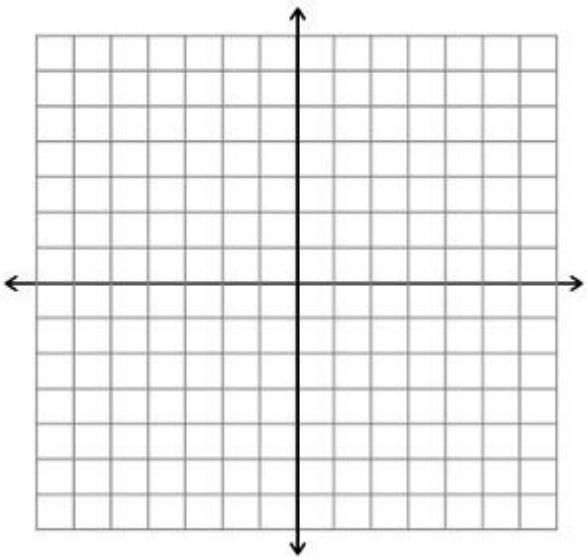
This is an example of a vertical shift. Explain what this means in mathematical terms:

Using a few different colors, sketch out what the graphs of

a. $f(x) = |x|$ versus b. $g(x) = |x - 1|$ versus c. $h(x) = |x + 2|$

look like all on the same graph below:



This is an example of a horizontal shift. Explain what this means in mathematical terms:



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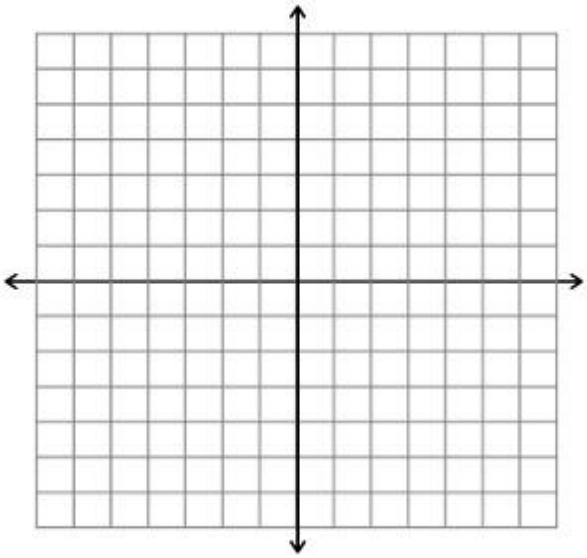
Using a few different colors, sketch out what the graphs of

a. $f(x) = |x|$ versus b. $g(x) = -|x|$

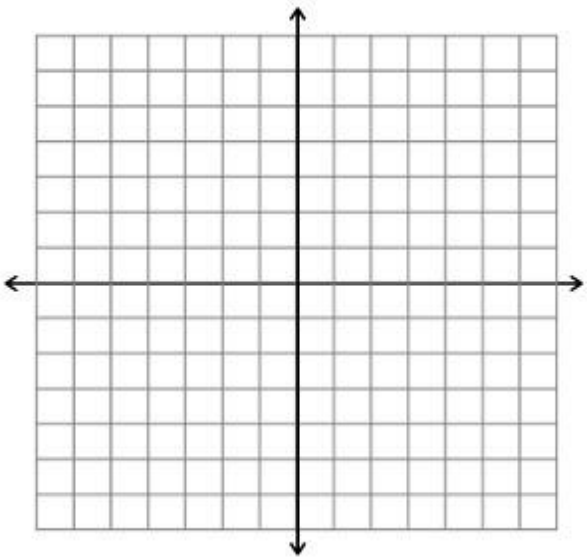
look like all on the same graph below:



This is an example of a reflection over the x-axis. Explain what this means in mathematical terms:

**<u>Example</u>** Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

$$f(x) = |x| + 3$$





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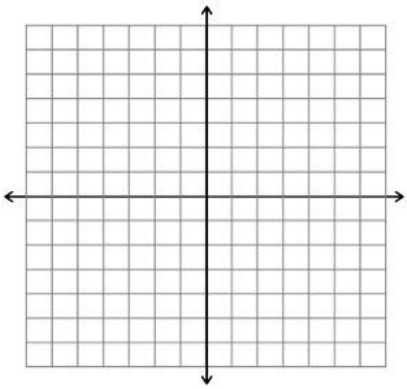
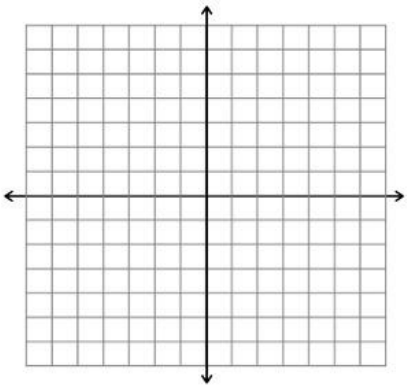
**Examples** Explain in plain language what types of shifts/reflections are occurring, and then graph the function.

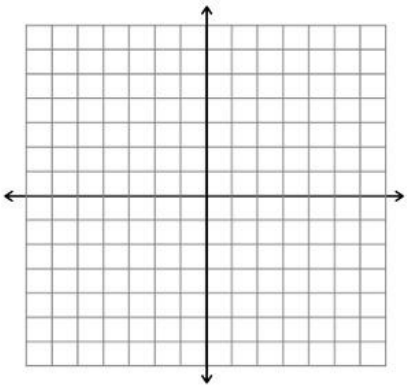
a. $f(x) = (x-1)^2$



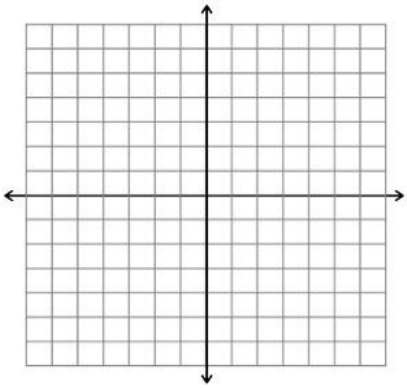
b. $g(x) = \sqrt{x} - 2$



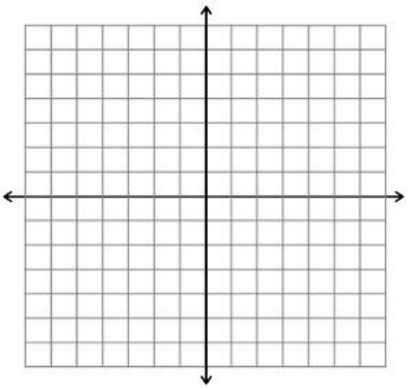
c. $h(x) = |x+1| - 2$



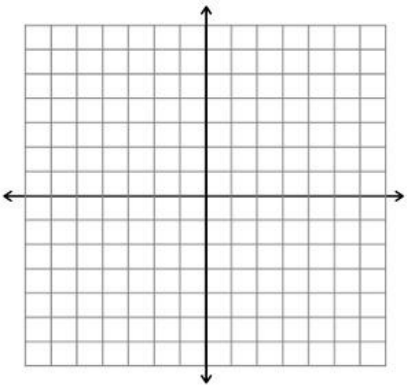
d. $k(x) = (x-2)^2 + 4$



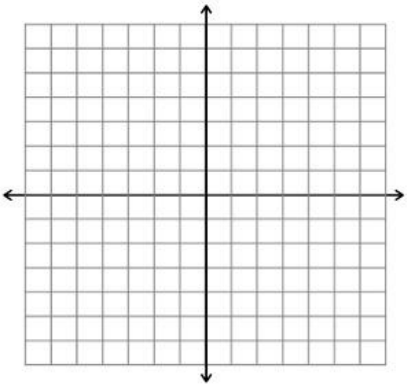
e. $f(x) = -(x+3)^2 - 2$



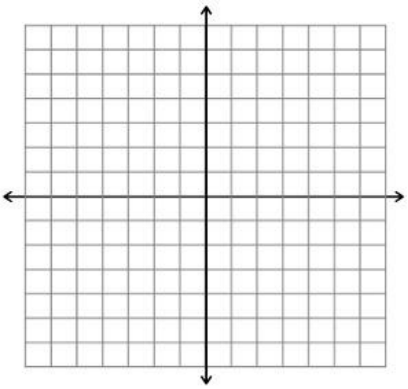
f. $g(x) = -|x+3| - 1$



g. $h(x) = -\sqrt{x-2}$



h. $k(x) = -\sqrt{x+1} - 3$





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**Examples** Write the equations that represent the listed transformations.

a. A function $f(x) = \sqrt{x}$ is shifted 7 units down.

b. A function $g(x) = x^3$ is shifted 3 units left and 2 units up.

c. A function $h(x) = |x|$ is shifted 2.5 units right and reflected over the x-axis.

d. A function $f(x) = e^x$ is shifted 4 units up and reflected over the x-axis.

e. A function $g(x) = \sqrt[3]{x}$ is shifted 3.2 units right, 5.7 units up, and reflected over the x-axis



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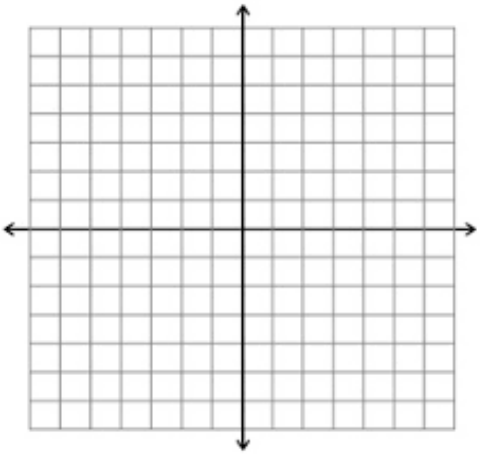
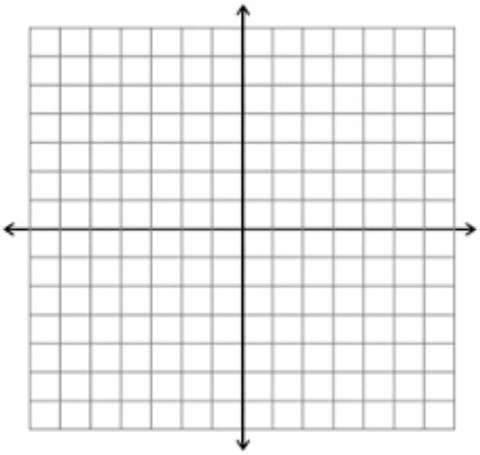
**Examples: Translations and Reflections of Functions**

Describe any translations / reflections / stretching / shrinking and sketch a graph of each of the following.

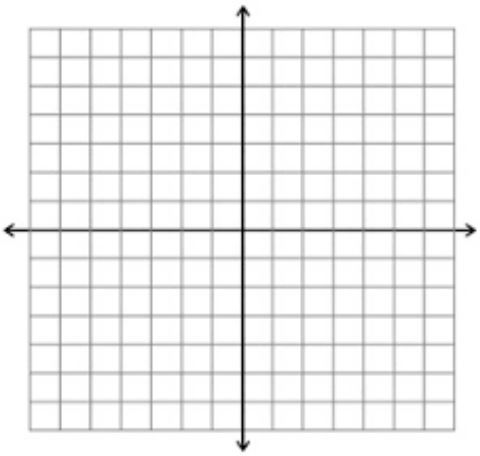
a. $f(x) = -(x+2)^2 - 3$



b. $g(x) = \frac{1}{2}(x-1)^3 + 2$



c. $h(x) = -\frac{1}{3}\sqrt{x+1} + 1$





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**Lesson: Symmetry and Even or Odd Functions**

Explain when a graph is symmetric with respect to the y-axis:

Explain when a graph is symmetric with respect to the x-axis:



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Explain when a graph is symmetric with respect to the origin:

What makes a function odd?

What makes a function even?



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So what is the test for even or odd functions?

**<u>Examples</u>** Determine what type of symmetry the following functions have:

a. $y = x^2 + 4$

b. $x^2 + y^2 = 25$



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**Examples** Determine what type of symmetry the following functions have:

a. $y = x^3 - 5$

b. $y = 7x^2 + 2x - 4$

c. $y = 2x^{\frac{1}{3}} - 3x$



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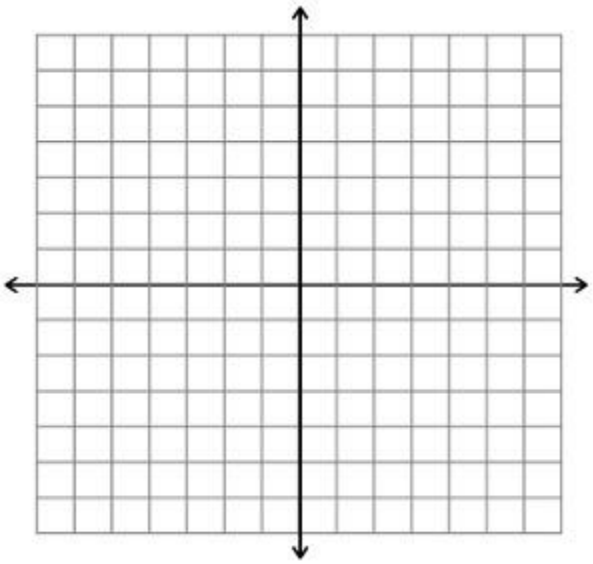
**Lesson: An Intro to Translations and Reflections of Graphs**

We are going to develop the intuition for this concept before we actually graph anything.

**Vertical Translations**

Draw the base graph of $f(x) = x^2$ in one color and then use some other colors to show how the translations work. Label the graphs.



Explain the general way that vertical shifts work:

**<u>Example</u>** How does $f(x) = \sqrt{x} + 6$ shift?



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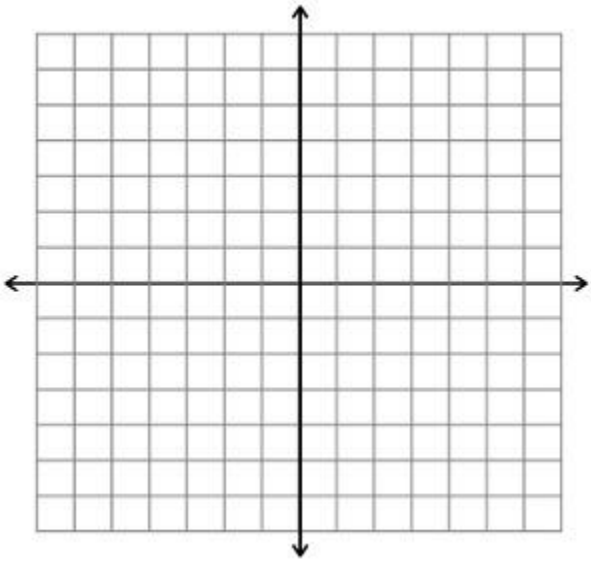
**Horizontal Translations**

Draw the base graph of $f(x) = |x|$ in one color and then use some other colors to show how the translations work. Label the graphs.



Explain the general way that horizontal shifts work:

**<u>Example</u>** How does $f(x) = \sqrt[3]{x+3}$ shift?



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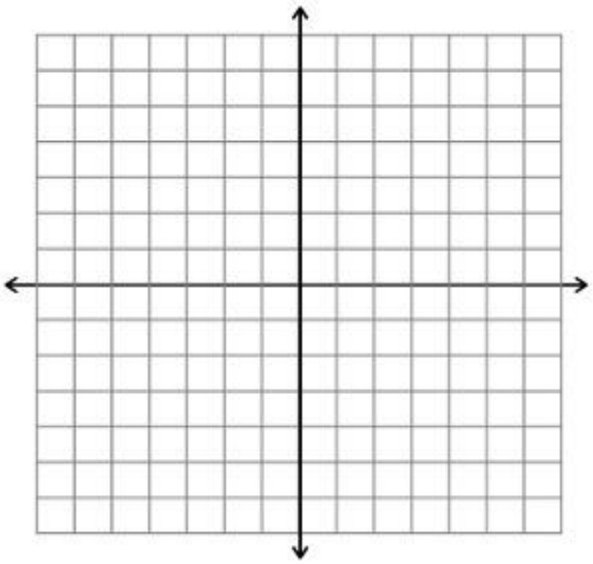
**Reflections Over the X-Axis**

Draw the base graph of $f(x) = \sqrt{x}$ and then graph the reflection in another color. Label the graph of the reflection. Label the graphs.



Explain the general way that reflections over the x-axis work:

**<u>Example</u>** How does $f(x) = -x^2$ look?



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**Example** Describe the translations and reflections of the following.

a. $f(x) = -x^2 + 3$

b. $g(x) = \sqrt{x+2} - 1$

c. $h(x) = (x-5)^3 + 2$

d. $f(x) = -|x+2| + 5$

e. $g(x) = -\sqrt[3]{x-1} - 4$



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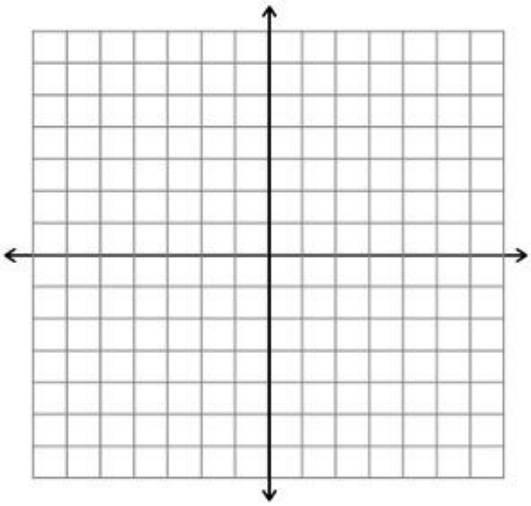
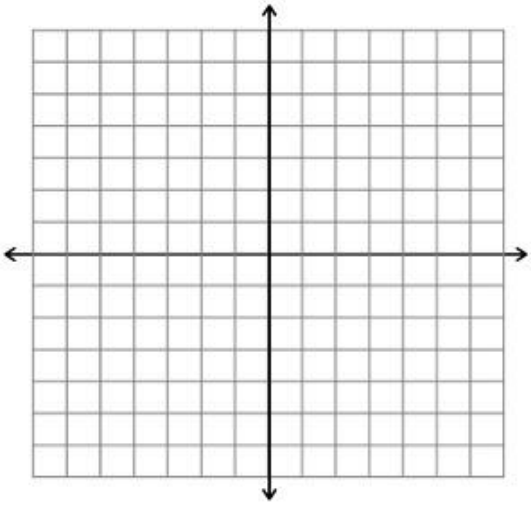
**Lesson: Graphing with Translations and Reflections**

**<u>Examples</u>** For each of the following, describe the translations and reflections. Then graph with 2-3 points.

a. $f(x) = x^2 + 1$
   Describe any translations or reflections:



b. $g(x) = \sqrt{x - 2}$
   Describe any translations or reflections:





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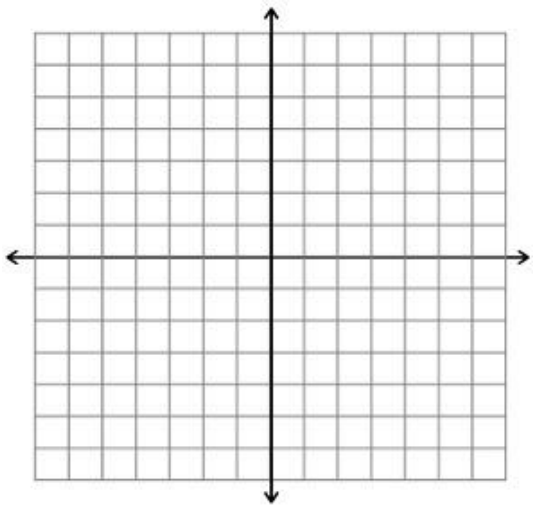
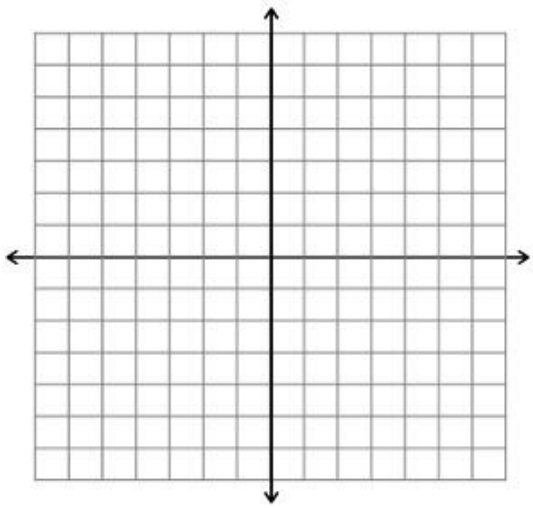
c. $f(x) = (x + 2)^3 - 1$
Describe any translations or reflections:



d. $h(x) = |x - 3| + 2$
Describe any translations or reflections:





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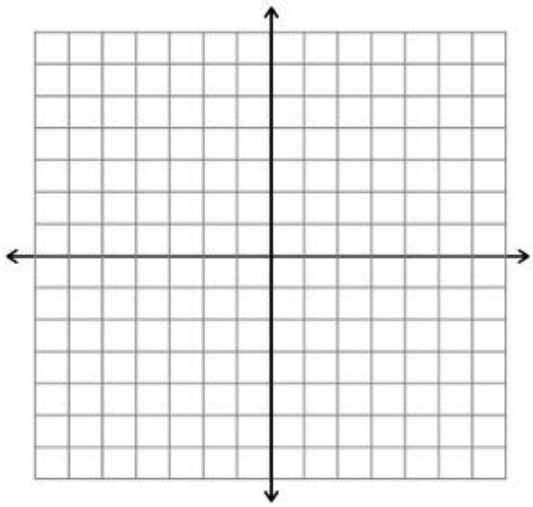
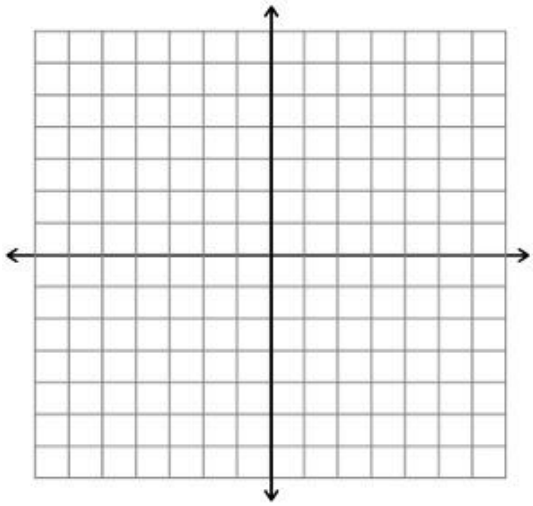
e. $f(x) = -|x - 3| - 2$
Describe any translations or reflections:



f. $f(x) = -\sqrt{x + 3} + 1$
Describe any translations or reflections:





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**Lesson: Writing Equations of Functions with Translations and Reflections**

**<u>Example</u>** Write the equation of a function $g(x)$ where the following transformations are performed on the graph of $f(x)$

a. $f(x) = \sqrt[3]{x}$ is shifted 3 units down and 2 units right

b. $f(x) = x^2$ is shifted 2.5 units up and $\pi$ units left

c. $f(x) = e^x$ is reflected about the x-axis, shifted 4 units down and 1 unit left



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**Lesson: Review of Quadratic Functions**

A polynomial function of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Where $a_n, a_{n-1}, \ldots, a_1, a_0$ are any complex or real number, $a_n \neq 0$, and $n$ is a nonnegative integer.

What do we call $n$ in this equation?

What do we call $a_n$ in this equation?

What do we call $a_n x^n$ in this equation?

**<u>Example</u>** What is the degree, leading coefficient, and dominating term in the following?

a. $f(x) = 4x^4 - 5x^2 + x^7 + 100$

b. $g(x) = x\sqrt{3} - 5$

c. $h(x) = 7$



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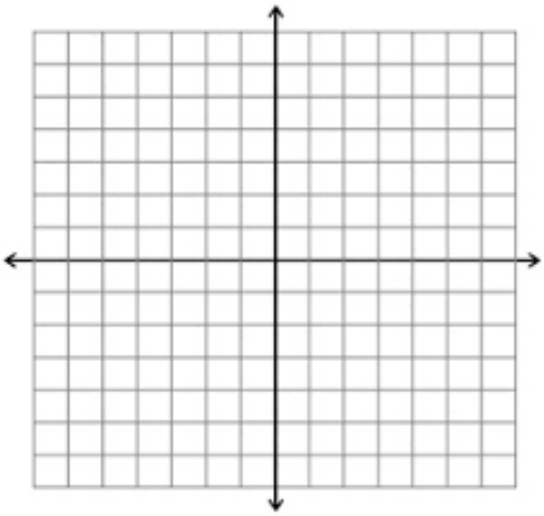
What is a quadratic function?

What do we call the graph of a quadratic function?

Draw and label the parts of a parabola.

**<u>Example</u>** Sketch the graph of $f(x) = -(x+4)^2 - 2$



What is the **vertex form** of the equation of the parabola?



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**Example** Show how to use completing the square to put $f(x) = x^2 + 6x - 4$ in vertex form. Write out clear directions.

**Example** Use completing the square to write $f(x) = 3x^2 + 5x + 2$ in vertex form, then sketch the graph.



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**<u>Example</u>** Find the intercepts of $f(x) = 3x^2 + 5x + 2$

How many x-intercepts can a parabola have? Draw a little sketch of each situation.

How can you tell how many intercepts a parabola will have if the parabola is in the form: $f(x) = ax^2 + bx + c$



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What is the vertex formula for a parabola in the form $f(x) = ax^2 + bx + c$?

**<u>Example</u>** Use the vertex formula to find the vertex of the following: $f(x) = 2x^2 - 4x + 1$



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**Examples: Finding Intercepts, Vertices, Etc of Quadratic Functions**

Find the vertex, axis of symmetry, domain, range, intercepts, and intervals of increasing and decreasing. If the function is in $f(x) = ax^2 + bx + c$ form, put it in vertex form. Sketch the graph. If the function is in $f(x) = a(x-h)^2 + k$ form, put it into standard form.

a. $f(x) = -\frac{1}{2}x^2 - 3x - \frac{1}{2}$



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b. $f(x) = 2(x-1)^2 + 3$



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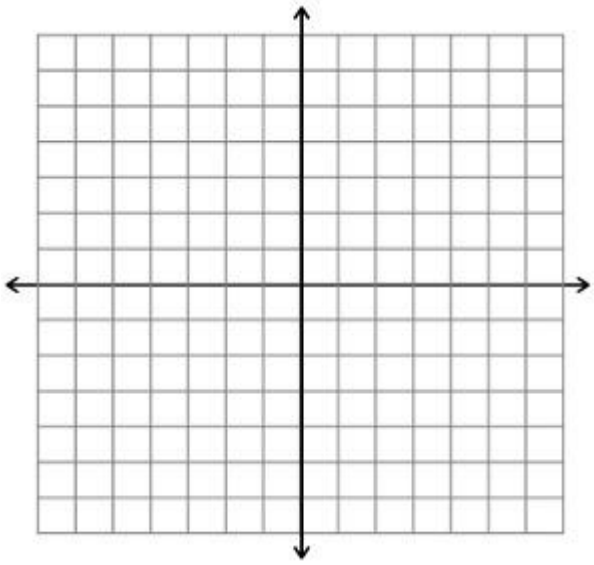
**Lesson: Understanding the Graphs of Quadratic Functions**

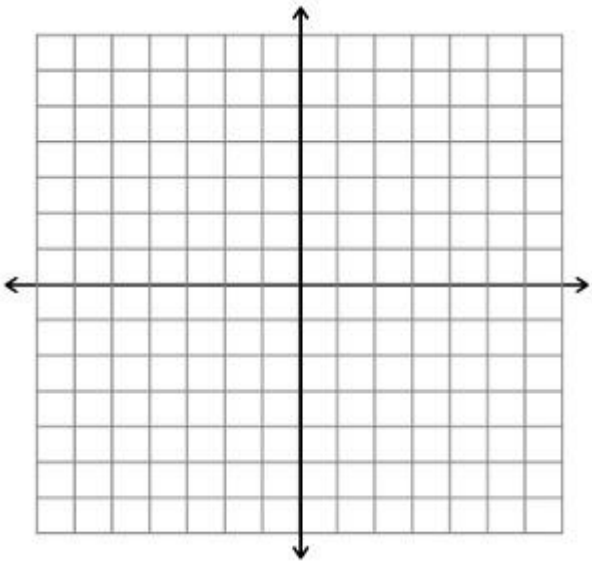
Quadratic functions are functions of the form _____, where $a \neq 0$. But in previous lessons we worked on graphing quadratic functions, ie parabolas in another way. Let's review that first.

**<u>Examples</u>** Graph the following:

a. $f(x) = (x+2)^2 - 3$

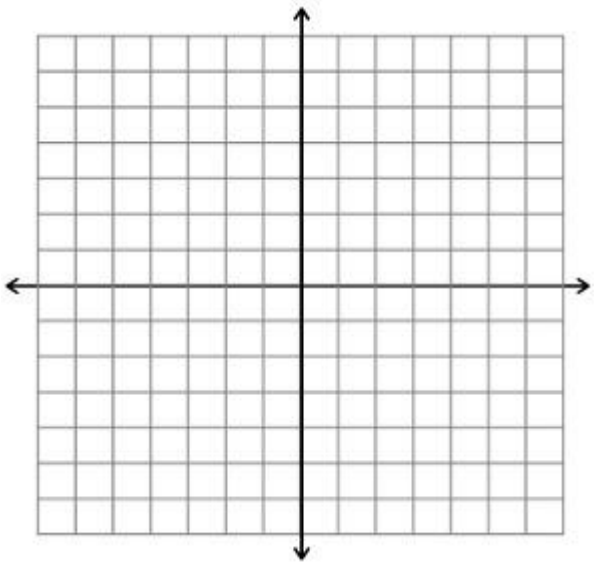


b. $g(x) = -(x-3)^2 + 1$

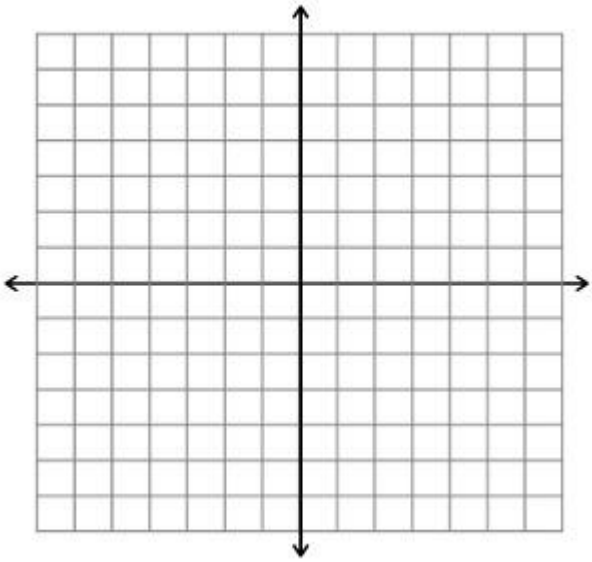


To begin, we will add on one more way to manipulate the graph of a parabola. Explain how to graph the follow:

a. $f(x) = 3x^2$
How do you interpret the 3?



b. $g(x) = -\frac{1}{2}(x-2)^2 + 1$
How do you interpret the $-\frac{1}{2}$?





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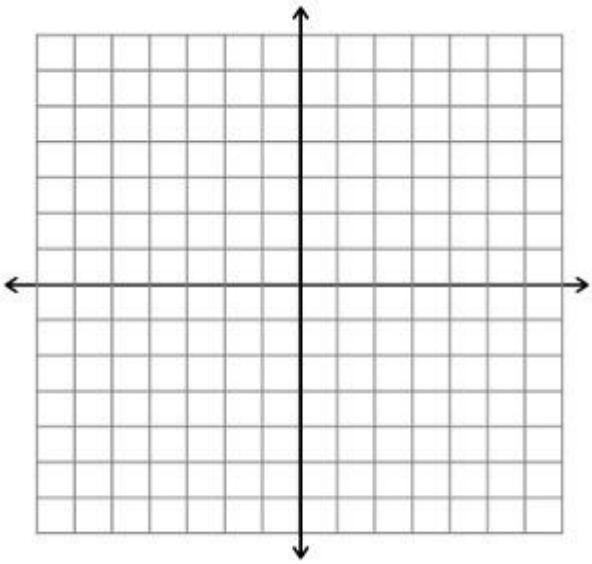
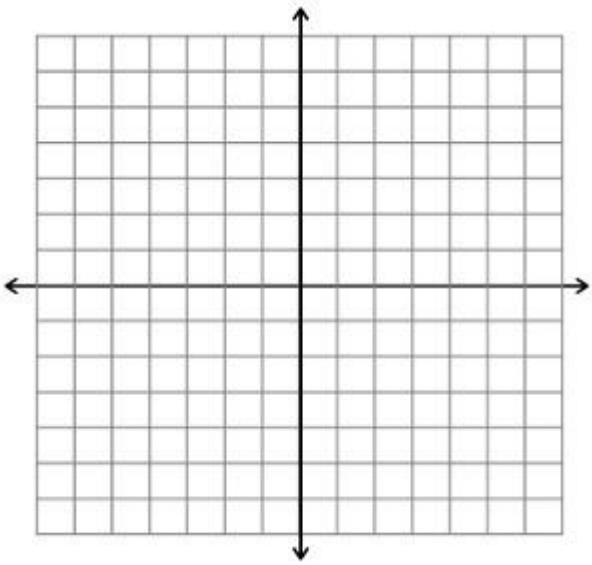
**Examples** Graph the following:

a. $f(x) = -2(x+1)^2 + 3$



b. $g(x) = \frac{1}{3}(x-2)^2 - 2$



Using all of the examples we have discussed, what is another equation form for the parabola?

Explain some of the characteristics of a parabola that we get from this format:



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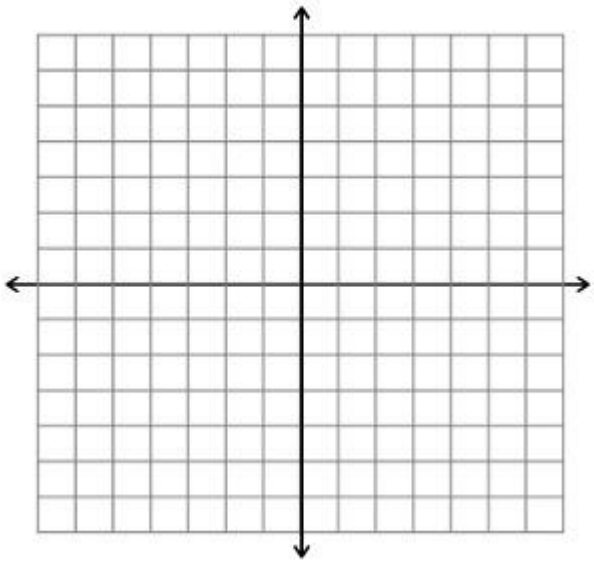
**Examples** For each of the following functions, find and state: (i) the vertex (ii) axis of symmetry (iii) x-intercepts (iv) y-intercepts (v) and then graph the function

a. $f(x) = (x-3)^2 - 4$



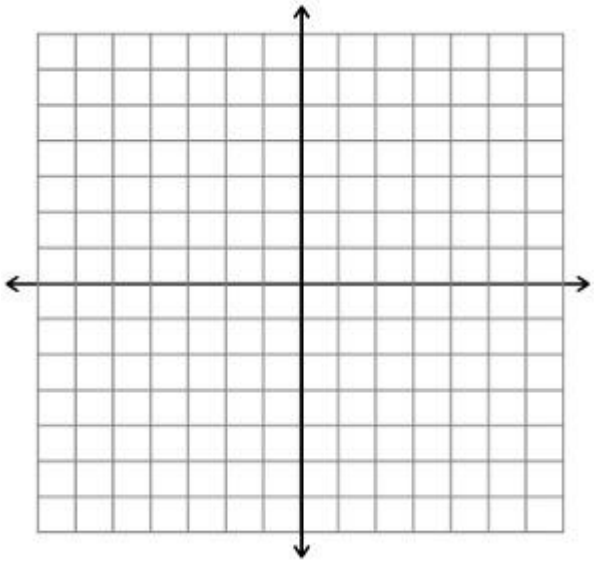
Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:

b. $f(x) = -2(x-3)^2 + 8$



Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:



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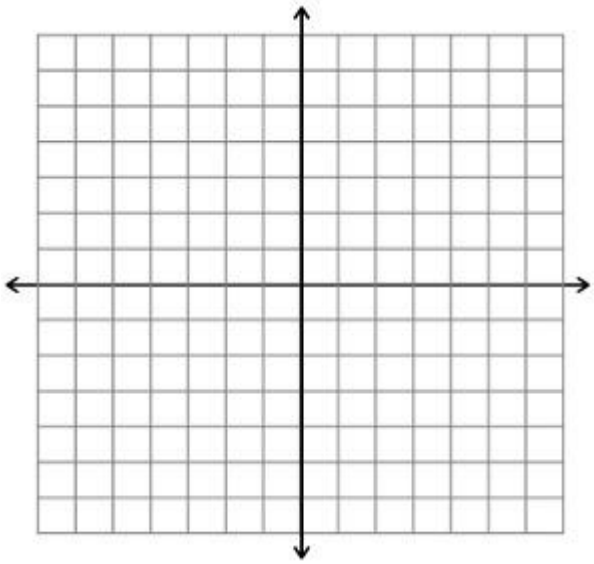
**Examples** For each of the following functions, find and state: (i) the vertex (ii) axis of symmetry (iii) x-intercepts (iv) y-intercepts (v) and then graph the function

a. $f(x) = \frac{1}{4}(x+2)^2 - 3$



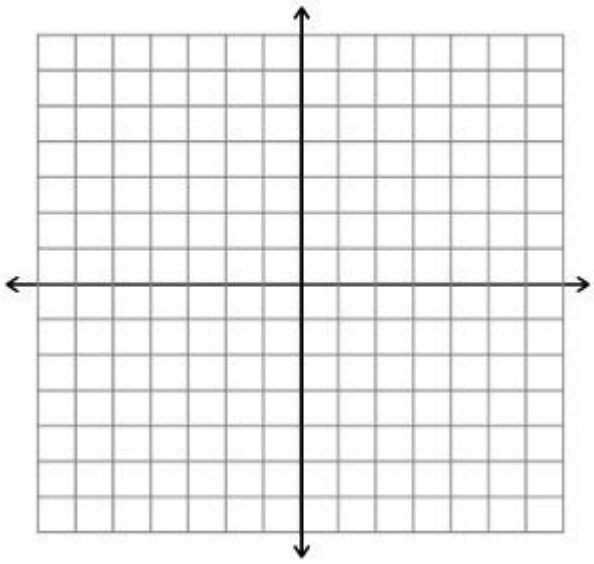
Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:

b. $f(x) = -\frac{1}{3}(x-1)^2 + 3$



Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:



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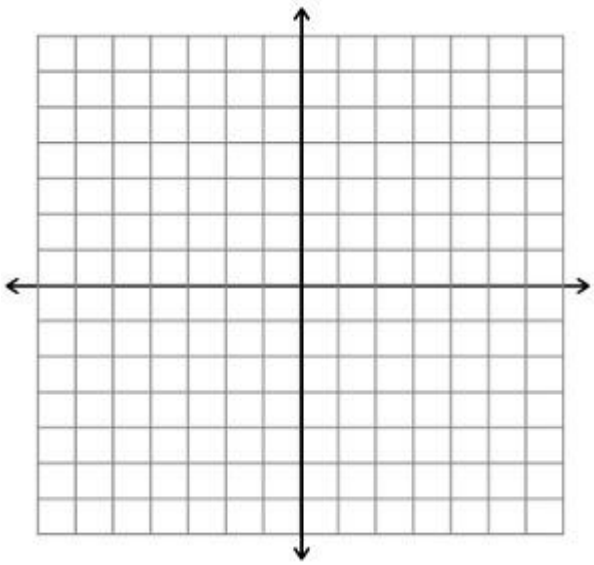
**Examples: Understanding How Many X-Intercepts There Are**

**<u>Examples</u>** For each of the following functions, find and state: (i) the vertex (ii) axis of symmetry (iii) x-intercepts (iv) y-intercepts (v) and then graph the function

a. $f(x) = \frac{1}{2}(x+1)^2 - 8$



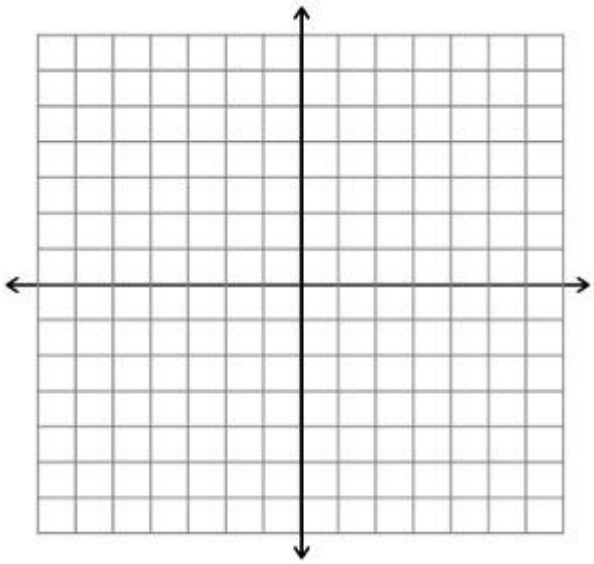
Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:

b. $f(x) = -2(x-3)^2$



Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:



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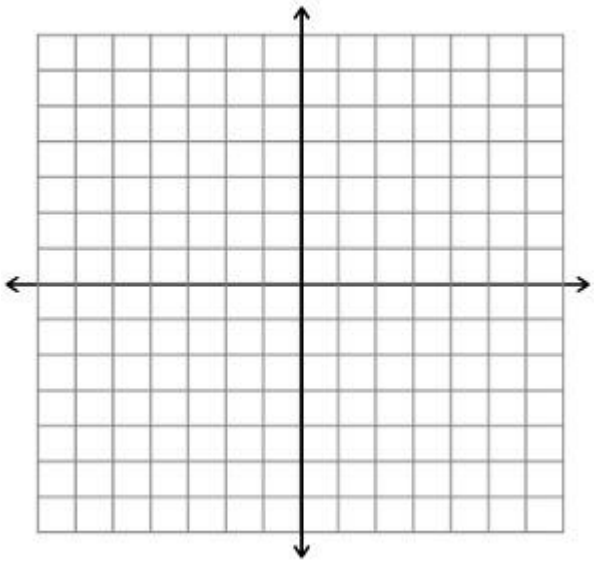
**Examples** For each of the following functions, find and state: (i) the vertex (ii) axis of symmetry (iii) x-intercepts (iv) y-intercepts (v) and then graph the function

a. $f(x) = (x-2)^2 + 1$



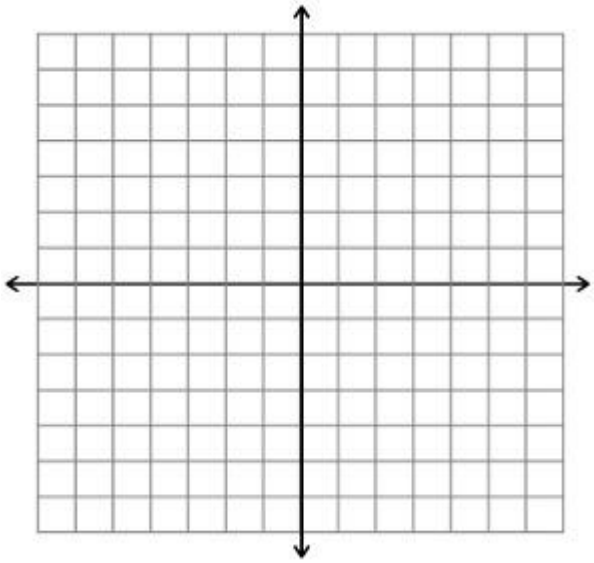
Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:

b. $f(x) = -(x+3)^2 - 1$



Vertex:

Axis of Symmetry:

X-intercepts:

Y-intercepts:



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**Lesson: Finding the Vertex of Quadratic Equations in the form $\boldsymbol{f(x) = ax^2 + bx + c}$**

What is the vertex formula?

Show how to use the vertex formula for $f(x) = 3x^2 + 6x - 2$

**<u>Examples</u>** Find the vertex of the following:

a. $f(x) = -x^2 + 4x - 2$

b. $f(x) = -\frac{1}{2}x^2 + 3x - 1$



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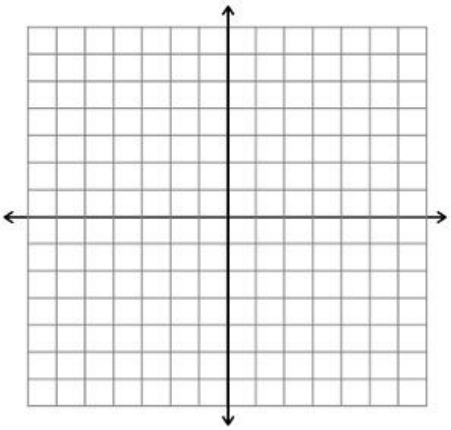
**Lesson: Putting Quadratics into the $f(x) = a(x-h)^2 + k$ Form**

Before you begin this lesson, you should already feel comfortable graphing an equation like $f(x) = 2(x-1)^2 + 3$. Graph this below:



Now show all the steps to put $f(x) = 2x^2 - 8x + 2$ into $f(x) = a(x-h)^2 + k$ form:



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**Examples** Put the following quadratic functions into $f(x) = a(x-h)^2 + k$ form:

a. $f(x) = x^2 + 4x - 2$

b. $g(x) = x^2 + 6x + 10$

c. $h(x) = -x^2 + 2x - 4$

d. $y = 3x^2 + 6x - 12$



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**Examples: Putting Quadratics into the $f(x) = a(x-h)^2 + k$ Form**

**<u>Examples</u>** Put the following quadratic functions into $f(x) = a(x-h)^2 + k$ form:

a. $f(x) = \frac{1}{2}x^2 + x - 3$

b. $g(x) = -\frac{1}{3}x^2 + 4x - 2$

c. $y = x^2 + 3x + 2$



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**Lesson: Determining Whether a Quadratic Function will have a Minimum or a Maximum**

Using the form $f(x) = ax^2 + bx + c$, explain what determines whether the parabola opens up or down:

**<u>Examples</u>** Determine whether the following quadratics open upwards or downwards, and then determine their vertex:

a. $f(x) = -2x^2 + 4x - 2$ upwards downwards

b. $g(x) = -\frac{1}{2}x^2 - 2x + 2$ upwards downwards

c. $h(x) = x^2 + 6x + 1$ upwards downwards



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**Lesson: Applications of Quadratic Functions 1**

A ball is thrown upwards from a cliff 24 feet about the ground. The height h of the ball, in feet, t seconds after it is thrown is given by:

$$h(t) = -16t^2 + 32t + 24$$

How long does it take the ball to reach its maximum height?

What is the maximum height attained by the ball?



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**Lesson: Quadratic Inequalities**

A quality inequality is an inequality in the form: $ax^2 + bx + c < 0$ where a, b, and c are real numbers. You can interchange $<$ with $\leq, >, \geq$.

There are 3 steps to solving a quadratic inequality. Write out each step as you solve the next example.

**<u>Example</u>** Write out the steps that you use to solve the following: $x^2 - x - 6 > 0$. Write your answer in interval notation.

Explain Step 1:

Explain Step 2:

Explain Step 3:



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**Example** Solve: $3x^2 - 4 \leq x$. Write your answer in interval notation.

**Example** Solve $x^2 \geq 25$. Write your answer in interval notation.



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**Lesson: Rational Inequalities**

Rational inequalities look like this:

$\frac{5}{x+3} > 3$ or $\frac{2x+5}{4x-1} \leq 2$

We will out the steps to solve a rational inequality by using the following example.

**<u>Example</u>** Solve $\frac{4}{x+2} \geq 6$ Put your final answer in interval notation.

Explain Step 1 and show how to do it:

Explain Step 2 and show how to do it:

Explain Step 3 and show how to do it:



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What do you have to watch out for with rational inequalities?

**<u>Example</u>** Solve $\frac{3}{x+6} \leq 2$ Put your final answer in interval notation.

**<u>Example</u>** Solve $\frac{x+3}{x-5} \geq 1$ Put your final answer in interval notation.



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**Lesson: Graphs of Basic Polynomial Functions and Translations of the Basic Graphs**

Explain the graphs of $f(x) = x^2, f(x) = x^4, f(x) = x^6$, etc.

Explain the graphs of $f(x) = x^3, f(x) = x^5, f(x) = x^7$, etc.



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If we multiply the polynomial like so: $f(x) = ax^n$, what happens to the graph?

What happens if we have $f(x) = a(x-h)^n + k$?

**<u>Example</u>** Sketch the graphs:

a. $f(x) = (x-5)^5 + 3$

b. $f(x) = -2(x+1)^6 - 4$



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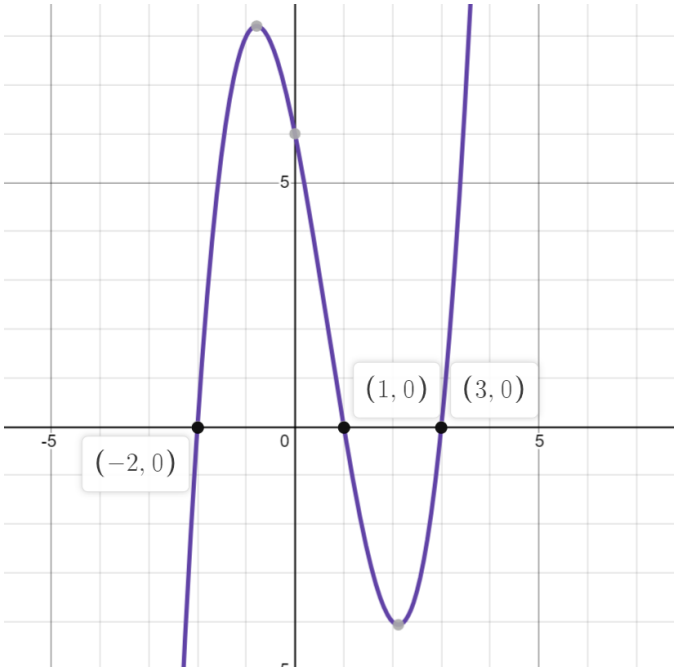
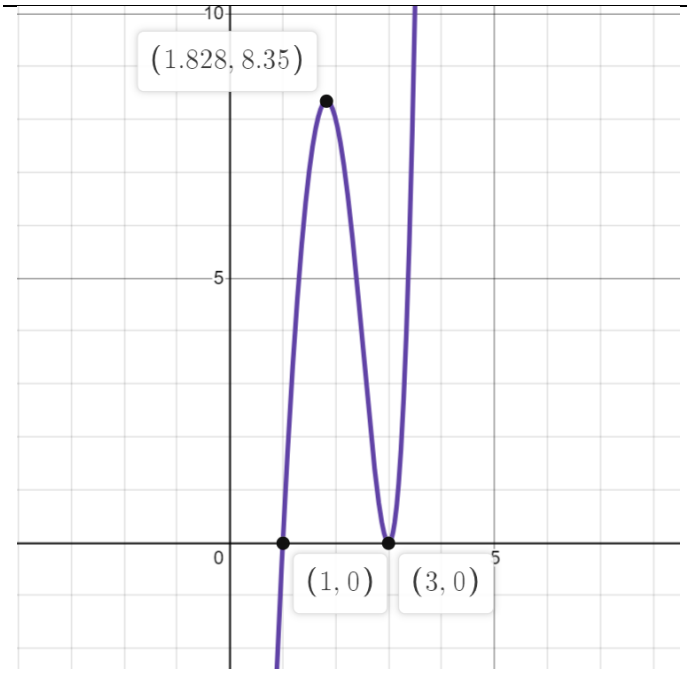
**Lesson: Graphs of Polynomial Functions, Turning Points, and End Behavior**

In the last video, we discuss the graphs of basic polynomials and translations. In this video, we will look at more complicated possibilities.

Below are the graphs of a few polynomials. What patterns do you notice?




$$f(x) = x^3 - 2x^2 - 5x + 6 = (x-1)(x+2)(x-3)$$




$$f(x) = x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36$$
$$= (x-3)^2(x-1)(x-2i)(x+2i)$$



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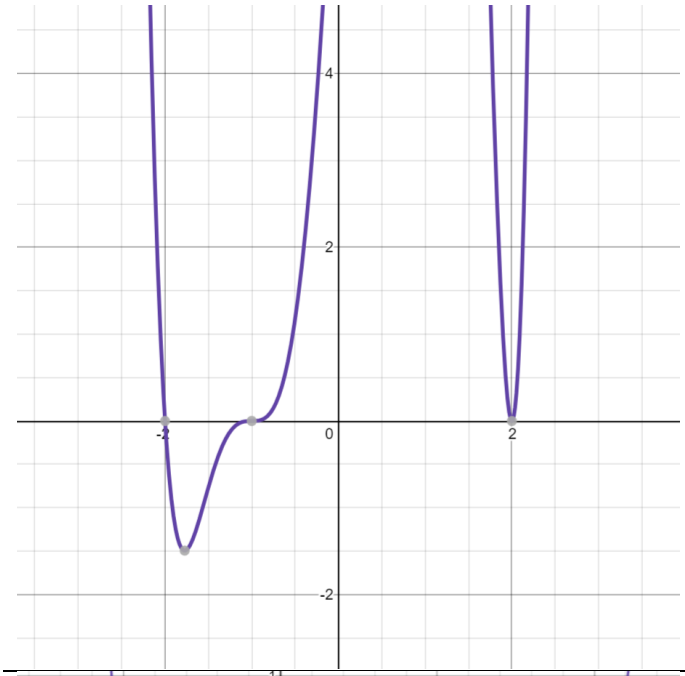
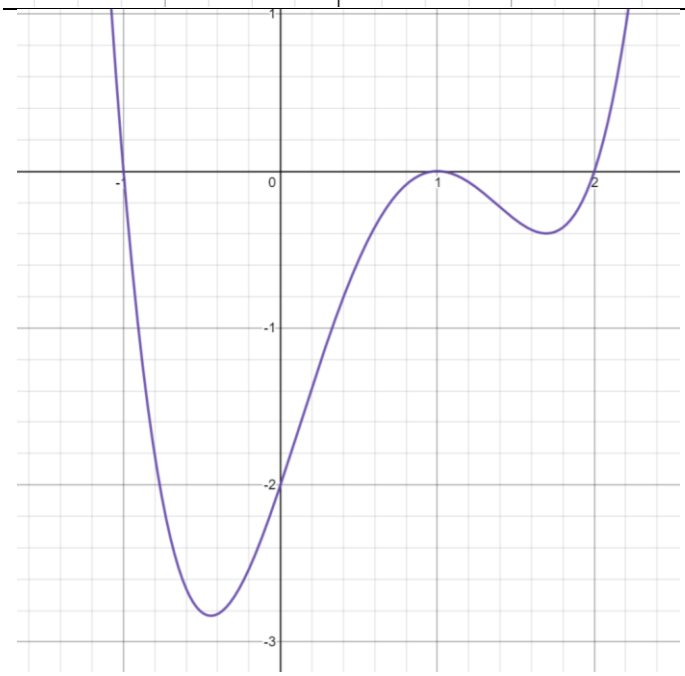



$f(x) = (x-2)^2(x+1)^3(x+2)$




$f(x) = (x-1)^2(x+1)(x-2)$

What do you notice about the behavior of these graphs at zeros? Summarize this in your own words, it was discussed as a general pattern in the video.



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What are turning points and how many will a degree n polynomial have?

What is the end behavior of a polynomial and how do you determine it?

**<u>Example</u>** What are the number of turning points of the following and what is the end behavior of each?

a. $f(x) = x^2 + 3x + 2$

b. $f(x) = (x+1)^2(x-2)$

c. $f(x) = -\frac{1}{2}x(x-3)^4(x+1)^3(x+2)$



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List the steps required to graph a polynomial:

**<u>Example</u>** Graph $f(x) = x^3 - x^2 - 4x + 4$



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**<u>Example</u>** Graph $f(x) = -(x+1)(x-2)^2(x-3)$



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What is the relationship between x-intercepts, zeros, solutions, and factors of polynomials????



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**Examples: Graphs of Polynomial Functions, Turning Points, and End Behavior**

Graph the following polynomials:

a. $f(x) = x^3 + 5x^2 + 2x - 8$



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b. $f(x) = 3x^4 + 5x^3 - 2x^2$



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c. $f(x) = (x-1)^3(x+2)^2$



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**Lesson: The Intermediate Value Theorem and the Boundedness Theorem**

State the Intermediate Value Theorem

**<u>Example</u>** Using synthetic division to show $f(x) = -x^3 + 2x^2 - 4x + 5$ has a real zero between 1 and 2.

**<u>Example</u>** Using synthetic division to show $f(x) = x^4 - 4x^3 - x + 3$has a real zero between 0.5 and 1.



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State the Boundedness Theorem

**<u>Example</u>** Show that $f(x) = x^4 + 5x^2 + 3x - 7$ has no real zero less than -3.

**<u>Example</u>** Show that $f(x) = 2x^5 - x^4 + 2x^3 - 2x^2 + 4x - 4$ has no real zero greater than 2



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**Lesson: The Basic Graphs of Rational Functions**

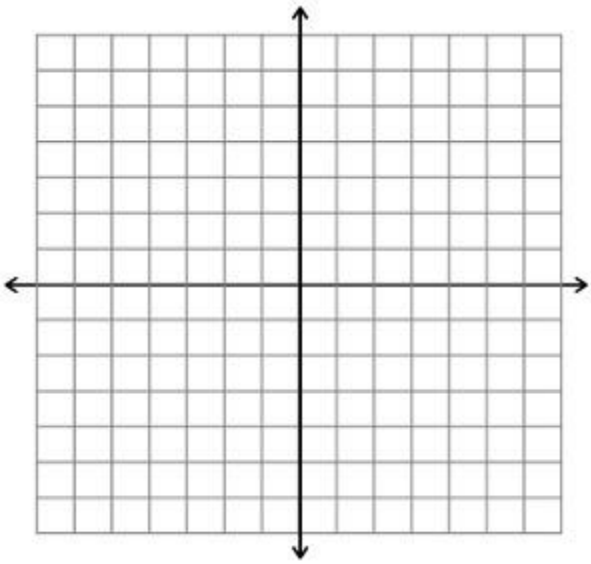
A rational function is a function of the form:

$\frac{f(x)}{g(x)}, g(x) \neq 0$

Where $f(x)$ and $g(x)$ are polynomials.

There are 2 graphs that we view as the most basic version of rational functions.

Make a table and graph $f(x) = \frac{1}{x}$



What is the domain and range?

Where does the graph not exist (ie where is it discontinuous)

What are the y-axis and x-axis for this graph?

What kind of symmetry does this graph have?



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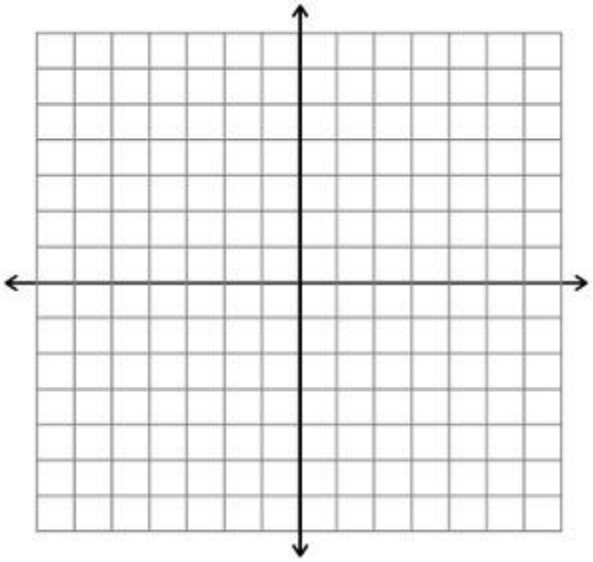
Make a table and graph $f(x) = \frac{1}{x^2}$



What is the domain and range?

Where does the graph not exist (ie where is it discontinuous)

What are the y-axis and x-axis for this graph?

What kind of symmetry does this graph have?



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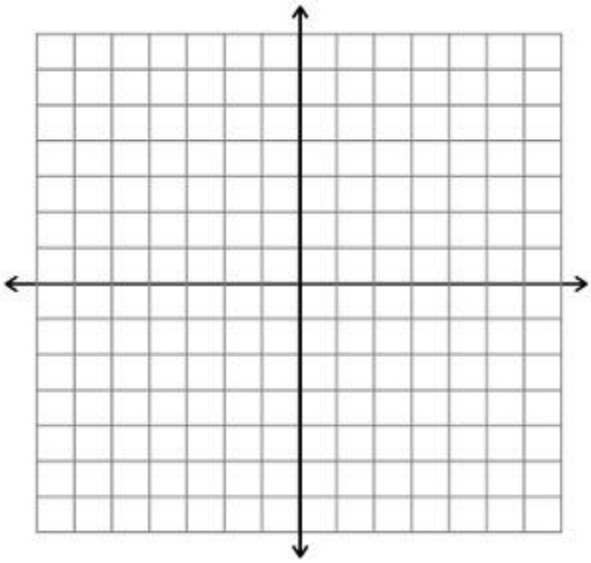
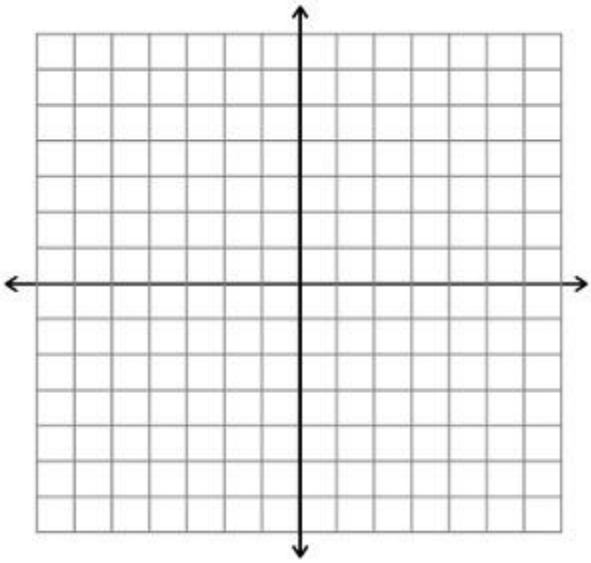
**Example** Describe the translations for the following graphs, and then draw an approximate sketch of the graph.

a. $f(x) = \frac{1}{(x+1)^2} - 2$



b. $g(x) = -\frac{2}{x} + 1$





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Sketch what happens for odd powers of n for: $f(x) = \frac{1}{x^n}$, ie just draw a sketch of $y = \frac{1}{x^3}, y = \frac{1}{x^5}, y = \frac{1}{x^7}$

Sketch what happens for higher even powers of n for: $f(x) = \frac{1}{x^n}$, ie just draw a sketch of $y = \frac{1}{x^4}, y = \frac{1}{x^6}, y = \frac{1}{x^8}$



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**Lesson: Asymptotes of Rational Functions**

What is an informal definition of asymptotes?

Is it possible that a graph would graph an asymptote?

Here are the formal definitions of asymptotes:

A graph $f(x) = \frac{p(x)}{q(x)}$ $(q(x) \neq 0)$ has a

- **Vertical asymptote** if $|f(x)| \to \infty$ as $x \to a$. The asymptote is x=a.
- **Horizontal asymptote** if $f(x) \to b$ as $|x| \to \infty$. The asymptote is y=b.

What is super important about asymptote?

Where do you typically find vertical asymptotes?



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How can you determine horizontal asymptotes? Describe the two ways, and also explain how oblique asymptotes. (In other words, explain what we talked about in the video).



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**<u>Examples</u>** Find any vertical, horizontal, or oblique asymptotes.

a. $f(x) = \frac{x+2}{x^2+7x+12}$



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b. $f(x) = \frac{3x+1}{4x-2}$



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c. $f(x) = \frac{x^2+3x+2}{x^2-9}$



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d. $f(x) = \frac{x^2+x}{x+3}$



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**Examples: Finding Oblique Asymptotes**

Find all the asymptotes for the following functions.

a. $f(x) = \frac{x^2+2x}{2x+1}$

b. $f(x) = \frac{x^2+2}{x+1}$



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**Lesson: Graphing Rational Functions**

To learn how to graph rational functions, we will outline the steps and then graph. **Clearly write directions for each step as we go through this process!!!**

Graph $f(x) = \frac{x+3}{x^2+7x+10}$. Outline all the relevant steps discussed in the video.



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Graph $f(x) = \frac{x+1}{x-3}$. Outline all the relevant steps discussed in the video.



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Graph $f(x) = \frac{x^2+1}{x-3}$. Outline all the relevant steps discussed in the video.



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**Examples: Graphing Rational Functions**

Graph $f(x) = \frac{(x+2)^2}{(x-1)(x+3)}$. Outline all the relevant steps discussed in the video.



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Graph $f(x) = \frac{2x^2-3x+1}{x-2}$. Outline all the relevant steps discussed in the video.



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**Examples: Equations of Rational Functions**

Copy the graphs from the video and then describe how to find the equations of those graphs.



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**Lesson: The Factor Theorem**

What does the factor theorem say?

**<u>Example</u>** Determine whether $(x-1)$ is a factor of each polynomial:

a. $f(x) = 9x^4 - 13x^2 + 4$

b. $f(x) = 4x^3 + 10x^2 + 2x - 4$

Given the above, what can we say is also a zero for example a?



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**Example** Factor $f(x) = 3x^3 + 4x^2 - 17x - 6$ into linear factors given that $-3$ is a zero.

**Example** Factor $f(x) = 2x^3 - x^2 - 8x + 4$ into linear factors given that $\frac{1}{2}$ is a zero.



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**Lesson: The Rational Zeros Theorem**

The rational zeros theorem is a way to determine *possible* candidates for *rational* zeros of a polynomial function.

State the Rational Zeros Theorem:

**<u>Example</u>** Find all the possible rational zeros for $f(x) = 2x^3 - x^2 - 8x + 4$ and then factor into linear factors.



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**<u>Example</u>** Find all the possible rational zeros for $f(x) = 5x^4 - 19x + x^2 + 31x + 6$ and then factor into linear factors.



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**Lesson: The Fundamental Theorem of Algebra and the Number of Zeros Theorem**

What does the Fundamental Theorem of Algebra state?

What is the number of zeros theorem?

**<u>Example</u>** What is the maximum number of zeros that each polynomial could possibly have?

a. $f(x) = x^2 - 3x + 1$

b. $f(x) = x^{17} - 3x + 1$

c. $f(x) = 9$



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**Example** Find a polynomial $f(x)$ that has zeros $-3, 3$, and $1$, where $f(2) = -10$

**Example** Find a polynomial $f(x)$ that has zeros $6, -3$, and $-2$, where $f(1) = 20$



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**Lesson: Conjugate Zeros Theorem**

Before we can discuss the theorem, let's review a few properties of conjugates.

What is the conjugate of the following:

a. $3 + 2i$ b. $5 - 3i$ c. $-7 + i$

What is another way that we denote conjugates? Summarize the explanation in the video. Give an example of how this notation works.

What are some properties of conjugates?



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What is the conjugate zeroes theorem?

**<u>Example</u>** Find a polynomial function of least degree with only real coefficients and zeros $2$ and $(3 - i)$.



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**Example** Suppose $k = -i$ is one zero of $f(x) = x^4 + 26x^2 + 25$. What are the other zeros?

**Example** Factor $f(x) = 2x^3 + (3 - 2i)x^2 + (-8 - 5i)x + (3 + 3i)$ into linear factors given $k = 1 + i$ is a factor.

Before you begin, what is unique about this problem?



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**Lesson: Finding Zeros of a Polynomial Function**

In previous videos we discussed:

- The factor theorem
- The rational zeros theorem
- The fundamental theorem of algebra
- The number of zeros theorem
- The conjugate zeros theorem

Using all of these theorems, we can combine them to find the zeros of *any* polynomial.

What can we say about a polynomial function of odd degree with real coefficients?

What can we say about a polynomial function of even degree with real coefficients?

What if a polynomial doesn't have only real coefficients?



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**Example** Find all zeroes of the following polynomials.

a. $f(x) = x^3 - 21x - 20$

b. $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$



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**Examples: Finding Zeros of a Polynomial Function**

Find all the zeros of the polynomials.

a. $f(x) = x^4 + 13x^2 + 36$

b. $f(x) = 2x^4 - 11x^3 + 4x^2 + 29x + 12$



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c. $f(x) = x^4 - 7x^3 + 17x^2 - x - 26$



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**Lesson: Descartes' Rule of Signs**

Let $f(x)$ be a polynomial function with real coefficients and a nonzero constant term, with terms in descending powers of x. Explain how to use Descartes' Rule of Signs:

**<u>Example</u>** Determine the number of different possibilities for the number of positive, negative, and nonreal complex zeroes:

a. $f(x) = 3x^5 + 4x^4 - x^3 + 2x - 1$



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b. $f(x) = x^6 + x^5 - x^4 + x^3 + x^2 - x + 1$

c. $f(x) = 3x^5 - 7x^3 + 6x + 8$



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**Lesson: Review of Function Operations**

Given that $f$ and $g$ are functions, how do we define the function operations?

$(f+g)(x) =$

$(f-g)(x) =$

$(fg)(x) =$

$\left(\frac{f}{g}\right)(x) =$

**<u>Example</u>** Let $f(x) = 3x+1, g(x) = x^2-4, h(x) = 3x^2+7x+2$, and $k(x) = x^2+6x+9$. Find each of the following:

a. $(f+g)(x) =$

b. $(f+g)(3) =$

c. $(g-k)(1) =$



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**<u>Example</u>** Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = 3x^2 + 7x + 2$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $\left(\frac{h}{f}\right)(x) =$

b. $(gh)(x) =$

c. $\left(\frac{k}{h}\right)(1) =$



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**Lesson: Review of Composition of Functions**

For two functions $f$ and $g$, what is the composition $(f \circ g)(x)$?

**<u>Example</u>** Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(f \circ g)(x) =$

b. $(g \circ f)(x) =$



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**<u>Example</u>** Let $f(x) = 3x + 1, g(x) = x^2 - 4, h(x) = \sqrt{x}$, and $k(x) = x^2 + 6x + 9$. Find each of the following:

a. $(k \circ f)(1) =$

b. $(g \circ h)(4) =$

c. $(h \circ k)(x) =$



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**Lesson: The Difference Quotient**

Explain what the difference quotient is and draw some sketches to explain how it works. In other words, watch the video explanation and then explain it in your own words on this page.



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State the formula for the difference quotient here:

**<u>Example</u>** Calculate the different quotient for the following.

a. $f(x) = 4x + 5$

b. $f(x) = x^2 - 1$



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**Lesson: The Difference Quotient**

Find the difference quotient.

a. $f(x) = \frac{1}{x+3}$

b. $f(x) = x^2 - 4x + 1$

c. $f(x) = -x^2 + 3$



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**Lesson: A Review of Inverse Functions**

What is a one-to-one function?

**<u>Examples</u>** Are the following one-to-one functions? If no, why not?

a. $f(x) = x^2$

b. $g(x) = 2x + 1$

What is the horizontal line test?



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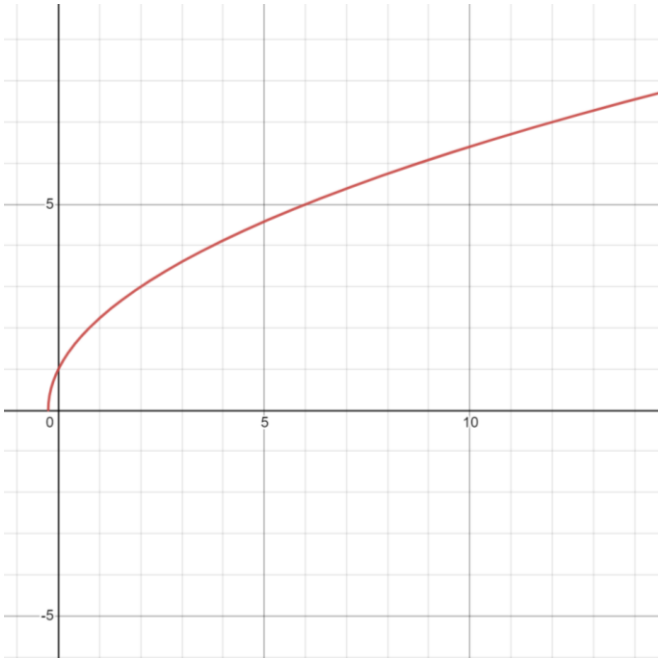
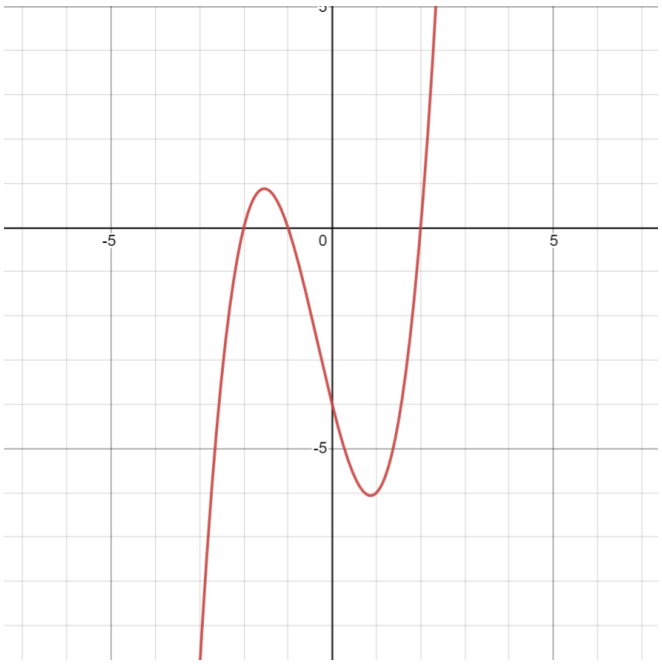
**Example** Use the horizontal line test to determine if the functions are one-to-one:

a.




b.




What is an inverse function?

What does an inverse function have to do with one-to-one functions?



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Describe how to use composition to determine if 2 functions are inverses of one another.

**Example** Use composition to determine if the functions $f(x) = 2x - 4$ and $g(x) = \frac{1}{2}x - 2$ are inverses.

**Example** Use composition to determine if the functions $f(x) = \frac{2}{x+6}$ and $g(x) = \frac{6+x}{2}$ are inverses.



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Using $f(x) = 4x + 2$, outline the steps for how to find the inverse of a function.



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**Example** Find the inverse of the following functions.

a. $f(x) = \sqrt[3]{3x+1}$

b. $f(x) = \frac{x+2}{x-1}$

c. $f(x) = \sqrt{x-5}$



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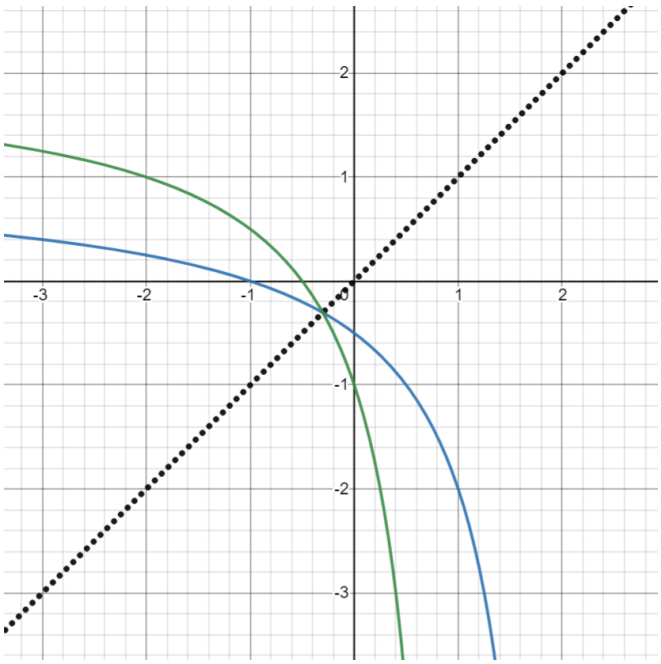
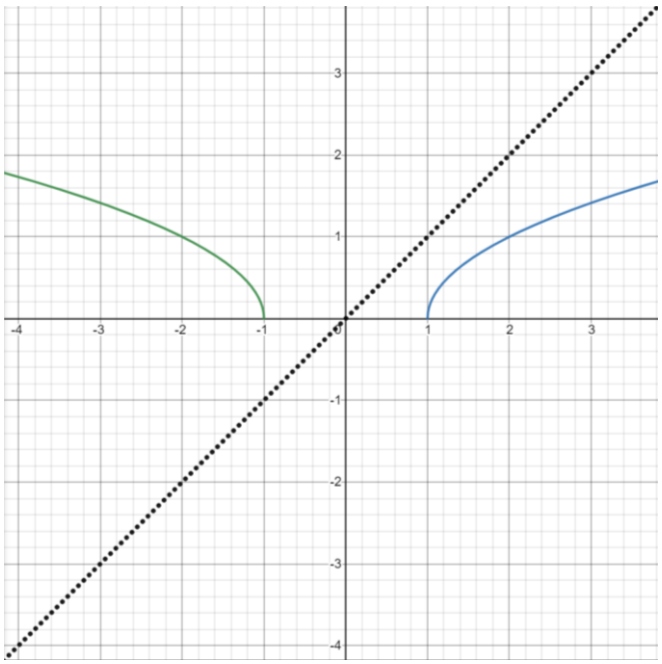
How can we tell on a graph if 2 functions are inverses of one another?

**<u>Examples</u>** Are the following graphs inverses of one another?

a.




b.



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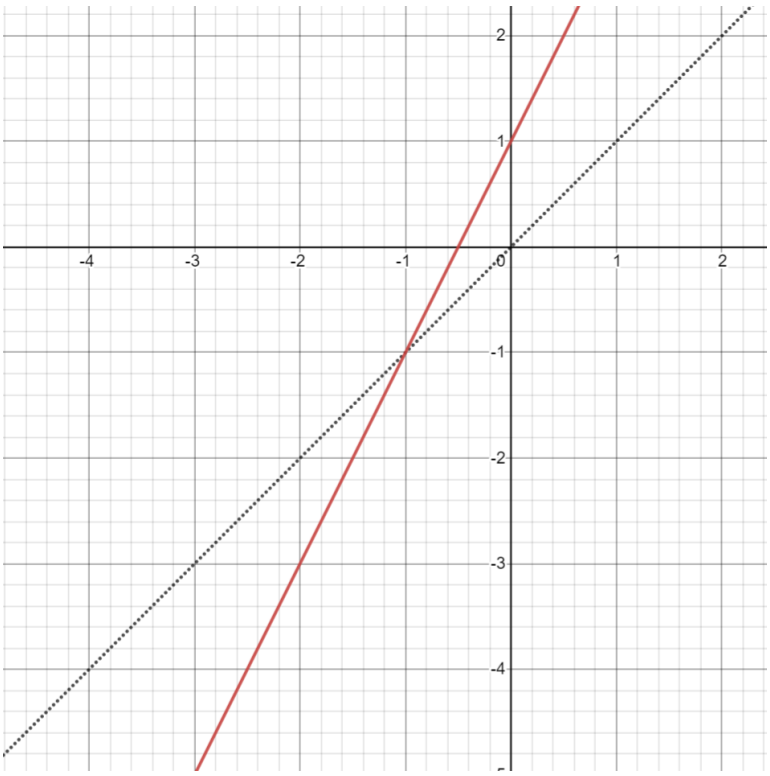
**Example** Draw the inverse of the following graph.



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**Lesson: Determining if Functions are One-to-One**

Before we begin, you should already be familiar with the concept of functions and the vertical line test. First let's do some review.

**<u>Review</u>** Determine whether the following relations are functions, then determine the domain and range.

a. $\{(2,3),(5,7),(2,4)\}$

b.

Now we will discuss a specific type of function: a one-to-one function. Define this below:

How can we visualize this?



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**Example** Determine if any of the following are i) functions and ii) one-to-one functions

a. $\{(2,4),(-1,7),(3,7)\}$

b. $\{(3,6),(-1,2),(-3,-4)\}$

Similar to how we have a test to determine if a graph represents a function, we have another test to determine if a function is one-to-one.

Describe the horizontal line test below:

**Examples** Determine whether the following graphs represent i) functions and ii) one-to-one functions (draw your own):

a. b. c.



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**Lesson: Finding the Inverse of Functions**

One-to-one functions have inverses. Describe what the inverse of a function is:

**<u>Example</u>** Find the inverse of the following function: $\{(1,4),(7,8),(3,-2)\}$

Describe the steps to find the inverse of a function:



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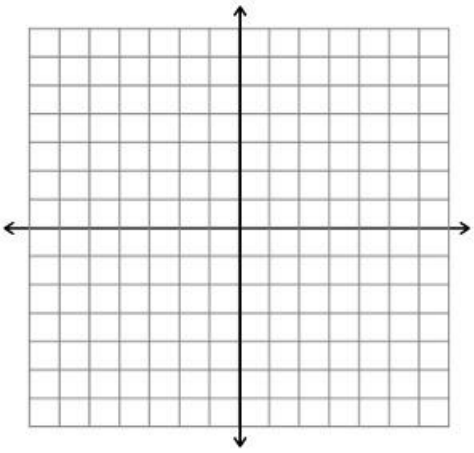
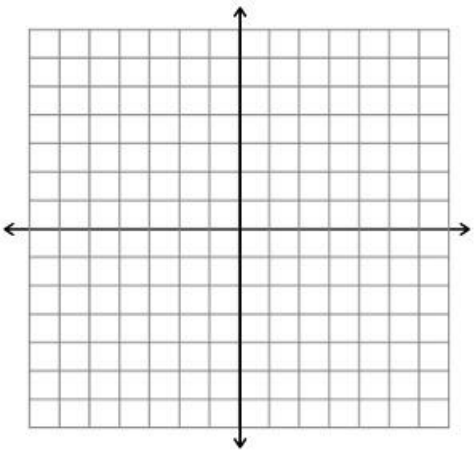
**<u>Examples</u>** Find the inverse of the following one-to-one functions. Then plot the original function on the graph with its inverse.

a. $f(x) = 3x + 1$



b. $g(x) = -\frac{1}{2}x + 4$



What is the relationship of a graph with its inverse?



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**<u>Examples</u>** Find the inverse of the following one-to-one functions.

a. $h(x) = \sqrt[3]{x}$

b. $f(x) = x^2, x \geq 0$



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**Lesson: A Review of Exponential Functions and their Graphs**

What is the form of a general exponential equation?

**<u>Example</u>** Let $f(x) = 3^x$ and $g(x) = \left(\frac{1}{2}\right)^x$. Find the following.

a. $f(-2)$

b. $f\left(\frac{3}{2}\right)$

c. $g(-3)$

d. $g(3.2)$



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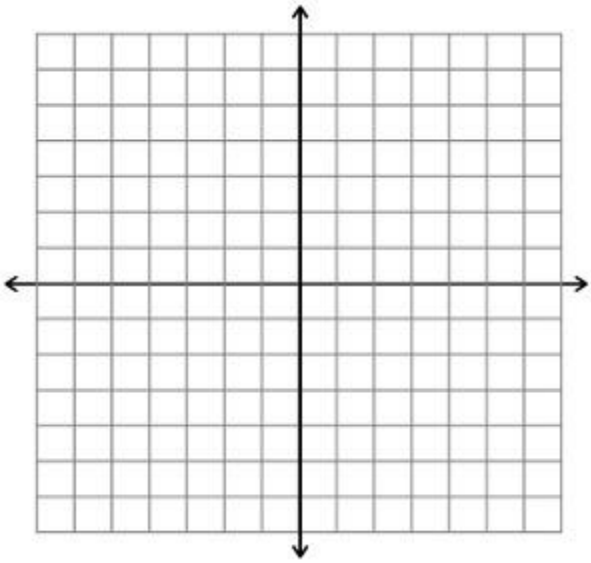
Sketch the graph of $f(x) = 2^x$ and create a table to go with it.



When will an exponential function $f(x) = a^x$ take this shape?

What is the x-axis for this graph?

What are 3 standard points to note here?

What is the domain and range?



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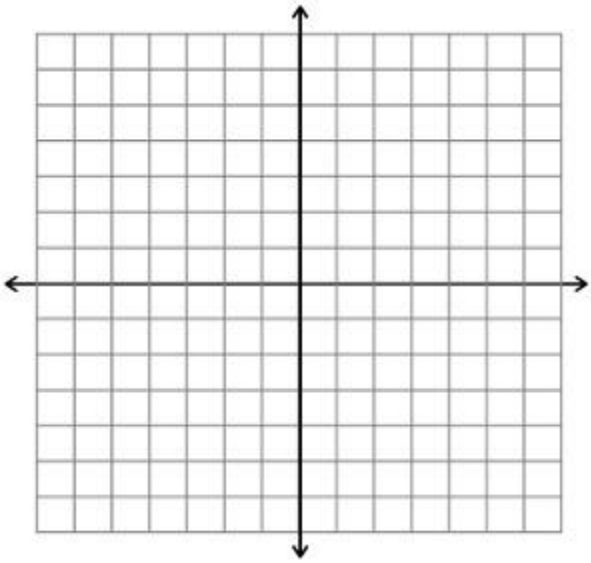
Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x$ and create a table to go with it.



When will an exponential function $f(x) = a^x$ take this shape?

What is the x-axis for this graph?

What are 3 standard points to note here?

What is the domain and range?



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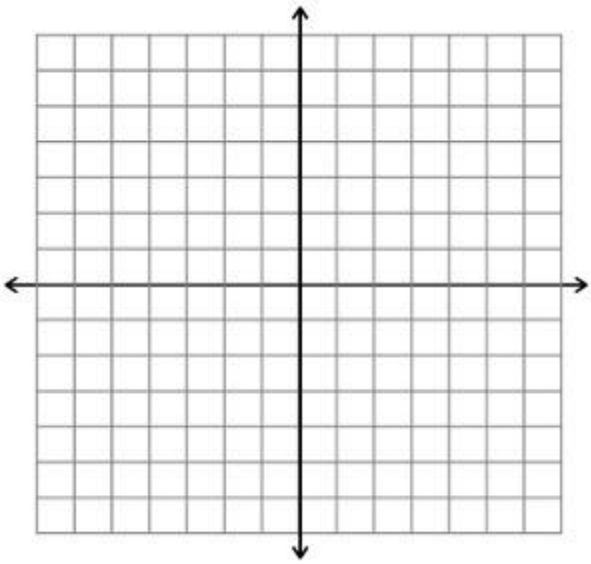
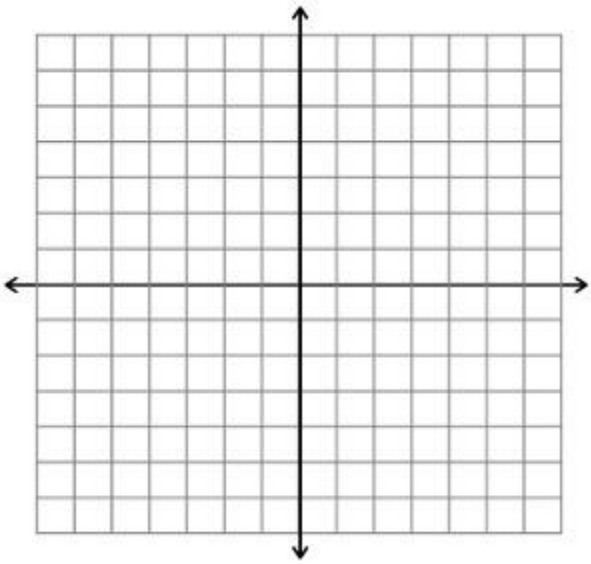
**Examples** Describe the translations and sketch the following graphs.

a. $f(x) = -3^x + 1$



b. $f(x) = \left(\frac{1}{2}\right)^{x+1} - 2$





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**Lesson: Solving Basic Exponential Equations**

Outline the general steps to solve $3^{x+1} = 81$.

Outline the general steps to solve $4^{x-2} = 8^{x+1}$.



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**<u>Examples</u>** Solve the following.

a. $27^{2x} = 9^{x-1}$

b. $e^{3-x} = (e^2)^{-x}$

c. $\left(\sqrt{3}\right)^{x+3} = 9^x$



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**Examples** Solve the following.

a. $x^{\frac{3}{2}} = 8$

b. $x^{\frac{5}{2}} = 4$

c. $x^{-4} = \frac{1}{16}$



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**Lesson: Compound Interest**

What is the formula for compound interest and what does it mean?

**<u>Example</u>** Suppose $2000 is invested into an account paying 2.5% interest per year compounded monthly. How much is in the account after 10 years, and how much interest is earned in that time?

What else do we call the A from the compound interest formula?

What else do we call the P from the compound interest formula?



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**Example** In 5 years, you want to have $8000 in cash to buy a car (including taxes). Let's use savings to help us reach this goal.

a. You find an account with a 3.25% interest rate compounded biannually (2x a year). What amount would you have to deposit today to ensure you have $8000 in 5 years?

b. Let's pretend that as of today, you have $5000. What interested rate would you need to ensure that this will grow to $8000 in 5 years?



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What is the number e?

What is the formula for continuous compounding?

**<u>Example</u>** Suppose $2000 is invested into an account paying 2.5% interest per year compounded continuously. How much is in the account after 10 years, and how much interest is earned in that time?



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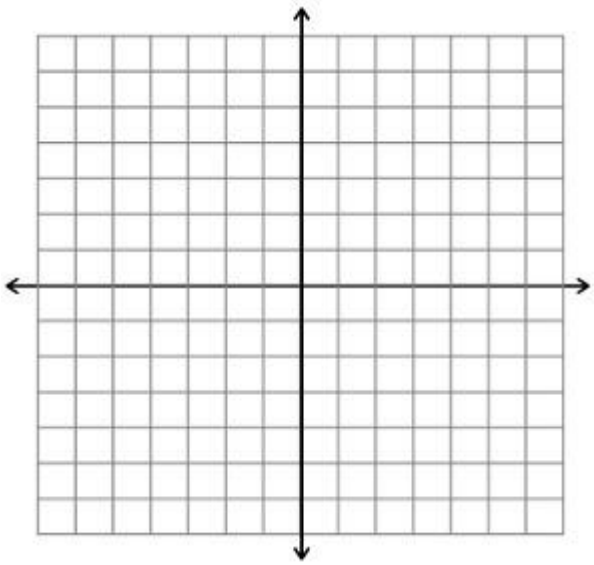
**Lesson: Exponential Functions**

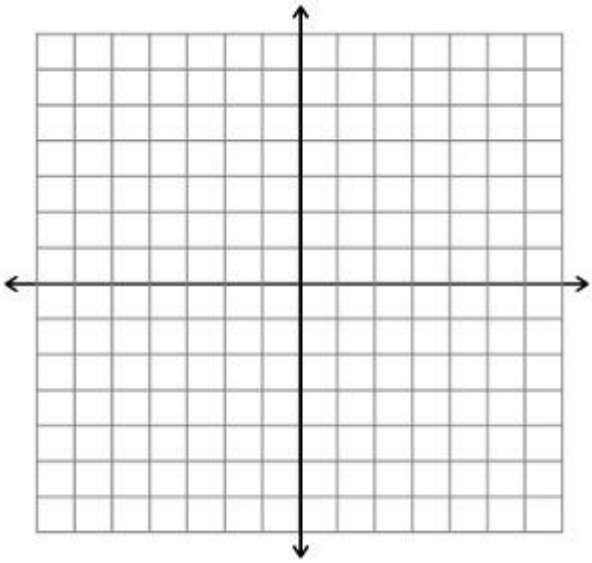
An exponential function is a function of the form _____ where $a > 0, a \neq 1$.

**<u>Examples</u>** Graph the following functions, then determine their domain and range.

a. $f(x) = 2^x$



b. $g(x) = \left(\frac{1}{2}\right)^x$



Using the graphs above, summarize some of the characteristics of these graphs:

The number e is an irrational number that is frequently used in math. What is an approximation of e?



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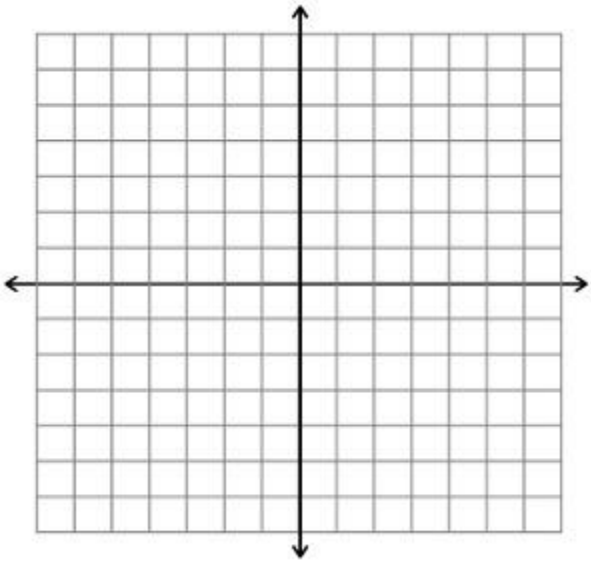
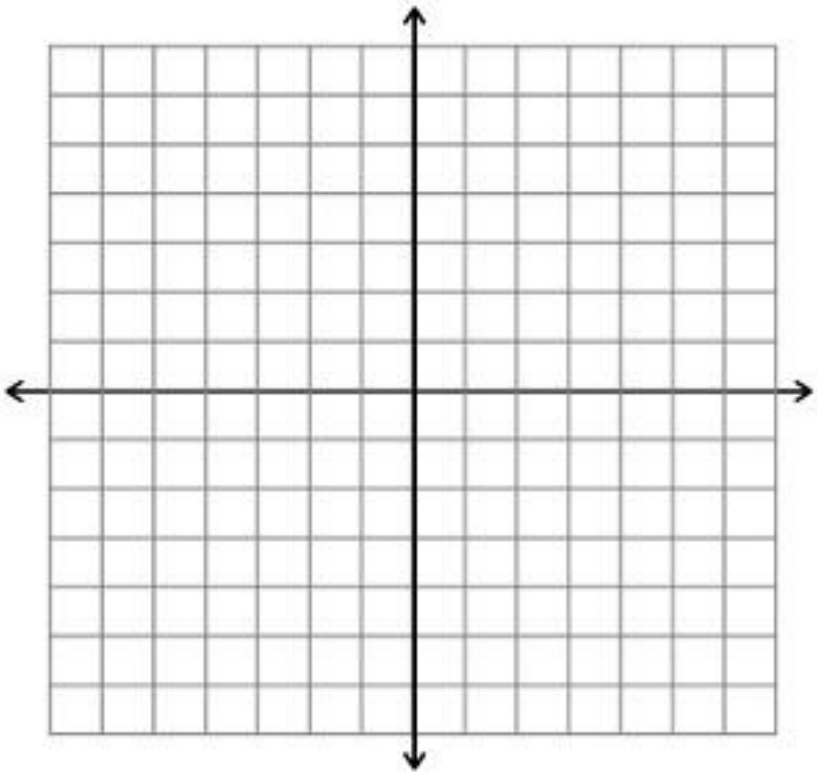
**<u>Example</u>** Graph $f(x) = 2^x, g(x) = e^x, h(x) = 3^x$ all on the same graph below for comparison:



Translations work similarly to other graphs we have discussed. Using the graph of $f(x) = 2^x$ as a "base graph", draw the following graphs together on the same graph: $f(x) = 2^x, g(x) = 2^x + 3, h(x) = 2^{x-1}$





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**Lesson: Solving Exponential Equations**

In this lesson, we will solve equations that look like this:

$$3^x = 27 \qquad \text{or} \qquad 4^{x+3} = 8^{x-2}$$

To do this, we will write directions around the following example: $2^{3x} = 8^{x+1}$

**<u>Examples</u>** Solve:

a. $49^{x-3} = 7^{2x}$

b. $9^{4x} = 27^{x+4}$



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**<u>Examples</u>** Solve:

a. $5^{x-2} = \frac{1}{25}$

b. $\left(\frac{3}{5}\right)^{2x} = \left(\frac{27}{125}\right)^{x+2}$

c. $27^{3x+1} = 9^{4x+2}$



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**Lesson: Review of the Basics of Logarithmic Functions**

Explain how to find the inverse of $f(x) = 3^x$, and explain how this relates to logarithms.

Define a logarithm:

**<u>Example</u>** Write the following in logarithmic form.

a. $3^2 = 9$ b. $5^0 = 1$ c. $4^{-2} = \frac{1}{16}$

**<u>Example</u>** Write the following in exponential form.

a. $\log_3 1 = 0$ b. $\log_5 125 = 3$ c. $\log_3 81 = 4$



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**<u>Example</u>** Show how to solve the following basic logarithmic equations.

a. $\log_x 8 = 3$

b. $\log_3 x = -2$

c. $\log_{49} 7 = x$



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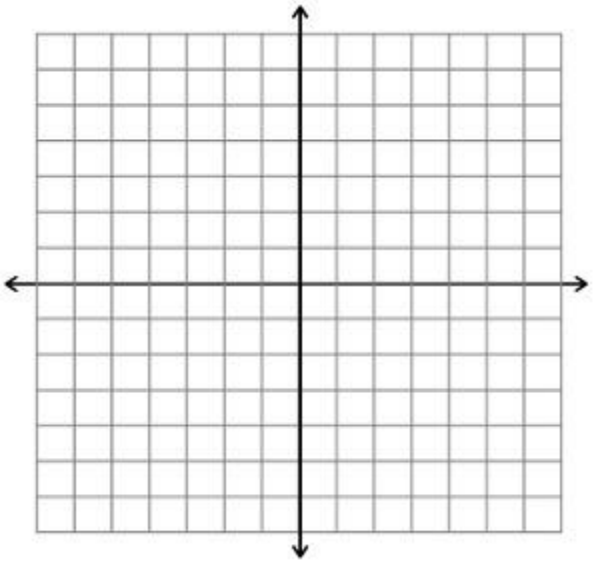
Create a table and show how to graph $f(x) = \log_2 x$



What 3 points can you easily find on the graph?

When does the graph take a shape like this for $f(x) = \log_a x$?

What is the y-axis for this graph?

What are the domain and range?



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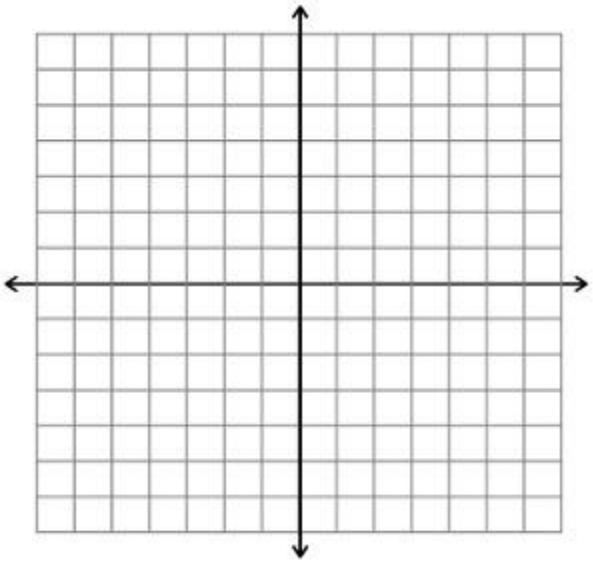
Create a table and show how to graph $f(x) = \log_{\frac{1}{2}} x$



What 3 points can you easily find on the graph?

When does the graph take a shape like this for $f(x) = \log_a x$?

What is the y-axis for this graph?

What are the domain and range?



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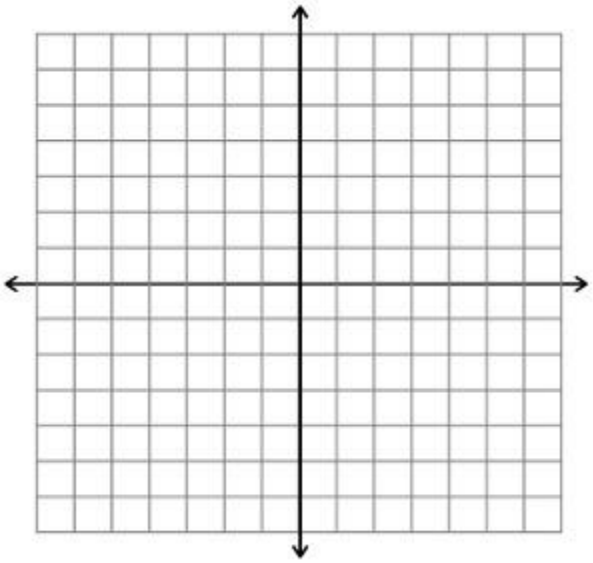
**<u>Examples</u>** Graph the following: $g(x) = \log_2(x-1) + 1$





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**Lesson: Review of the Basics of Properties of Logarithsm**

List the major properties of logarithms.

Explain what the common logarithms mean:

$\log x =$

$\ln x =$



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**Examples** Expand the following as much as possible.

a. $\log_3\left(\frac{3x^3y^7}{\sqrt{z}}\right)$

b. $\ln\left(\sqrt[3]{x^2y}\right)$

c. $\log_3\sqrt{6}$

d. $\log_5\frac{25x^3y^2}{p^7r^3}$



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**Example** Write as a single logarithm.

a. $\log(x+2)+\log(x-2)-\log(x+3)$

b. $3\log_2 3+2\log_2 y-5\log_2 x$

c. $\frac{1}{2}\log_5 16+3\log_5 y^2-\log_5 x^7-\log_5 y$

d. $4\ln 2+\ln x^6-2\ln y^3-3\ln x^2$



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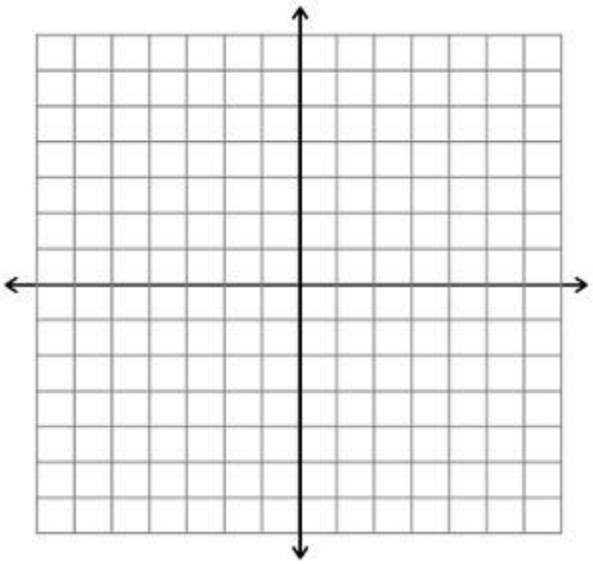
**Lesson: Intro to Logarithmic Functions**

First, we want to review how to function inverses of functions. Find the inverse of: $f(x) = 2x + 8$

Graph the exponential function $f(x) = 3^x$ below:



Recall the exponential functions are one-to-one, which means that they have inverses. Attempt to find the inverse of $f(x) = 3^x$ below:



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As we can see, there is an issue with finding the inverse – we have no way to solve for y! As such, we need new mathematics to help solve this.

Write down the definition of a logarithm:

**Examples** Write the following in exponential form:

a. $\log_3 81 = 4$

b. $\log_2 \frac{1}{8} = -3$

c. $\log_5 125 = 3$

d. $\log_9 9 = 1$

**Examples** Write the following in logarithmic form:

a. $2^4 = 16$

b. $3^0 = 1$

c. $6^{-2} = \frac{1}{36}$

d. $8^{\frac{1}{3}} = 2$



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Now we can finally determine what is the inverse of $f(x) = 3^x$:

There are a few common logarithms:

- $\log x$ means:

- $\ln x$ means:

There are also 2 key properties of logarithms:

1. $\log_a a = 1$: explain why this makes sense

2. $\log_a 1 = 0$: explain why this makes sense:



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**Lesson: Basics of Solving Logarithmic Equations and Evaluating Logarithms**

We want to solve equations that look like:

$$\log_3 x = 4 \qquad \text{or} \qquad \log_x 36 = 2$$

Before you begin, you should already be able to do the following:

a. Put $3^5$ in logarithmic form:

b. Put $\log_5 5 = 1$ in exponential form:

Now we can explain how to solve logarithmic equations using the example $\log_3(2x + 1) = 3$

**<u>Example</u>** Solve: $\log_x 16 = 4$



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**Example** Solve:

a. $\log_9 x = 2$

b. $\log_6(2x + 4) = 2$

Now we can also discuss how to evaluate logarithms. Explain how to evaluate: $\log_3 27$:

**Examples** Evaluate:

a. $\log_5 \frac{1}{25}$

b. $\log 100$



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**Examples** Evaluate:

a. $\ln e$

b. $\log_3 27$

c. $\log_{25} 5$

d. $\log \frac{1}{10}$



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**Lesson: How to Graph a Logarithm**

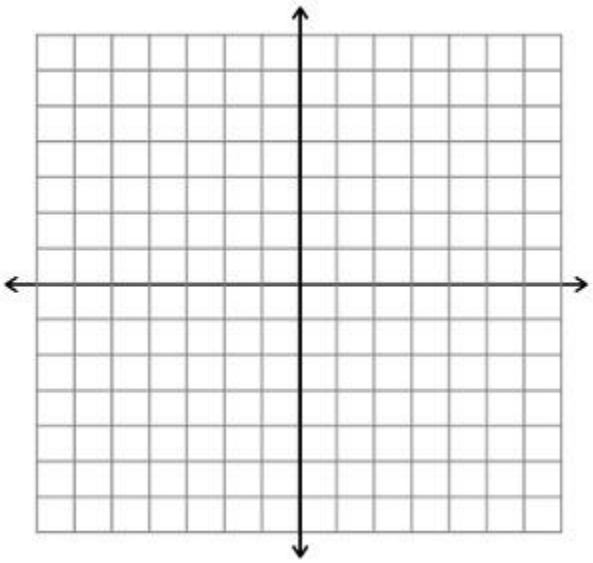
Before we begin, you should be able to do the following:

a. Convert to logarithmic form: $25 = 5^2$

b. Convert to exponential form: $\log_2 8 = 3$

c. Find the inverse of $y = 2^{\mathrm{x}}$

Now we can explain how to graph $y = \log_3 x$



For comparison, also graph $f(x) = 3^x$ on the same graph above.



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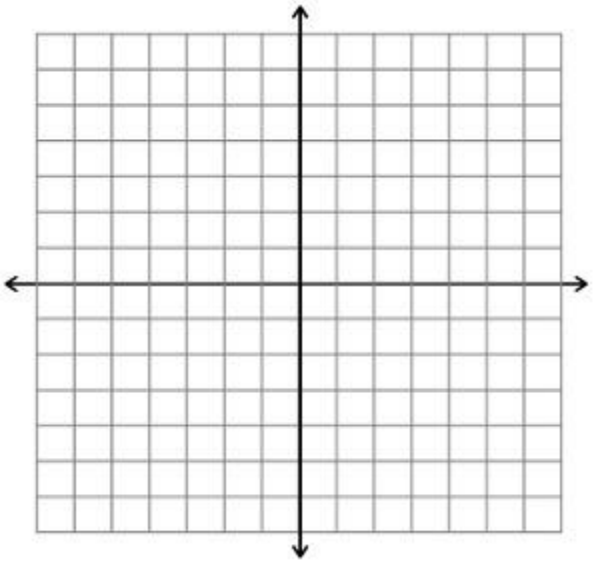
**<u>Example</u>** Graph $y = \log_{\frac{1}{2}} x$ below:





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**Lesson: Change of Base Theorem**

State the change of base theorem:

**<u>Example</u>** Show how to evaluate the following:

a. $\log_5 17$

b. $\log_2 3$

c. $\log_{\frac{1}{2}} 5$



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**Lesson: Solving Exponential Equations**

Outline the steps to solve: $2^{x+3} = 7^{3x}$

**<u>Examples</u>** Solve the following.

a. $e^{x^2} = 100$

b. $e^{2x-3} * e^{3x} = 5e$



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**Examples** Solve.

a. $3(2)^{x+2} + 6 = 105$

b. $e^{2x} + 8e^{x} + 15 = 0$

c. $3^{2x} + 2(3^{x}) = 24$



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**Lesson: Solving Logarithmic Equations**

Outline the steps to solve: $\log(3 - x) = 0.75$

Compare these steps to: $\log(x + 20) = \log(x + 10) + \log(6)$. How are these problems different?



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**Examples** Solve the following.

a. $\log_4(x^3 + 37) = 3$

b. $\log x + \log x^3 = 4$



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**<u>Examples</u>** Solve the following.

a. $\log(x^2 - 4) - \log(x - 2) = \log(8)$

b. $\log(5 + 4x) - \log(3 + x) = \log 3$



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**Lesson: Solving Exponential and Logarithmic Formulas**

**<u>Examples</u>** Solve for the desired variable.

a. for b: $p = a + \frac{c}{\ln b}$

b. for n: $A = \frac{Pr}{1-(1+r)^{-n}}$



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c. for A: $\log A = \log B - C \log D$

d. for x: $D = 100 + 5 \log x$



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**Lesson: Models of Exponential Growth and Decay**

Explain the equation for the general model:

Some examples of growth can come from populations or investments. Some examples of decay can come from half-life/radioactive decay, or carbon dating.

Define the following:

Doubling time:

Half-life:

What is the formula for continuous compounding?



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**Example** A sample of 400g of radioactive lead decays to polonium according to the function: $A(t) = 400e^{-0.03t}$.

a. Find the amount of lead remaining after 5 years.

b. Find the half-life.

**Example** The amount of carbon-14 present in a paint after t years if given by $y = y_0 e^{-0.0002t}$. The paint contains 37% of its carbon-14. How old are the paintings?



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**Example** Find the doubling time of an investment earning 5% interest if interest is compounded continuously.

**Example** During an epidemic, the number of people who have never had a particular virus and are not immune decreases exponential, where this decline is modeled by the function: $f(x) = 20{,}000e^{-0.025t}$, where t is time in days.

a. How many people were susceptible at the beginning of the epidemic?

b. How many people are susceptible after 2 weeks?



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**Lesson: A Review of Angles and Degrees**

Draw, label, and define how to make an angle:

Show how to create an angle of positive versus negative measure:

**Degree** is generally how we measure angles.

Draw the complete rotation of a ray. How many degrees is this?



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Define:

- Acute angle

- Right angle

- Obtuse angle

- Straight angle

Define:

- Complementary angles

- Supplementary angles



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**Example** Find the complement and supplement of a $55°$ angle.

**Example** Find the measures of the marked angles shown in the video.



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**Examples: Converting Between Minutes and Seconds**

**<u>Examples</u>** Convert the following to degrees.

a. $23°15'32''$

b. $38°27'15''$

**<u>Examples</u>** Convert the following to degrees, minutes, and seconds.

a. $32.713°$

b. $72.17°$



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**Examples: Adding and Subtracting Degrees, Minutes, and Seconds**

Add or subtract.

a. $45°15' + 23°57'$

b. $23°18'29'' + 14°19'47''$

c. $94°16' - 27°29'$

d. $72°18'15'' - 15°20'30''$



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**Examples: Applications of Minutes and Seconds**

**Example** A turntable makes 45 revolutions in a minute. How many revolutions is that per second?

**Example** A windmill makes 80 revolutions per minute. How many revolutions is that per second?

**Example** Wind power is caused by a turbine spinning 10 to 20 revolutions per minute. How many revolutions does it make per second?



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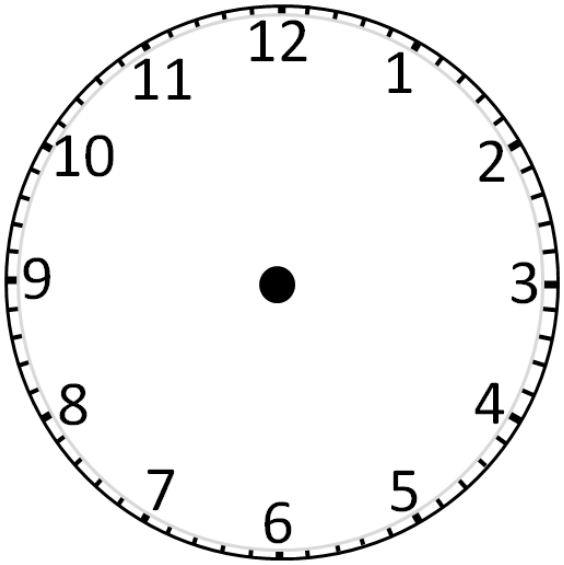
**Lesson: Angle Measure with a Clock**

Explain the basics of angle measure, ie where is $90°, 180°$?

Each number represents how many degrees? SHOW how to actually determine this?

**<u>Example</u>** Find in the clock below, and show how to find the angle measure for 5:00. Draw pictures to help illustrate how this works, pictures are key to remembering!!!



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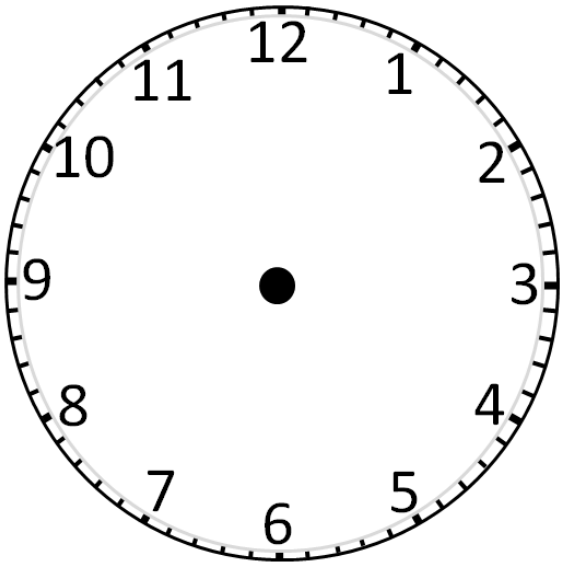
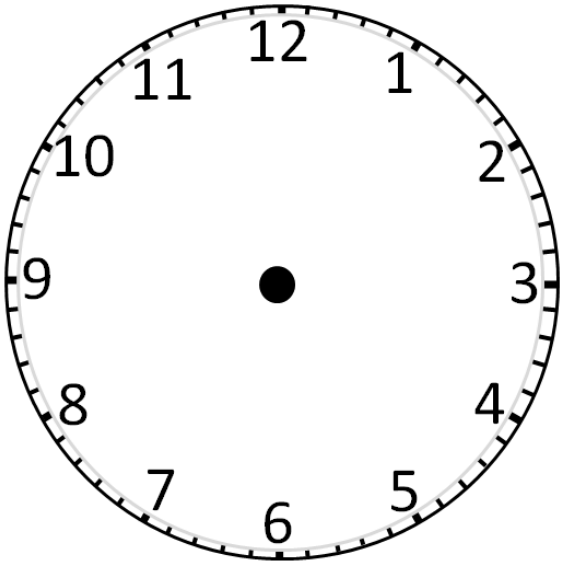
**Example** Find in the clock below, and show how to find the angle measure for 8:20. Draw pictures to help illustrate how this works, pictures are key to remembering!!!




**Example** Find in the clock below, and show how to find the angle measure for 10:05. Draw pictures to help illustrate how this works, pictures are key to remembering!!!



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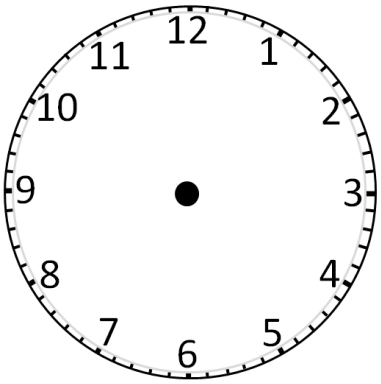
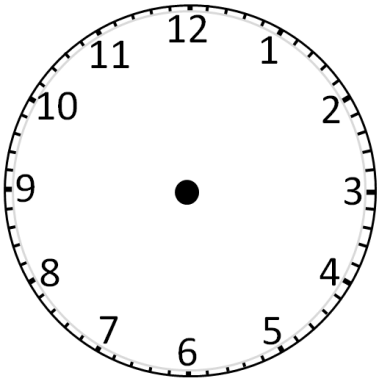
**Examples: Angle Measure from a Clock**

Find the angles associated with each time.

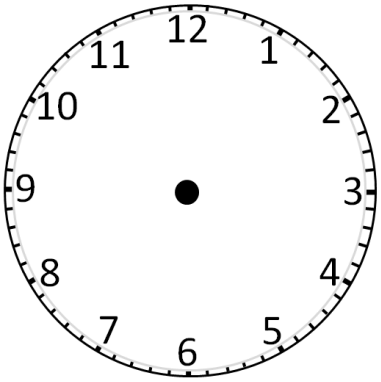
a. 6:25




b. 3:15




c. 2:38



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**Lesson: Standard Position and Coterminal Angles**

When is an angle in standard position? Draw an example.

Show some examples of how we can determine angle measure by quadrant.

What are quadrantal angles? Draw a picture to illustrate.



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What makes two angles coterminal? Draw a picture to explain.

**Example** Find the angle of least positive measure that is coterminal with each angle:

a. $715°$

b. $-80°$

**Example** How can you find all angles coterminal with $45°$? Prove it by filling in the following table.

| Value of n | Angle coterminal with $45°$ |
|---|---|
| 2 | |
| 0 | |
| -1 | |



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**Examples** Fill in the table below for which angles are coterminal with the quadrantal angles

| Quadrantal Angle | Coterminal Angles |
|---|---|
| $0°$ | |
| $90°$ | |
| $180°$ | |
| $270°$ | |



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**Lesson: Angle Relationships**

Draw and explain what vertical angles are:

Draw and explain what makes a transversal line. Draw a big picture, you will use it for the next part of your notes:

Label the diagram and note the relationships between the different angles below:



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**Example** Draw and label the lines from the video. Find the measures of angles 1, 2, 3, and 4.

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**Lesson: Review of Triangles and Similar Triangles**

The sum of angle measures of any triangle is:

Draw the different types of triangles:

**<u>Examples</u>** Draw the triangles in the example, then classify the triangles as acute, right, or obtuse. Also classify as equilateral, isosceles, or scalene.



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What makes 2 triangles similar?

What makes 2 triangles congruent?

What is the key relationship between similar and congruent triangles?

**<u>What are the 2 conditions for similar triangles?</u>**

In other words, triangle ABC is similar to triangle DEF if:



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**Example** Draw the similar triangles in the video, and then find any missing angle measures.

**Example** Draw the similar triangles in the video, and then find any missing side lengths.

**Example** Draw the similar triangles in the video, and then find any missing measures.



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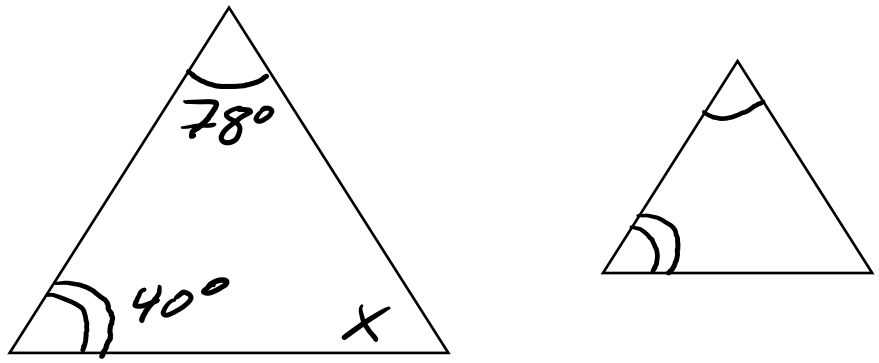
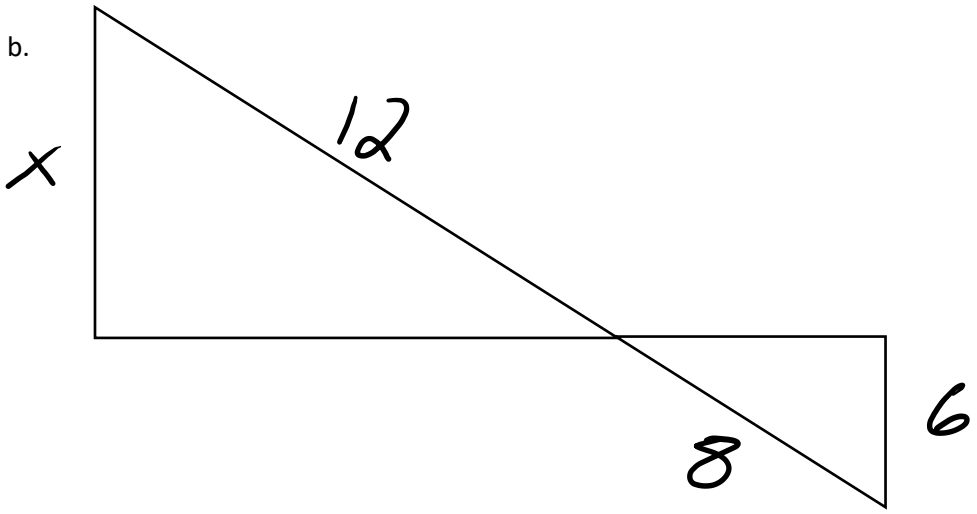
**Examples: Similar Triangles**

For each example, assume the triangles are similar. Triangles are not drawn to scale. Find any missing angle or side measures.

a.




b.



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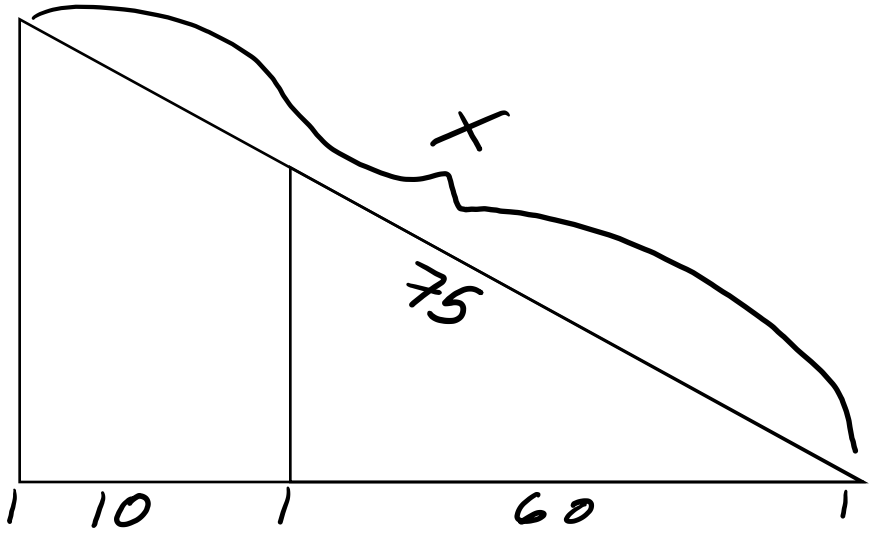
c.



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**Lesson: Radian Measure**

Radian measure is another way to describe angle measure, as opposed to degrees.

What does radian measure allow us to do?

What is the definition of a radian?

Are radians a unit of measure like feet or meters? Circle: Yes No

What is the formula for the distance around a circle?

What does this formula show?

Write a way to relate radians to degrees:



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So what is $\pi$ radians?

Show a conversion rate between radians and degrees:

**<u>Examples</u>** Convert each degree to radians:

a. $35°$

b. $-175°$

c. $27.8°$

Convert each radian measure to degrees:

a. $\frac{7}{4}\pi$

b. $-\frac{2\pi}{3}$

c. $7$



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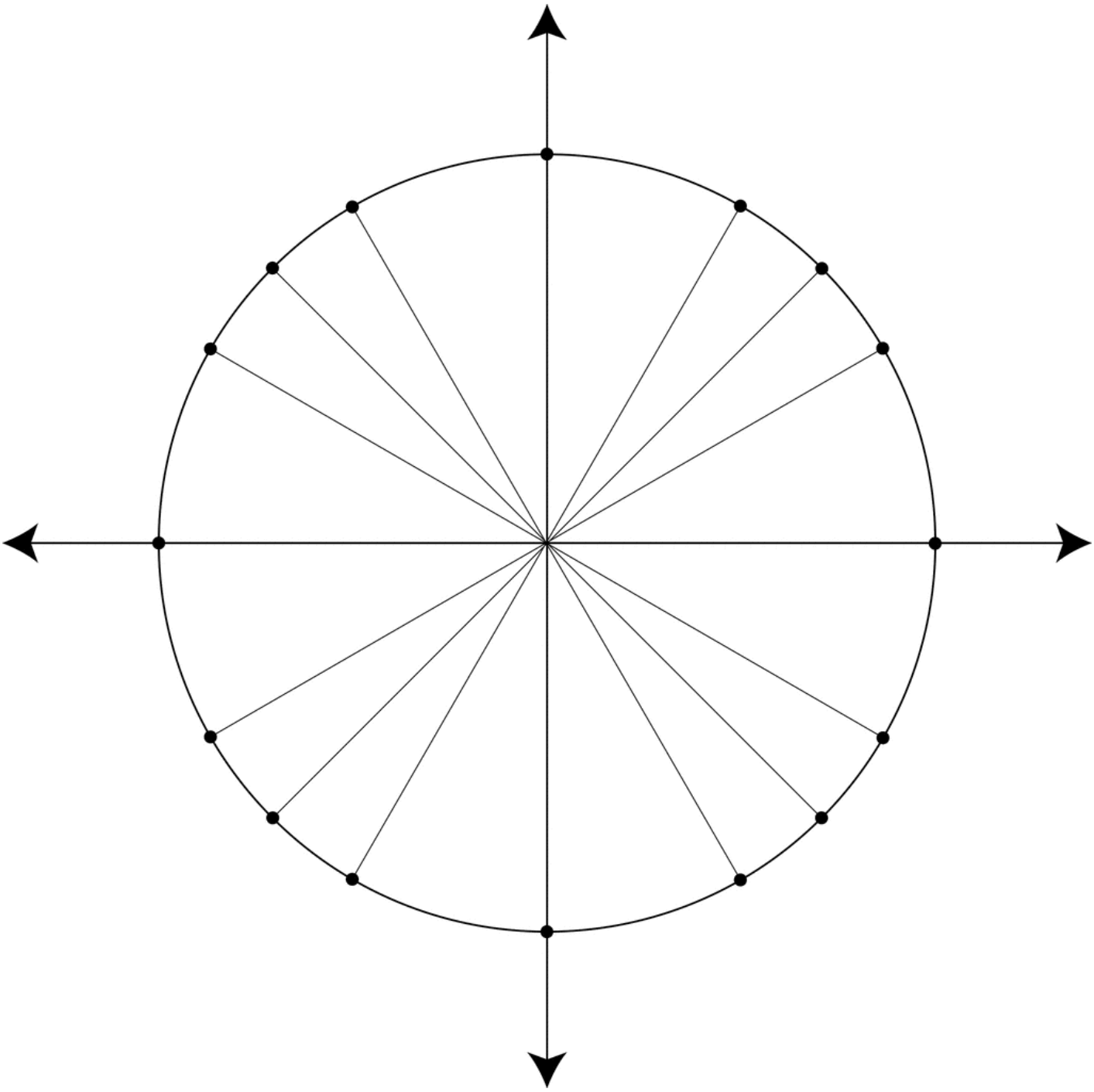
If no unit of angle measure is specified, what should you assume?

Below we will construct an illustration of all the common degree measures we have discussed, and then write their radian equivalent. Take your time to copy this perfectly!





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You will need to know all of those common degrees AND radian measures for the exam!

You may want to reference cheat sheets to help you with the next few problems.

**<u>Examples</u>** Find the value of each function.

a. $\sin\frac{5\pi}{6}$

b. $\tan\frac{3\pi}{4}$

c. $\csc\frac{5\pi}{3}$

d. $\cos\left(-\frac{2\pi}{3}\right)$



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**Lesson: Arc Length on a Circle and Some Applications**

We can determine the formula for the arc length of a circle from the definition of radians.

Regarding radians, what does $\theta$ equal? Show how this relates to arc length.

What is key about the arc length formula?

**<u>Examples</u>** A circle has radius 22.30 cm. Find the length of the arc intercepted by a central angle having each of the following measures.

a. $\frac{5\pi}{6}$ radians

b. $66°$



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**Examples** A circle has radius 15.22 cm. Find the length of the arc intercepted by a central angle having each of the following measures.

a. $\frac{3\pi}{8}$

b. $125°$

What is the definition of latitude?

**Example** The latitude of Panama City, Panama is $9°$ N and the latitude of Pittsburgh, PA is $40°$ N. Find the north-south distance of these 2 cities. The radius of the Earth is 6400 km.



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**Example** A rope is being wound around a drum with a radius of 0.7525 ft. How much rope will be wound around the drum if the drum is rotated through an angle of $42.25°$?

**Example** Two gears are adjusted so that the smaller gear drives the larger one. If the smaller gear rotates through an angle of $150°$, through how many degrees will the larger gear rotate? Refer to the photo in the video.



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**Examples: Applications using Arc Length**

Charleston, SC and Toronto, Canada, are 1100 km apart and lie on the same north-south line. If the latitude of Charleston is $33°$ N, what is the latitude of Toronto?

Find the radius of the larger wheel in the figure if the small wheel rotates $75.0°$ when the large wheel rotates $45°$.



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How many inches will a small weight rise if a pulley is rotated through an angle of $65°15'$? Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 4 inches?



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**Lesson: Area of a Sector of a Circle and Some Applications**

What do we define as a sector of a circle?

What is the formula for the area of a section of a circle with radius r?

**<u>Examples</u>** Find the area of a sector of a circle having the given radius and central angle $\theta$. Answers should be to the nearest tenth.

a. $r = 11.5$ ft, $\theta = \frac{5\pi}{6}$ radians

b. $r = 13.5$ m, $\theta = 125°$



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**Example** Find the measure (in radians) of a central angle of a sector of a circle of area $25\ in^2$ in a circle of radius 2.5 in.

**Example** A center-pivot irrigation system provides water to a sector-shaped field as shown in the figure in the video. Find the area of the field if $\theta = 35.0°$ and $r = 155$ yd.

**Example** A circular sector has an area of $60\ in^2$. The radius of the circle is 6 inches. What is the arc length of the sector?



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**Lesson: The Unit Circle and Circular Functions**

Explain how we create the unit circle:

What is the equation for the unit circle?

Recall the formula for arc length is $s = r\theta$. When the radius of the circle is 1, what does this formula become?

Why does this matter / what is cool about this?



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Since r=1 on the unit circle, what can we say about our previously studied trig functions? Just summarize what we discussed in the video as best as you can.

Write out the **circular functions**

1.

2.

3.

4.

5.

6.



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Explain the symmetry of the unit circle:

If a point $(a, b)$ in on the unit circle, what other points are guaranteed to be on the unit circle?

How can we use symmetry with drawing the unit circle? How is it handy?



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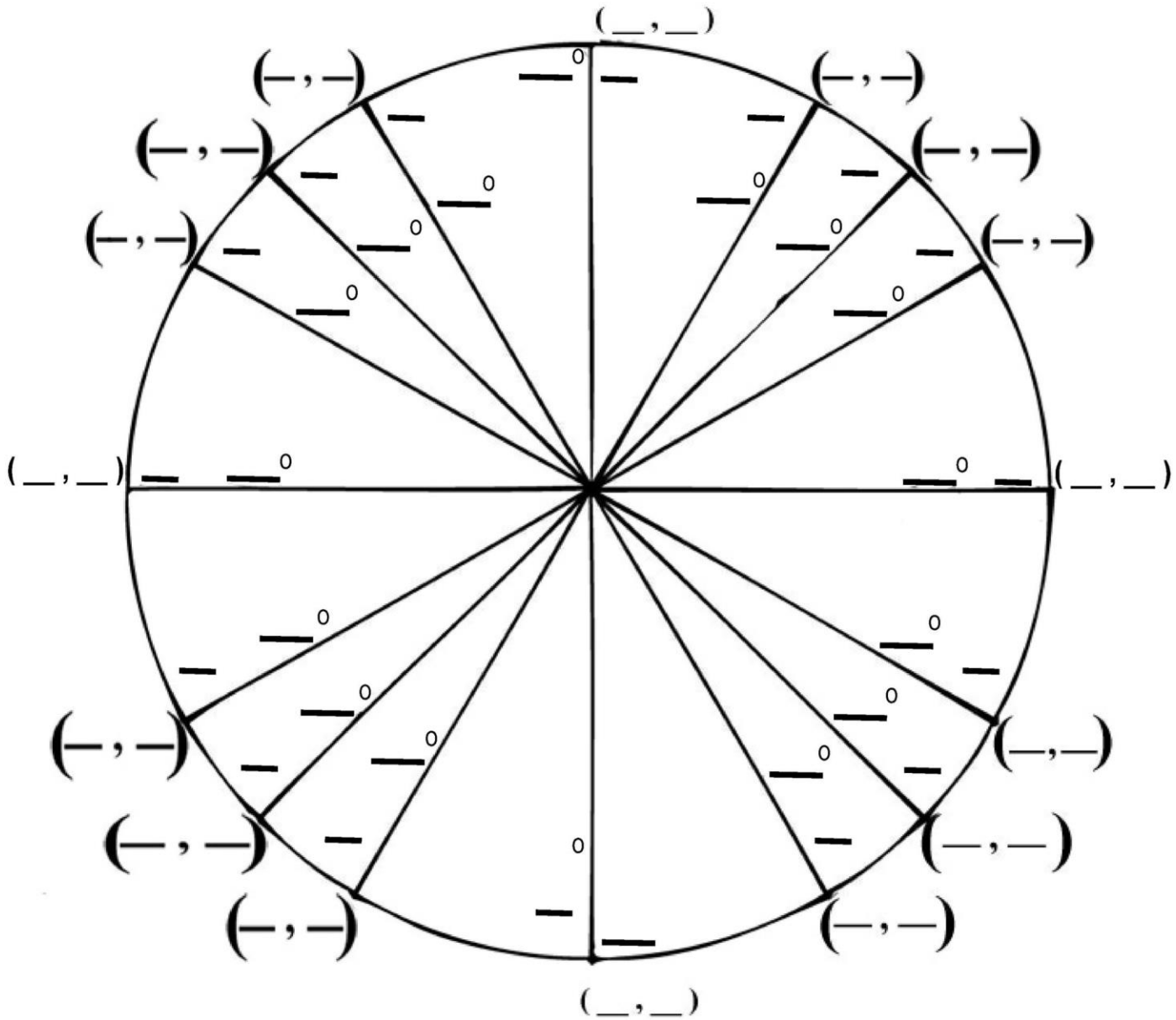
Label the unit circle. Note: you will have to have this memorized! These are the standard angle measures that you'd be expected to do without a calculator!




Now, going back to the circular functions we discussed, how do you view the x-coordinates?

How do you view the y-coordinates?



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**Examples** Use the unit circle to find the values of the following:

a. $\sin\frac{\pi}{3}$

b. $\cos\frac{5\pi}{4}$

c. $\sin\frac{11\pi}{6}$



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**Lesson: Arc Length on a Circle and Some Applications**

Using the unit circle, we can also determine the domains of our circular functions.

What is the domain of sine and cosine, and why?

What is the domain of tangent and secant, and why?

What is the domain of cotangent and cosecant, and why?



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**Lesson: Values of Circular Functions**

We have previously discussed the unit circle (you will want to have it handy). One the most important things about the unit circle is that it will quickly tell you the value of the sine and cosine functions.

On the unit circle, what does the y-coordinate represent?

On the unit circle, what does the x-coordinate represent?

When we think about the values of these functions, what do we assume?

What does this mean for your calculator?

**<u>Examples</u>** Use the unit circle to find the values of:

a. $\sin\frac{\pi}{2}$ b. $\cos\frac{\pi}{2}$ c. $\tan\frac{\pi}{2}$



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**Examples** Use the unit circle to find the values of:

a. $\sin\frac{5\pi}{4}$ b. $\cos\frac{5\pi}{4}$ c. $\tan\frac{5\pi}{4}$

**Example** Compare methods. Use reference angles and degrees to find the exact value of $\sin\frac{2\pi}{3}$, then confirm your answer with the unit circle.

For approximate values, we still must use a calculator. Remember, when you are not specifically told you are using degrees, then you assume you are using radians.

What must you ensure with your calculator?

**Examples** Use a calculator to approximate the function values.

a. $\cos(1.72)$

b. $\sin(-3.54)$

c. $\cot(1.438)$



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**Lesson: Values of Inverse Circular Functions**

What do the functions $\sin^{-1} x$ , $\cos^{-1} x$, $\cot^{-1} x$, etc. represent?

Your calculator makes a few key assumptions if it uses these functions (the reasons why are explained later):

1. For all x in $[-1, 1]$, using $\sin^{-1} x$ with a calculator will return a value between:

2. For all x in $[-1, 1]$, using $\cos^{-1} x$ with a calculator will return a value between:

3. For all real numbers x, using $\tan^{-1} x$ with a calculator will return a value between:



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Sometimes you will be asked to find answers in different quadrants than what your calculator will return!

**<u>Example</u>** Solve for s, given that s is in the interval $\left[0, \frac{\pi}{2}\right]$: $\sin s = 0.3311$

**<u>Example</u>** Solve for s, given that s is in the interval $\left[0, \frac{\pi}{2}\right]$: $\tan s = 0.2155$



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**Example** Solve for s, given that s is in the interval $\left[\pi, \frac{3\pi}{2}\right]$: $\tan s = \sqrt{3}$

**Example** Solve for s, given that s is in the interval $[0, 2\pi)$: $\cos^2 s = \frac{1}{2}$



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**Lesson: The Unit Circle and Circular Functions as Line Segments**

The unit circle is really just an extension of how we have previously thought about trig functions as line segments / right triangles.

Draw the illustration in the video, then show how these sides relate to our trig functions.

From your illustration, how do we define the following?

$\cos\theta =$ $\tan\theta =$ $\csc\theta =$

$\sin\theta =$ $\cot\theta =$ $\sec\theta =$



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**Example** Using your illustration from the previous page, suppose the measurement of TVU is $45°$. Find the lengths of the following line segments:

OQ

OU

OV

VR

US

PQ



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**Lesson: Linear Speed and Angular Speed and Applications**

What is the formula for linear speed?

When are some situations where you might use linear speed?

What is the formula for angular speed?

When are some situations where you might use angular speed?



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**Examples** Use the formula $\omega = \frac{\theta}{t}$ to find the value of the missing variable.

a. $\omega = \frac{\pi}{3}$ radians per second, $t = 6$ seconds

b. $\theta = 3.721$ radians, $t = 18.34$ seconds

**Examples** Use the formula $v = r\omega$ to find the value of the missing variable.

a. $r = 10\ m$, $\omega = \frac{9\pi}{5}$ radians per second

b. $v = 10$ m per second, $\omega = \frac{3\pi}{2}$ radians per second



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**<u>Example</u>** Suppose a point P is on a circle with a radius of 5 cm, and the ray OP Is rotating with an angular speed of $\frac{\pi}{10}$ radian per second.

a. Find the angle generated by P in 3 seconds.

b. Find the distance traveled by P along the circle in 3 seconds.

c. Find the linear speed of P in cm per second.



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The formula $\omega = \frac{\theta}{t}$ can be rewritten as $\theta = \omega t$. We can then use the formula $s = r\theta$ to create a different formula. Show the new formula we can create:

**<u>Examples</u>** Use the formula above to find the value of the missing variable.

a. $r = 5\ cm, \omega = \frac{2\pi}{3}$ radians per second, $t = 8$ seconds

b. $s = \frac{3\pi}{2}$ km, $r = 4$ km, $t = 3$ seconds



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**Example** The orbit of the earth around the sun is almost circular. Suppose that the orbit has a radius of 93 million miles.

a. We know that a year is 365 days. Find the angle formed by the Earth's movement in one day.

b. Give the angular speed in radians per hour.

C. Find the linear speed of the Earth in miles per hour.



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**Examples: Applications of Linear Speed and Angular Speed**

**Example** Find the angular speed of the minute hand of a clock.

**Example** Find the linear speed of the tip of the second hand of a clock, if the hand is 25 mm long.

**Example** A bicycle has tires with a 12-inch radius. The tires are turning at the rate of 200 revolutions per minute. How fast if the bike traveling in miles per hour?



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**Example** A pulley has a radius of 13.23 cm. It took 15 seconds for 45 cm of belt to go around the pulley.

a. Find the linear speed of the belt in cm per sec.

b. Find the angular speed of the pulley in radians per second.

**Example** A thread is being pulled off a spool at a rate of 55.4 cm per second. Find the radius of the spool if it makes 155 revolutions per minute.



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**Lesson: The Pythagorean Theorem, The Distance Formula, and Intro to Trig Functions**

State the Pythagorean theorem:

State the distance formula:

How does the Pythagorean theorem relate to the distance formula?

Now we will introduce **trigonometric functions!** Explain what makes trig functions unique:



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How do trig functions relate to the distance formula? Draw the picture associated with this.

Explain the 6 trig functions and how they relate to what we discussed above.

Sometimes you will see expressions such as $\sin^2 90°$. What does this mean?



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**Example** The terminal side of an angle $\theta$ in standard position passed through the point (-12, -5). Find the values of the 6 trig functions of angle $\theta$.

**Example** Find the 6 trig function values of an angle $\theta$ in standard position if the terminal side of $\theta$ is defined by

$3x + 5y = 0, x \geq 0$



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**Example** Suppose the point $(x, y)$ is in the indicated quadrant. Determine whether given ratio is positive or negative.

$$III, \frac{y}{r}$$



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**Examples: Finding Values of Trig Functions**

For each given point, sketch the angle of $\theta$ in standard position such that $\theta$ has the least positive measure and the given point is on the terminal side. The find the values of the 6 trig functions.

a. $(-3, -4)$

b. $(7, 4)$

c. $\left(-\sqrt{2}, \sqrt{2}\right)$



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**Examples: Finding the 6 Trig Functions Given an Equation of the Terminal Side**

An equation of the terminal side of an angle $\theta$ (in standard position) is given with a restriction on x. Sketch the situation with least positive angle $\theta$, and find the values of the 6 trig functions.

a. $-6x - y = 0, x \geq 0.$

b. $x + y = 0, x \geq 0$

c. $x = 0, y \geq 0.$



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**Lesson: Quadrantal Angles and Trig Functions**

What were the quadrantal angles?

If one of the quadrantal angles is involved, how does this impact how we think about the trig functions? Draw a sketch to help illustrate this point. Explain how these are different from non-quadrantal angles, and how we choose points in these cases.



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**Example** Find the indicated value of: $\sin 90°$.

How should we think about something like $\sin^2 90°$?

**Example** Use trigonometric function values of quadrantal angles to evaluate: $\tan 0° - 6 \sin 90°$.

**Example** Use trigonometric function values of quadrantal angles to evaluate: $\sin^2 180° + \cos^2 180°$.



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**Examples: Quadrantal Angles and Trig Functions**

Find the function values:

a. $\cot 90°$

b. $\sec 180°$

c. $\sin(-270°)$

d. $3\sec 180° - 2\tan 360°$

e. $\sin^2 90° - 3\cos^2 90°$



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**Lesson: Right-Triangle Based Definitions of Trigonometric Functions and Cofunctions**

Instead of using standard position to describe trig functions, we can alternatively use right triangles.

Draw a right triangle to orient yourself.

What does A represent?

Show how to define the 6 trig functions with this right triangle.

1.

2.

3.

4.

5.

6.



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What mnemonic device can help you remember the trig functions?

**Example** Draw the triangle in the video, and then show how to find the values of sine, cosine, and tangent.

**Example** Using the same example above, show how to find the values of cosecant, secant, and cotangent.



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Draw a new right triangle, and show an alternative angle measure we can find from this picture:

These relationships are called the **cofunction identities,** summarize these below:



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**Examples** Write each function in terms of its cofunction

a. $\sin 10°$

b. $\cot 65°$

c. $\csc 34°$

**Example** Find one solution for each equation. Assume all angles involved are acute angles.

a. $\cos\theta = \sin(2\theta - 30°)$

b. $\sec(\theta + 15°) = \csc(2\theta + 25°)$



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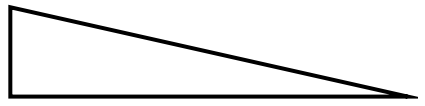
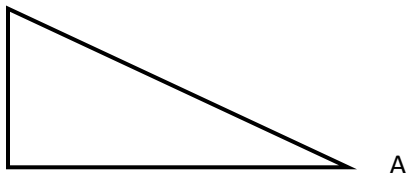
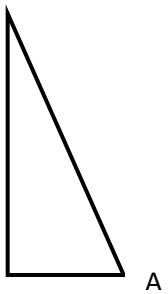
**Lesson: More on Trig Function Values as Angles Change and Special Angles**




Explain how angle A changes in these pictures:

**<u>Examples</u>** Determine whether each statement is true or false.

a. $\tan 25° < \tan 22°$

b. $\cos 25° < \sin 25°$

There are a few angles that occur frequently in trig and have some *nice* properties that we like to call them out. Draw out a triangle where each angle is $60°$ and describe any other properties:



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If you bisect this triangle, what happens? Draw a picture to illustrate, and then describe the trig functions associated with it:



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Finally, draw an isosceles right triangle and describe any special characteristics with it:



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We have now examined some *extremely* common trig measures that will come up all the time in trig. They are so important that we will create a separate reference sheet for them.

**Read this!** I highly recommend that you remove the last page from this packet, and instead put it at the very front of your notes so that you can easily reference it. If you are moving on to calculus, KEEP THIS PAGE IN GOOD CONDITION FOR THE NEXT FEW YEARS. You will thank yourself later that you have this reference as these angles will come up over and over again, and students frequently forget them. I will occasionally create these handouts and make these suggestions to you as we go along in trig – you will thank yourself (and me) later if you keep these. It is extremely common for students to forget some of KEY parts of trig and then struggle in calculus – I don't want this to happen to you!

# Some Function Values of Special (Common) Angles

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|
| $0°$ | | | | | | |
| $30°$ | | | | | | |
| $45°$ | | | | | | |
| $60°$ | | | | | | |
| $90°$ | | | | | | |



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**Lesson: Sine and Cosine as Periodic Functions**

What is a periodic function?

What are some examples?

What are 2 known periodic functions?

What can we say about the behavior of these functions as we move along the unit circle?

| AS $s$ CHANGES FROM | THE VALUE OF $\sin s$ | THE VALUE OF $\cos s$ |
|---|---|---|
| $0$ TO $\frac{\pi}{2}$ | | |
| $\frac{\pi}{2}$ TO $\pi$ | | |
| $\pi$ TO $\frac{3\pi}{2}$ | | |
| $\frac{3\pi}{2}$ TO $2\pi$ | | |



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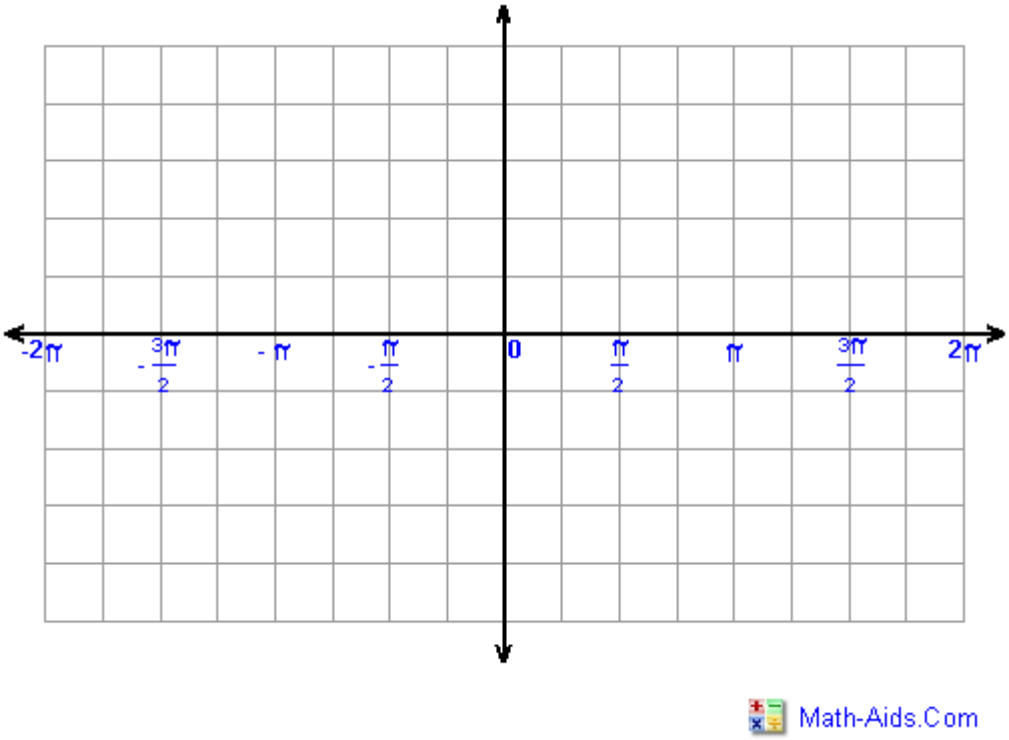
But let's change s to x going forward so we don't get confused.

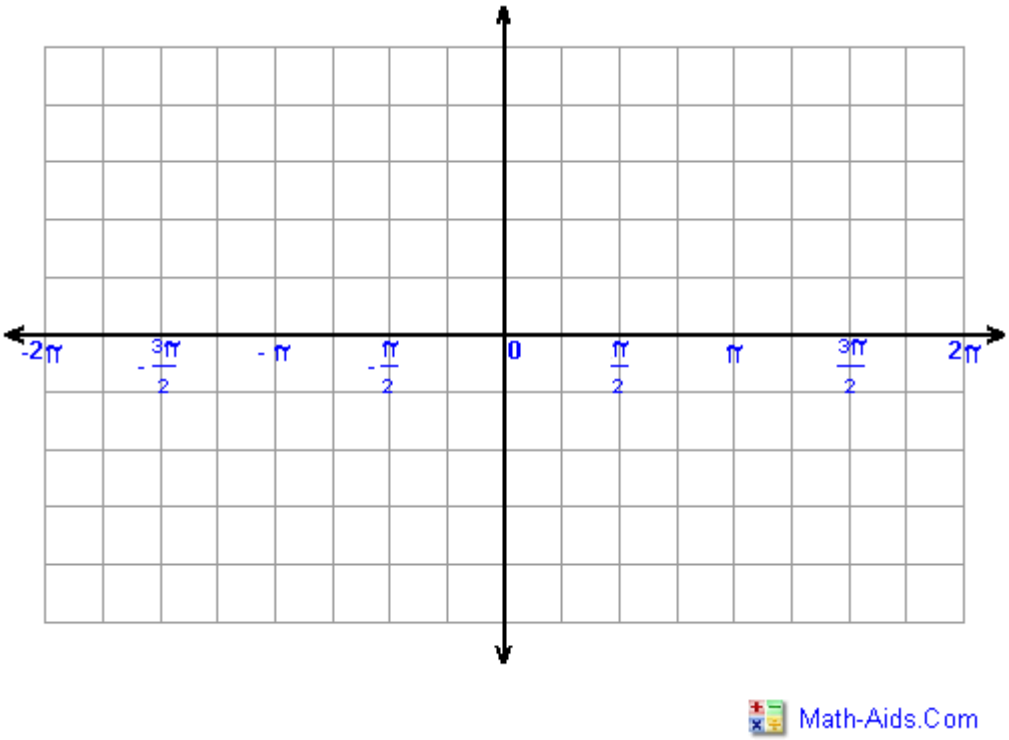
**Draw the graph of sine, filling in the table and then using the graph paper.**

| $x$ | $y = \sin x$ |
|---|---|
| $0$ | |
| $\frac{\pi}{4}$ | |
| $\frac{\pi}{2}$ | |
| $\pi$ | |
| $\frac{3\pi}{2}$ | |
| $2\pi$ | |




**Draw the graph of cosine, filling in the table and then using the graph paper.**

| $x$ | $y = \sin x$ |
|---|---|
| $0$ | |
| $\frac{\pi}{4}$ | |
| $\frac{\pi}{2}$ | |
| $\pi$ | |
| $\frac{3\pi}{2}$ | |
| $2\pi$ | |



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What term do we use to describe the graph of sine?

How do we describe the graph of cosine?

What is the period of both the sine and cosine functions?

Where are the graphs continuous?

What is the domain and range of the graphs?

How can we describe the x-intercepts of sine?

How can we describe the x-intercepts of cosine?



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What is the test for even or odd functions?

What does it mean to be an even function?

What does it mean to be an odd function?

How do you know that $y = \sin x$ is odd? How would you show it?

How do you know that $y = \cos x$ is even? How would you show it?



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**Lesson: Using Amplitude and Period to Graph Sine and Cosine**

What is amplitude?

Using the equation $y = a \sin x$, when will this graph stretch or shrink? Draw 2 comparison sketches to help you remember (just a quick sketch so you remember how these visually compare)

Using the equation $y = a \cos x$, when will this graph stretch or shrink? Draw 2 comparison sketches to help you remember (just a quick sketch so you remember how these visually compare)



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What happens when $a$ is negative? Demonstrate this with some brief sketches so that you can remember.

What is period?

Using the equation $y = \sin bx$, watch the video and write an explanation with sketches to demonstrate period.



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Using the equation $y = \cos bx$, watch the video and write an explanation with sketches to demonstrate period.

What formula can you use to determine the period of a function?

**<u>Examples</u>** Determine the periods of the following functions:

a. $y = \sin 3x$

b. $y = \cos \frac{1}{2} x$

c. $y = \sin 2\pi x$



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Summarize amplitude and period when the function is of the form $y = a \sin bx$ and $y = a \cos bx$

**<u>Examples</u>** Graph the following over 2 period intervals, completing the table with strategic points to help you graph. Give the period and amplitude in each. You will have to sketch your own graph and create your own axes.

$y = \cos \frac{1}{3} x$

**Period:** **Amplitude:**

| $\boldsymbol{x}$ | $\boldsymbol{y = cos \frac{1}{3} x}$ |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |



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$y = 2\sin x$

**Period:** **Amplitude:**

| $\boldsymbol{x}$ | $\boldsymbol{y = 2\,sin\,x}$ |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |



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$y = 3\cos 2\pi x$

**Period:** **Amplitude:**

| $x$ | $y$ |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |



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**Examples: Using Amplitude and Period to Graph Sine and Cosine**

Graph the following over two-period intervals.

a. $y = -\frac{1}{2}\sin x$



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b. $y = 3\cos 2x$

d. $y = -\sin \pi x$



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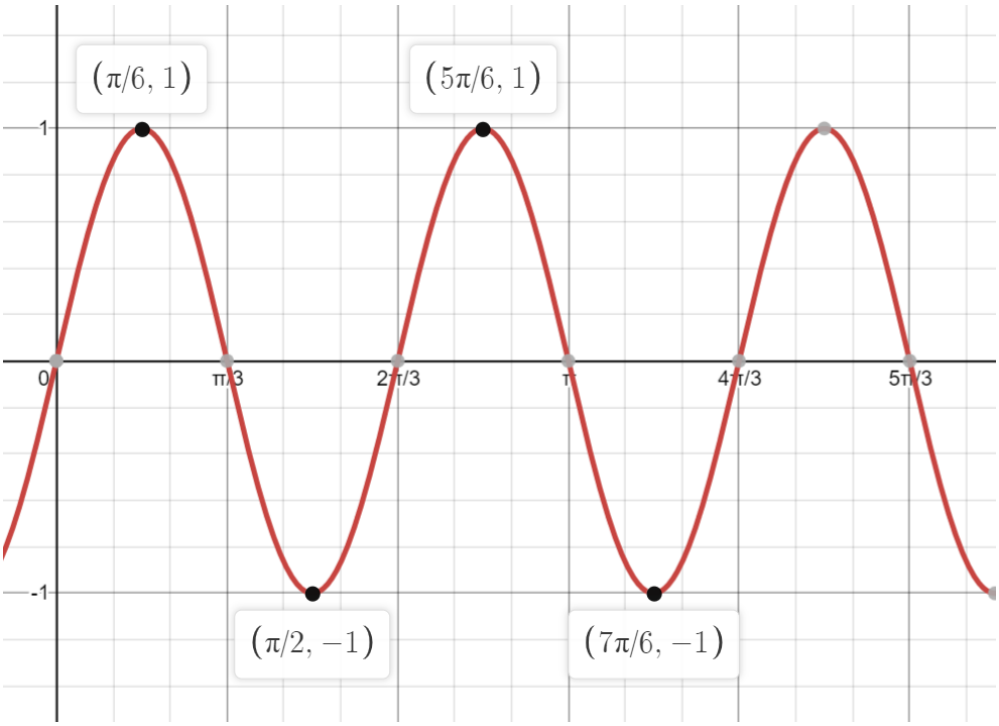
**Lesson: Determining the Equations of Sine and Cosine from Graphs**

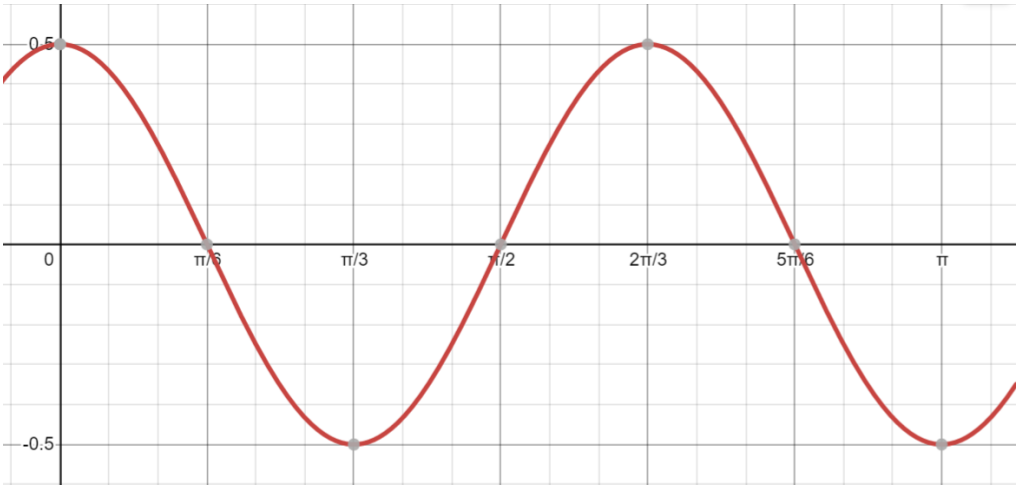
In each graph, determine the equation of the form $y = a\cos bx$ or $y = a\sin bx$, with $b > 0$, for each graph.

Desmos.com was used to create these graphs.

a.




b.



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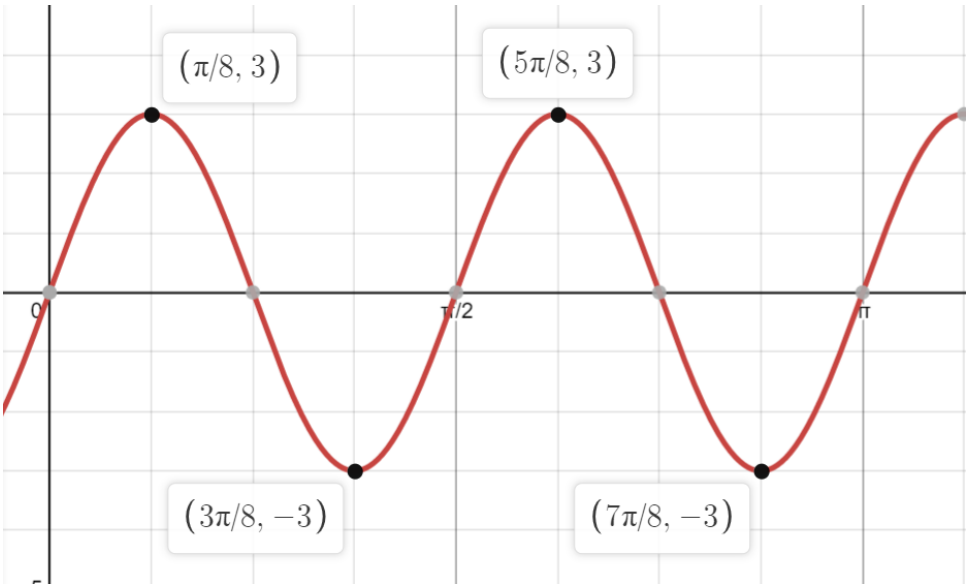
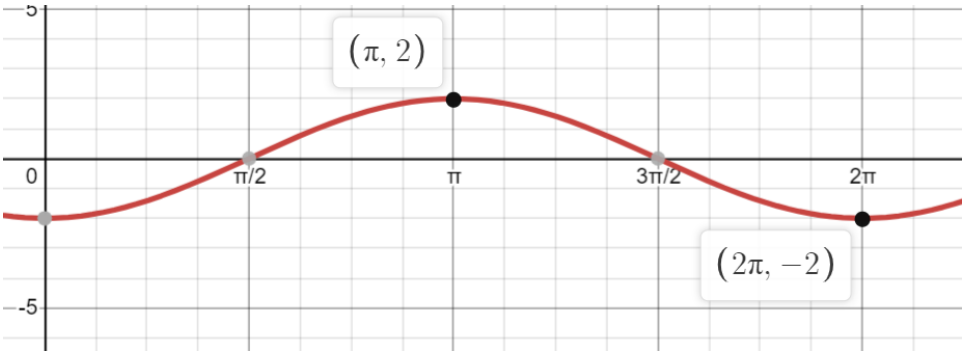
c.




d.



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**Lesson: Translations of the Sine and Cosine Graphs**

How do you describe a horizontal translation? In your own words / explanation, summarize with sketches how the video show these translations work.

What is phase shift?

What is an argument?

How do you describe a vertical translation? In your own words / explanation, summarize with sketches how the video show these translations work.

What happens when a function is of the form: $f(x) = a\cos(bx + bd)$ with the phase shift?



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**Examples** Identify the amplitude, period, phase shift, argument, and vertical translation.

a. $y = \sin\left(x + \frac{\pi}{4}\right)$

b. $y = 2 - \sin x$

c. $y = 3 + \sin(x + 2)$

d. $y = -4 + \frac{1}{2}\cos x$

e. $y = 2 + \sin(2x - 2\pi)$

f. $y = 1 - 2\cos\left(3x + \frac{\pi}{2}\right)$

g. $y = 2 - 4\cos(3x + 1)$

h. $y = 3\pi - 2\sin(\pi x - 3)$



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**Examples** Show how to find the start and end points of an interval for us to graph.

a. $y = \sin\left(x + \frac{\pi}{2}\right)$

b. $y = 1 - 2\sin\left(3x - \frac{\pi}{2}\right)$

c. $y = 3\pi - 2\sin(\pi x - 3)$



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**<u>Example</u>** Using the graph $y = \sin\left(x + \frac{\pi}{4}\right)$, and *clearly write out the steps* required to graph any sine or cosine graph. Identify the amplitude, period, phase shift, and vertical translation.



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**Example** Graph $y = 1 + \cos x$. Identify the amplitude, period, phase shift, and vertical translation.



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**<u>Example</u>** Graph $y = -1 + \frac{1}{2}(\cos 2x - \pi)$. Identify the amplitude, period, phase shift, and vertical translation.



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**Examples: Translations with Sine and Cosine**

Graph the following: $y = \frac{5}{2}\sin\left(4\left(x + \frac{\pi}{4}\right)\right)$



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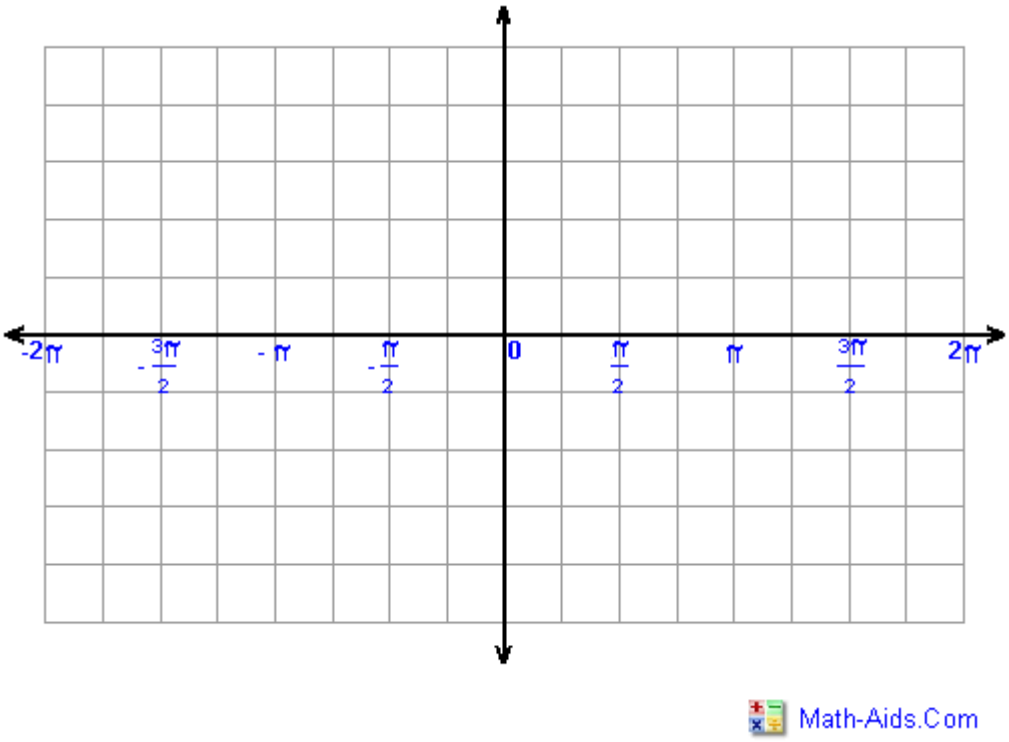
**Lesson: The Graphs of Tangent and Cotangent**

What is an asymptote? Draw a sketch to help explain.

**Draw the graph of tangent, filling in the table and then using the graph paper.**

| $x$ | $y = \tan x$ |
|---|---|
| $-\frac{\pi}{2}$ | |
| $-\frac{\pi}{4}$ | |
| $0$ | |
| $\frac{\pi}{4}$ | |
| $\frac{\pi}{2}$ | |
| $\frac{3\pi}{2}$ | |



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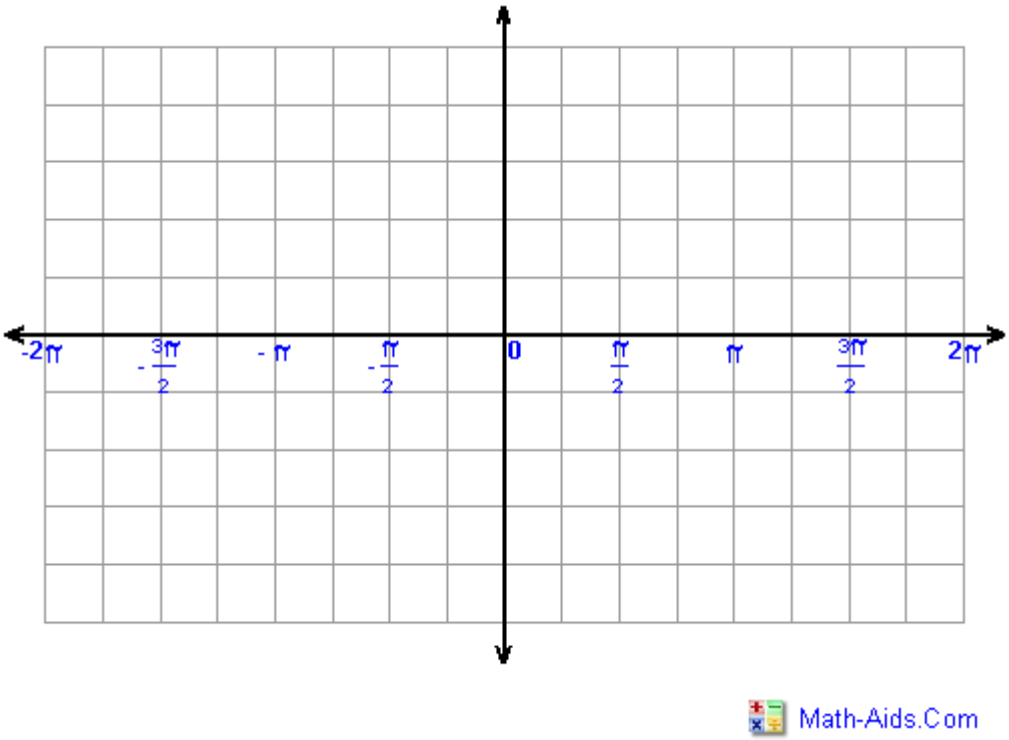
**Draw the graph of cotangent, filling in the table and then using the graph paper.**

| $x$ | $y = \cot x$ |
|---|---|
| $0$ | |
| $\frac{\pi}{4}$ | |
| $\frac{\pi}{2}$ | |
| $\frac{3\pi}{2}$ | |
| $\pi$ | |
| $\frac{5\pi}{4}$ | |




How would we describe the period for these functions?

Where are the graphs continuous (and discontinuous)?



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What is the domain and range of the graphs?

How can we describe the x-intercepts of tangent?

How can we describe the x-intercepts of cotangent?

Are either function even or odd?

How do you interpret amplitude with these functions and why?

What is the key for graphing? (Be specific with the equations you need to leverage)



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**<u>Example</u>** Graph over a one-period interval: $y = \tan 2x$



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**<u>Example</u>** Graph over a two-period interval: $y = -\frac{1}{2}\cot 4x$



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**Lesson: Translations with The Graphs of Tangent and Cotangent**

Tangent and cotangent can experience the same types of translations that sine and cosine did (which we discussed in a previous lesson).

**<u>Examples</u>** Describe the period, phase shift, and vertical translations for each function.

a. $y = 3\tan 4x$

b. $y = -\cot(x + \pi)$

c. $y = 2\tan\left(3x - \frac{\pi}{2}\right)$

d. $y = 3 + \cot(x - 4)$

e. $y = -2 + \tan\left(2\pi x - \frac{\pi}{4}\right)$

f. $y = \frac{1}{2} - \cot(5x + 2)$

**<u>Examples</u>** Show how to find 2 adjacent asymptotes for graphing each of the following:

a. $y = 3\tan 4x$

b. $y = 2\tan\left(3x - \frac{\pi}{2}\right)$

e. $y = -2 + \tan\left(2\pi x - \frac{\pi}{4}\right)$

f. $y = \frac{1}{2} - \cot(5x + 2)$



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**<u>Example</u>** Graph $y = 3 + \cot\left(x + \frac{\pi}{4}\right)$



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**<u>Example</u>** Graph $y = -1 + 2\tan\,(4x + \pi)$



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**Examples: Translations with The Graphs of Tangent and Cotangent**

Graph over a two-period interval.

a. $y = \tan\left(3x + \frac{\pi}{4}\right)$



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b. $y = -2 + 3\cot(4x - \pi)$



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c. $y = 1 - \tan(2\pi x - \pi)$



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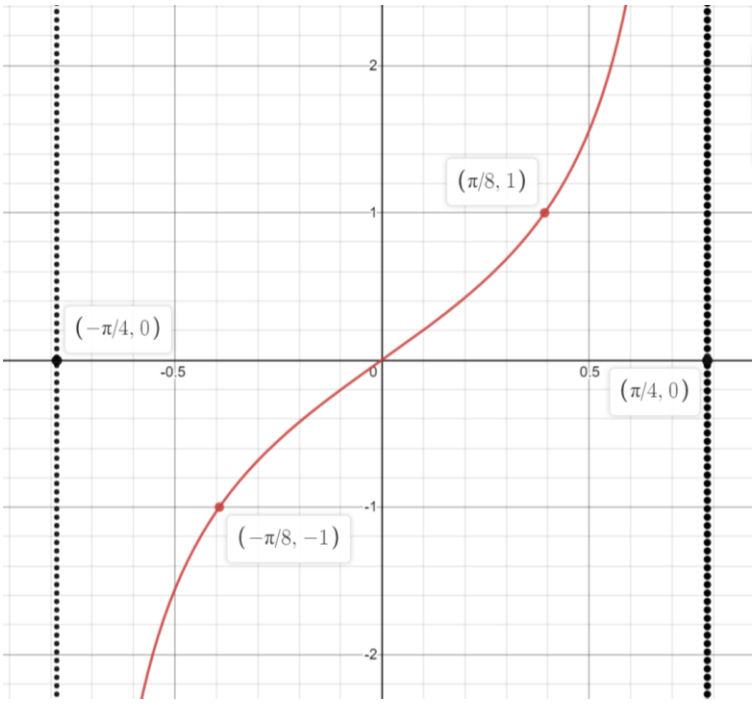
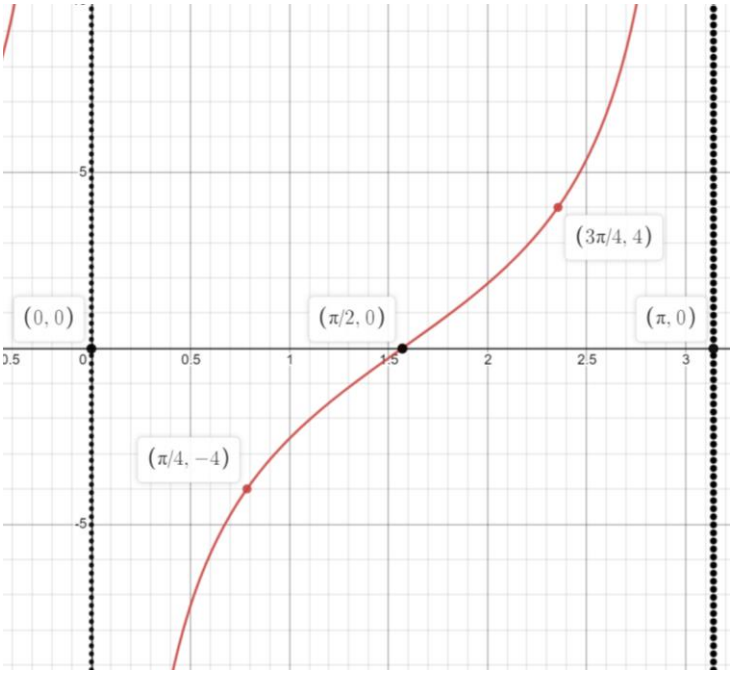
**Lesson: Determining the Equations of Tangent and Cotangent from Graphs**

Graphs from desmos.com of course!

a.




b.



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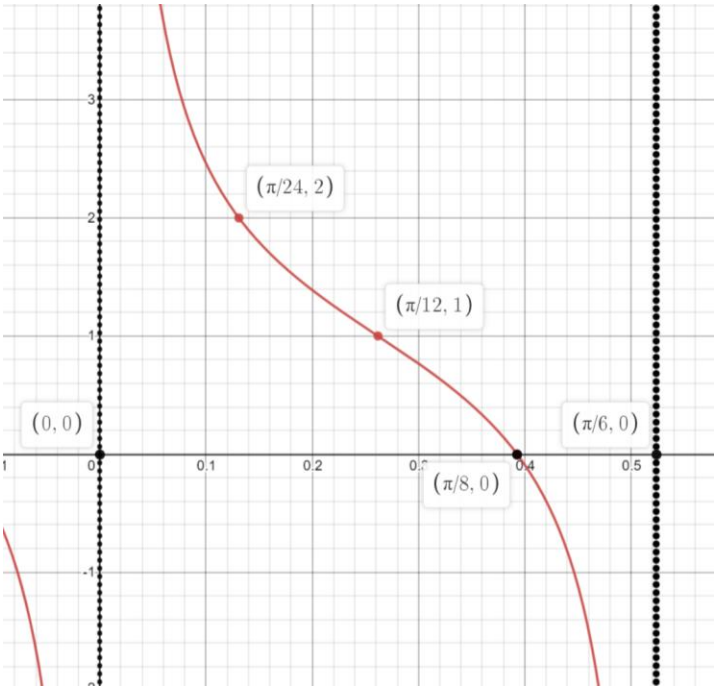
c.



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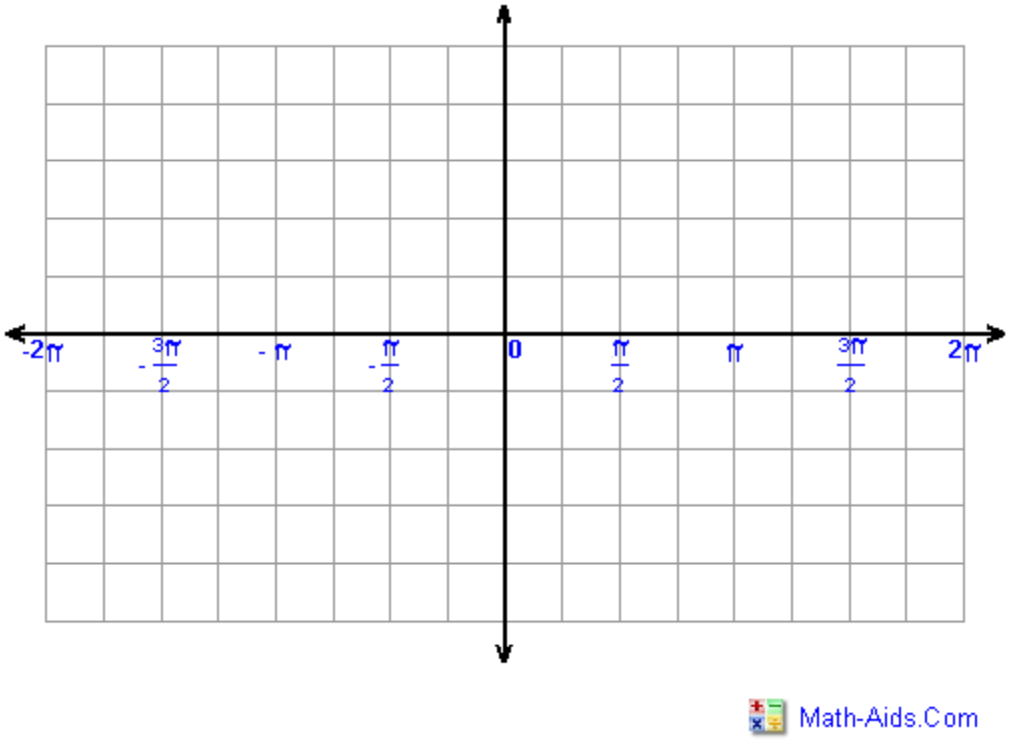
**Lesson: Translations of The Graphs of Secant and Cosecant**

**Draw the graph of secant, filling in the table and then using the graph paper.**

| $x$ | $y = \sec x$ |
|---|---|
| $-\frac{\pi}{2}$ | |
| $-\frac{\pi}{4}$ | |
| $0$ | |
| $\frac{\pi}{4}$ | |
| $\frac{\pi}{2}$ | |
| $\frac{3\pi}{4}$ | |
| $\pi$ | |
| $\frac{3\pi}{2}$ | |



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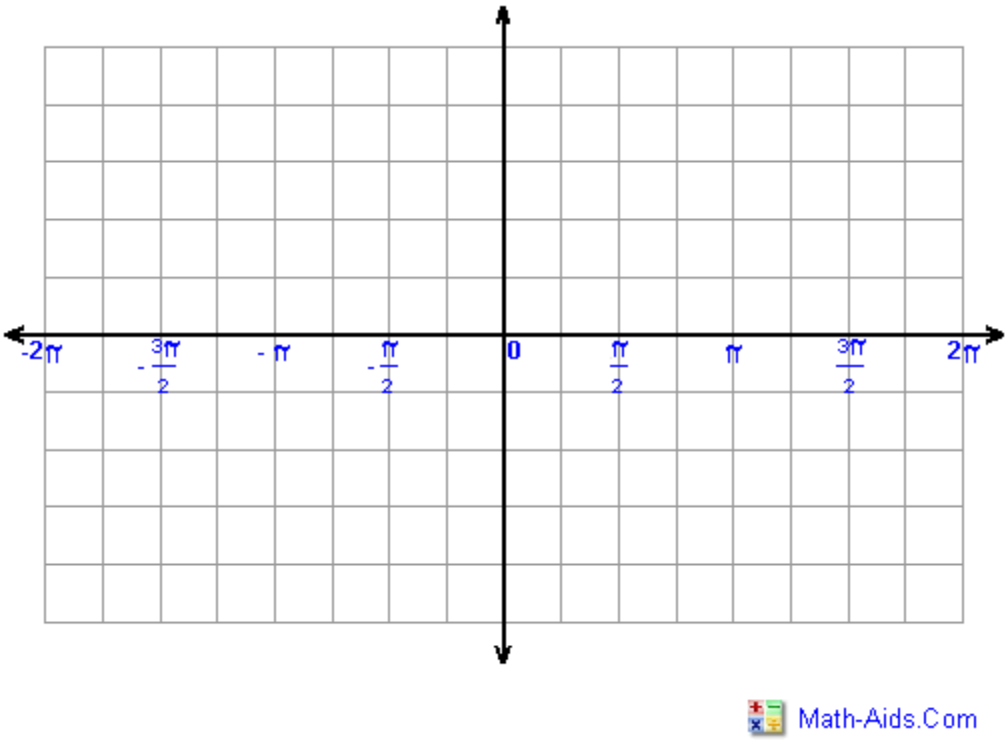
**Draw the graph of cosecant, filling in the table and then using the graph paper.**

| $x$ | $y = \sec x$ |
|---|---|
| $0$ | |
| $\frac{\pi}{6}$ | |
| $\frac{\pi}{3}$ | |
| $\frac{\pi}{2}$ | |
| $\frac{2\pi}{3}$ | |
| $\pi$ | |
| $\frac{3\pi}{2}$ | |
| $2\pi$ | |



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How would we describe the period for these functions?

Where are the graphs continuous (and discontinuous)?

What is the domain and range of the graphs?

How can we describe the x-intercepts of both graphs?



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Are either function even or odd?

How do you interpret amplitude with these functions?

What is the key for graphing? (Be specific with the equations you need to leverage)



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**<u>Examples</u>** Graph $y = 2\csc\frac{1}{2}x$



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**<u>Examples</u>** Graph $y = 1 + \frac{1}{2}\sec\left(x - \frac{\pi}{2}\right)$



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**Examples: Translations of The Graphs of Secant and Cosecant**

Graph the following over a one-period interval.

a. $y = -2\sec\left(x + \frac{3\pi}{4}\right)$



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b. $y = \frac{1}{2}\csc(2x - \pi)$



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c. $y = 2 + \csc\left(\frac{1}{2}x - \pi\right)$



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d. $y = 2 - 3\sec\left(2\pi x - \frac{\pi}{2}\right)$



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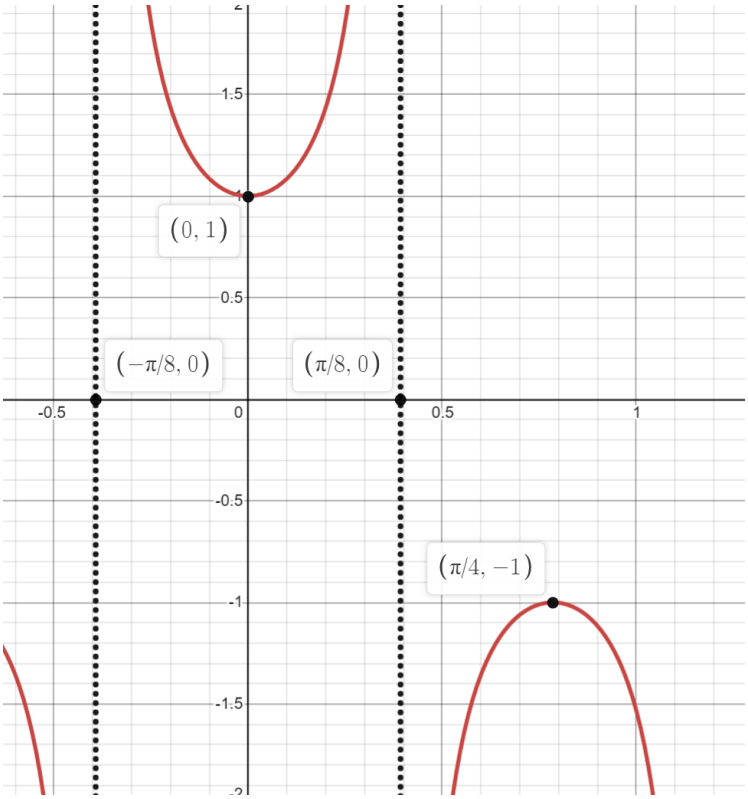
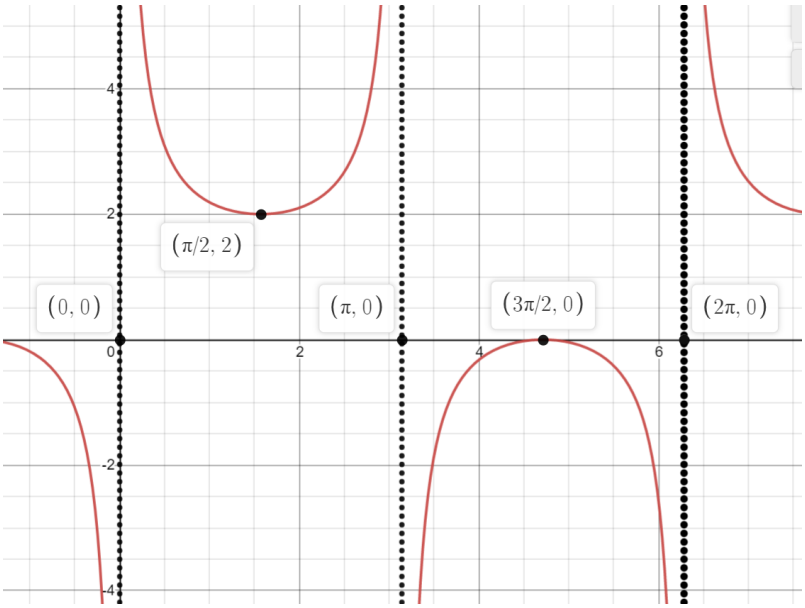
**Lesson: Equations of Secant and Cosecant from Graphs**

Determine the equation of secant or cosecant from the graphs.

a.




b.



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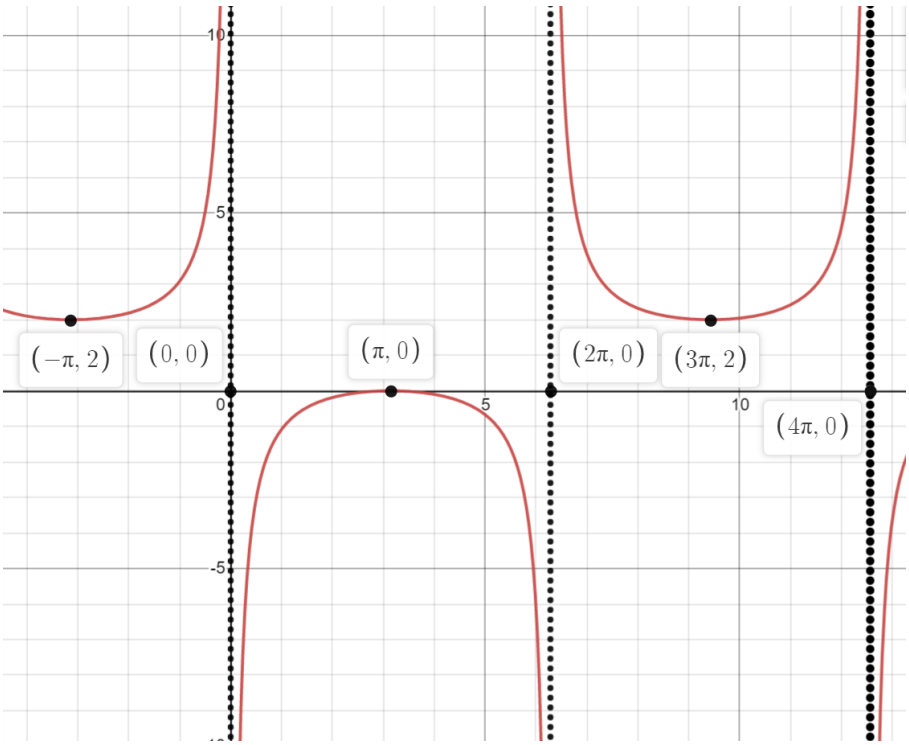
c.



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**Lesson: Inverse Functions (a review)**

What does it mean to be an "inverse function" ?

**<u>Example</u>** If a function includes the points $\{(2,3), (5,4), (-7,2), (-1,5)\}$, then what points will the inverse have?

What is the horizontal line test?

**<u>Example</u>** Draw the graphs from the video, then determine if they have inverses:



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How do you find the inverse of a function $f(x)$?

**<u>Example</u>** Find the inverse of the following:

a. $f(x) = 2x + 1$

b. $f(x) = \sqrt{x}$



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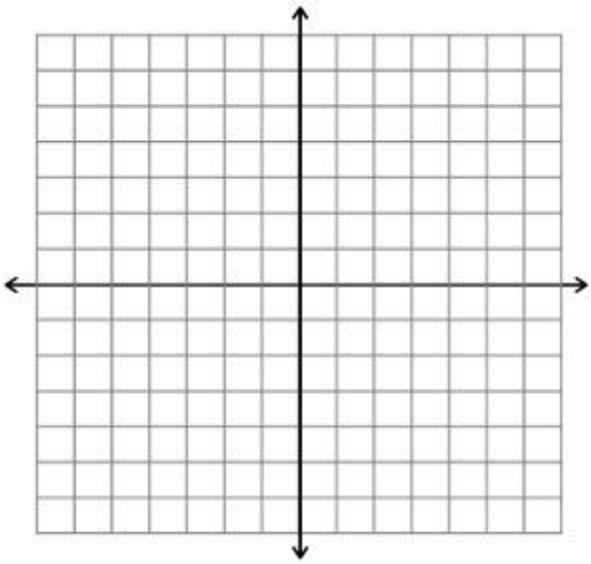
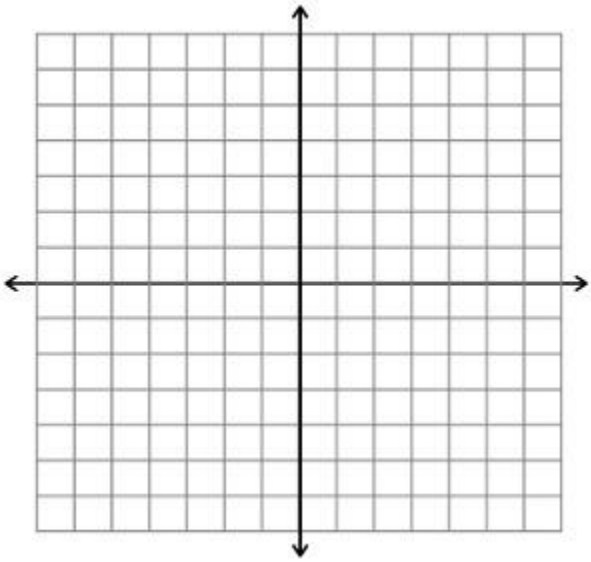
**<u>Examples</u>** Graph the following functions with their inverses:

a. $f(x) = 2x + 1$



b. $f(x) = \sqrt{x}$



From these graphs, we can see that inverses of functions are really just what?



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What can we say about the domain and range inverses of functions?



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**Lesson: Inverse Trig Functions**

Does the sine function have an inverse? Why or why not?

How can we manipulate the graph of the sine function to determine an inverse?

What are two ways that we denote the inverse sine function?

How do we “formally” define the inverse sine function?



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Show how to graph the inverse sine function. Draw both the table and graph discussed from the video:

**<u>Example</u>** Find the value of each of the following:

a. $y = \arcsin\frac{\sqrt{3}}{2}$

b. $f(x) = \sin^{-1}\left(-\frac{\pi}{4}\right)$

c. $y = \sin^{-1} -\sqrt{2}$



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Does the cosine function have an inverse? How should we think about this?

How do we denote the inverse of cosine?

How do we define the inverse cosine function?

Show how to graph the inverse cosine function. Draw both the table and graph discussed from the video:



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**<u>Examples</u>** Find the value of each of the following:

a. $y = \arccos 1$

b. $f(x) = \cos^{-1}\left(\frac{1}{2}\right)$

How do we denote the inverse of tangent?

How do we define the inverse tangent function?



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Show how to graph the inverse tangent function. Draw both the table and graph discussed from the video:

Define the other inverse circular functions:



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Summarize the domains, ranges, and quadrants of the inverse trig functions below (this is important, you'll need to study this for the next exam):

| Function | Domain | Interval | Quadrants |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

**<u>Examples</u>** Find the degree measure of $\theta$:

a. $\theta = \arctan\sqrt{3}$

b. $\theta = \csc^{-1}(-\sqrt{2})$



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**Lesson: Finding Values of Inverse Trig Functions**

Here we will just examine a wealth of examples of how to find different values of inverse trig functions.

**<u>Examples</u>** Use a calculator to determine the appropriate values.

a. Find y in radians if $y = \sec^{-1}(-4)$

b. Find $\theta$ in degrees if $\theta = \operatorname{arccot}(-0.25431)$

Before you evaluate those, you will have to watch some explanation of how / why we evaluate these the way that we do in the calculator. Write out the explanation of how/why we evaluate $\csc^{-1}(2)$ in a calculator below:



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**Cheat Sheet – How to Evaluate Inverse Trig Functions That Don't Have a Button on a Calculator**

**Cheat Sheet (for Comparison) – How to Evaluate Regular Trig Functions That Don't Have a Button on a Calculator**

**<u>This might get confusing so maybe rip this sheet out and put it someplace where you can easily reference it??</u>**



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**Examples** Use a calculator to find each value.

a. $\sin(\cos^{-1} 0.25)$

b. $\cot\,(\cos^{-1}(0.56331233)$

**Examples** Evaluate each expression without using a calculator:

a. $\cos\left(sin^{-1}\left(\frac{2}{3}\right)\right)$

b. $\sec\,(\sec^{-1} 2)$

c. $\sin\left(tan^{-1}\frac{4}{3} - cos^{-1}\frac{12}{13}\right)$



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**<u>Examples</u>** Write each expression as an algebraic expression in terms of u.

a. $\cot\left(\sec^{-1} u\right)\ , u > 0$

b. $\sec\left(\operatorname{arccot}\left(\frac{\sqrt{4-u^2}}{u}\right)\right.$



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**Lesson: Equations Involving Inverse Trig Functions**

**<u>Example</u>** Solve $y = 4\tan 3x$, where x is restricted to $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$.

**<u>Example</u>** Solve $y = \tan\,(2x - 1)$ for x in $\left(\frac{1}{2} - \frac{\pi}{4}, \frac{1}{2} + \frac{\pi}{4}\right)$



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**Example** Solve $3\tan^{-1} x = \pi$

**Example** Solve $6\arccos\left(x - \frac{\pi}{3}\right) = \pi$

**Example** Solve $\cot^{-1} x = \tan^{-1}\frac{4}{3}$



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**Example** Solve $\arccos x + 2\arcsin\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

**Example** Solve $\sin^{-1} x + \tan^{-1} x = 0$



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**Lesson: Solving Trig Equations, part 1 – linear methods**

Recall how linear methods work first by solving: $2x + 4 = 8$

Now compare this approach with the equation: $3\tan\theta - \sqrt{3} = 0$, over the interval $[0°, 360°)$

How does this answer change if we want to solve for *all* solutions?

**<u>Example</u>** Solve: $2\sin x + 3 = 4$ over $[0, 2\pi)$



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**Lesson: Solving Trig Equations, part 2 – quadratic methods**

Using quadratic methods to solve trig equations relies on using the zero-factor property. Show how the zero-factor property works by solving $(x+5)(x-3)=0$.

Compare this approach by solving $\cos\theta\sin\theta = 0 = -\cos\theta$, over $[0°, 360°)$.

**<u>Example</u>** Solve $(\csc x + 2)\left(\csc x - \sqrt{2}\right) = 0$ for all solutions.



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Recall how to use quadratic methods by solving $x^2 + 5x + 6 = 0$.

Compare this approach by solving $\cos^2 x + 2\cos x + 1 = 0$ over $[0°, 360°)$.

**<u>Example</u>** Solve $2\cos^2 x + \cos x - 1 = 0$ for all possible solutions.



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**Example** Solve $2\cos^2 x - \sqrt{3}\cos x = 0$ over $[0, 2\pi)$

**Example** Solve $\tan x \csc x - \sqrt{3}\csc x = 0$ for all solutions.



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**Example** Solve $\csc^2 x - 2\cot x = 0$ over $[0°, 360°)$

**Example** Solve $\tan x - \sqrt{3} = \sqrt{3}\cot x - 1$ for all possible solutions.



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**Lesson: Solving Trig Equations, part 3 – equations with half-angles**

**<u>Example</u>** Solve $2\cos\frac{x}{2} - \sqrt{2} = 0$ over the interval $[0, 2\pi)$

How would the answers change if you needed to find all solutions?



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**Example** Solve $2\sqrt{3}\sin\frac{x}{2} = 3$ for all possible solutions.

**Example** Solve $\sin^2\frac{x}{2} - 2 = 0$ for all possible solutions.



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**Lesson: Solving Trig Equations, part 4 – equations with multiple angles**

**<u>Example</u>** Solve $\cos 2x = \sin x$ over $[0, 2\pi)$

**<u>Example</u>** Solve $2\cos^2 x - 2\sin^2 x + 1 = 0$ over $[0°, 360°)$

How would this change for all solutions?



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**Example** Solve $\cot 2x - \csc 2x = 1$ over $[0, 2\pi)$

**Example** Solve $\tan 2x - \sec 2x = 2$ over $[0, 2\pi)$



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**Example** Solve $\cos x - 1 = \cos 2x$ for all solutions.

**Example** Solve $\sin 2x = 2\cos^2 x$ for all solutions.



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**Lesson: Reciprocal Identities of Trig Functions**

First, what is a reciprocal? Give an example.

What are the 6 reciprocal identities?

1.

2.

3.

4.

5.

6.

You should NOT write out something like _____ or _____as a shortcut for a reciprocal identities. Why not?

How can you shortcut the reciprocal identity then with notation?



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What are some other ways we can relate the reciprocal identities to one another?

**<u>Examples</u>** Find the function values.

a. $\tan\theta$ given that $\cot\theta = 3$

b. $\sec\theta$ given that $\cos\theta = -\frac{5}{\sqrt{50}}$

c. $\sin\theta$ given that $\csc\theta = 1.25$



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**Lesson: Signs and Ranges of Trig Functions**

Review: what does r represent when we are talking about trig functions?

Since r is a distance, we know _____.

What determines if a trig function has a positive or negative value?

Draw a picture to help breakdown and analyze the signs of theta:

Now fill in this table to help you remember the sign of each function:

| $\theta$ IN QUADRANT | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|
| I | | | | | | |
| II | | | | | | |
| III | | | | | | |
| IV | | | | | | |

One way to remember the signs is with the phrase All Students Take Calculus. Break down this this is helpful.



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**Examples** Determine the signs of the trig functions of an angle in standard position with the given measure.

a. $23°$

b. $255°$

c. $-30°$

**Examples** Determine which quadrant of an angle $\theta$ that satisfies the given conditions.

a. $\sin\theta > 0, \csc\theta > 0$

b. $\sec\theta > 0, \csc\theta > 0$



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Now we will show how to get ranges of the six trig functions.

Draw a picture to demonstrate how angle $\theta$ increase in measure from $0° - 90°$.

As the measure of an angle _____, _____ _____ but never exceeds _____. Therefore:

Let's repeat this idea but in quadrant IV. Follow along with the video below and write down the observations again.



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What can we conclude from these sketches and observations?

What can we conclude about the values of $\tan\theta$ and $\cot\theta$? And why?



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What can we conclude about the values of $\sec\theta$ and $\csc\theta$? Why?

We have now discovered the ranges of all the trig functions. Let's summarize this in the table.

| TRIG FUNCTION | RANGE (WRITE IN BOTH SET-BUILDER NOTATION AND INTERVAL NOTATION) |
|---|---|
| $\sin\theta, \cos\theta$ | |
| $\tan\theta, \cot\theta$ | |
| $\sec\theta, \csc\theta$ | |

**Example** Determine if each statement is possible or impossible.

a. $\cot\theta = -0.98$ b. $\cos\theta = -1.25$ c. $\sec\theta = 0.72$



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**Lesson: Pythagorean and Quotient Identities**

Show how we can build the first Pythagorean identity. These are actually pretty darn important, so you really do need to understand this argument!!

Now let's build another one:

And now let's build the last one:



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**Summarize the Pythagorean identities:**

What is only key about these identities?

Now let's show how to build the quotient identities:



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By the way, similar reasoning holds for $\cot\theta$.

**Summarize the Quotient Identities below:**

**<u>Example</u>** Find the value of $\cos\theta$ and $\tan\theta$, give that $\sin\theta = -\frac{\sqrt{2}}{2}$ and $\cos\theta > 0$.

**<u>Example</u>** Find $\sin\theta$, given that $\cos\theta = \frac{4}{5}$ and $\theta$ is in Quadrant IV.



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**<u>Example</u>** Give all 6 trig function values given that $\cot\theta = \frac{\sqrt{3}}{8}$ and $\theta$ is in quadrant I.



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**Examples: Using Pythagorean and Quotient Identities to Find Trig Function Values**

Find all 6 trig function values given the information.

a. $\cos\theta = -\frac{3}{5}$, $\theta$ is in quadrant III.

b. $\csc\theta = 2, \theta$ is in quadrant II.

c. $\cos\theta = 1$



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**Lesson: Fundamental Identities of Trig**

Review the test for symmetry (and the results):

We know that the sine function is odd, show what this means:

We know that the cosine function is even, show what this means:

These are called **identities.**



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Using these identities, what other identity can we determine about the tangent function:

These are called the **even-odd identities.**

We have already seen a handful of identities, and we call these the fundamental identities. Summarize them below.

**Reciprocal Identities**

**Quotient Identities**

**Pythagorean Identities**

**Even-Odd Identities**



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**Example** If $\cos\theta = \frac{3}{4}$ and $\theta$ is in quadrant I, find $\sin\theta$.

**Example** If $\tan\theta = -\frac{1}{4}$ and $\theta$ is in quadrant IV, find the values of all the other 5 trig functions.



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We use the identities to work on problems called transformations.

**Example** Write $\tan x$ in terms of $\sec x$.

**Example** Write the following in terms of sine and cosine, and then simplify until no quotients appear and all functions are of $\theta$ only.

a. $\csc\theta \cos\theta \tan\theta$

b. $\cos\ \theta(\cos\theta - \sec\theta)$



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**Examples: Using Fundamental Trig Identities to Find Values of Other Trig Functions**

Find the five remaining trig functions given the following information.

a. $\csc\theta = -\frac{5}{2}$, $\theta$ in quadrant III.

b. $\sec\theta = \frac{7}{2}$, $\tan\theta < 0$



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**Examples: Writing Trig Functions in Terms of Other Trig Functions (Fundamental Identities)**

a. Write $\cot x$ in terms of $\sin x$

b. Write $\csc x$ in terms of $\cos x$



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**Examples: Using Fundamental Identities to Write Expressions in Terms of Sine and Cosine**

Write each expression in terms of sine and cosine, then simplify so that no quotients appear and all functions are in $\theta$ only.

a. $\cos\theta \csc\theta$

b. $\frac{(1+\cot\theta)}{\cot\theta}$

c. $\frac{\sec^2\theta-1}{csc^2\,\theta-1}$

d. $\frac{\csc\theta}{\cot(-\theta)}$



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**Lesson: Verifying Trigonometric Identities**

As you progress through higher mathematics, sometimes you will need to manipulate equations in very creative ways. One way to learn this level of creativity is by verifying trigonometric identities. We will go through a few techniques of how to do this.

**<u>Strategy 1: Rewrite the More Complicated Side / Work from One Side</u>**

a. Verify that $\frac{1-\sin^2 x}{\cos x} = \cos x$. Show *why.*

b. Verify that $\sin^2 x + \tan^2 x + \cos^2 x = \sec^2 x$



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**Strategy 2: Express the Functions in Terms of All Sine and Cosine**

a. $\frac{\tan x}{\sec x} = \sin x$

b. $\sec x\,(\sin x + \cos x) = 1 + \tan x$



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**Strategy 3: Factor or Complete Indicated Algebraic Operations**

**Examples** Factor.

a. $x^2 - 4$

b. $\sin^2 x - 4$

c. $(\tan x + \cot x)^2 - (\tan x - \cot x)^2$

d. $2\sin^2 x + 3\sin x + 1$



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**Examples** Verify the following identities.

a. $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

b. $\frac{1-\cos x}{1+\cos x} = (\cot x - \csc x)^2$



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**Strategy 4: Manipulate Both Sides**

a. $\frac{\cos x}{\sec x} + \frac{\sin x}{\csc x} = \sec^2 x - \tan^2 x$

b. $\tan^2 x \ \ sin^2 x = tan^2 x + cos^2 x - 1$



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**Strategy 5: Use the Conjugate**

What is the conjugate of the following:

a. $x - 4$

b. $\sqrt{2} - \sqrt{3}$

c. $\sin x + 1$

Verify the following identities: $\frac{1-\cos x}{1+\cos x} = \csc^2 x - 2\csc x \cot x + \cot^2 x$



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**Examples: Verifying Trigonometric Identities**

a. $\frac{\cos x}{\cot x \sin x} = 1$

b. $\frac{1}{\sec x - \tan x} = \sec x + \tan x$



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c. $\frac{1+\sin t}{1-\sin t} - \frac{1-\sin t}{1+\sin t} = 4\tan t \sec t$

d. $(\cos\theta - \sin\theta)(\sec\theta + \csc\theta) = \cot\theta - \tan\theta$



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**Lesson: Proofs of the Sum and Difference Identities for Cosine**

Write out the proof for $\cos(A - B) = \cos A \cos B + \sin A \sin B$. Do your best to understand the proof and then explain it in a way that you will understand.



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Write out the identity for $\cos\,(A - B)$ below (yes I know I stated it above, but YOU should write it so you have a better chance of remembering it):

Write out the proof for $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Write out the identity below (yes, write it out so you don't forget!):



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**Lesson: Using Sum and Difference Identities for Cosine**

**<u>Examples</u>** Show how to use the cosine of a sum or difference to find the exact value of the expressions:

a. $\cos 105°$

b. $\cos \frac{\pi}{12}$

c. $\cos 40° \cos 50° - \sin 40° \sin 50°$

Show how the cofunction identities are an extension of the cosine of a difference:



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Write out all the **cofunction identities**

**<u>Examples</u>** Find one value of $\theta$ that satisfies the following:

a. $\sec\theta = \csc 72°$

b. $\tan\theta = \cot(-44°)$

c. $\cos\theta = \sin\frac{5\pi}{6}$



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**Example** Write $\cos(180° - \theta)$ as a trigonometric function of $\theta$ alone.

**Example** Suppose $\sin s = \frac{3}{5}$ and $\sin t = -\frac{12}{13}$, where s is in quadrant I and t is in quadrant III. Find $\cos\,(s + t)$ and $\cos(s - t)$.

**Example** Verify that $\sec(\pi + x) = -\sec x$



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**Lesson: Sum and Difference Identities for Cosine**

Write out the proof for $\sin(A + B)$

Write out the proof for $\sin(A - B)$

Write out the identities for the sum and difference of sine below:

Write out the proof for $\tan\,(A + B)$



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Write all identities for the sum and difference of tangent below:

**<u>Examples</u>** Find the exact value of each expression.

a. $\sin(-15°)$

b. $\tan\left(-\frac{5\pi}{12}\right)$

c. $\sin\left(\frac{5\pi}{9}\right)\cos\left(\frac{\pi}{18}\right) + \cos\left(\frac{5\pi}{9}\right)\sin\left(\frac{\pi}{18}\right)$



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**<u>Examples</u>** Write each function as an expression involving functions of $\theta$ alone.

a. $\cos(30° + \theta)$

b. $\sin\,(45° + \theta)$

c. $\tan\left(\frac{\pi}{4} + x\right)$



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**Examples** Suppose A and B are angles in standard position such that $\sin A = -\frac{3}{5}\ \frac{3\pi}{2} < A < 2\pi$ and

$\cos B = -\frac{7}{25}$, $\pi < B < \frac{3\pi}{2}$. Find the following.

a. $\sin(A + B)$

b. $\tan(A - B)$

c. the quadrant of A-B



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**<u>Examples</u>** Verify the identities.

a. $\sin(x+y)+\sin(x-y)=2\sin x\cos y$

b. $\frac{\sin(x-y)}{\sin(x+y)}=\frac{\tan x-\tan y}{\tan x+\tan y}$



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**Examples: Sine of a Sum and Tangent of a Sum**

Let $\cos s = \frac{5}{12}$, $\sin t = \frac{4}{5}$ where s and t are in quadrant I. Find:

$\sin(s + t)$

$\tan\,(s + t)$

The quadrant of $(s + t)$



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**Lesson: Double-Angle Identities**

Show how we derive the first identity for $\cos 2A$:

Show how we derive the second identity for $\cos 2A$:

Show how we derive the third identity for $\cos 2A$:



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Show how we derive the identity for $\sin 2A$:

Show how we derive the identity for $\tan 2A$:

Summarize the Double-Angle Identities Below:



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**Example** Give $\sin\theta = \frac{5}{8}$ and $\cos\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

**Example** Find the values of the six trig functions give $\cos 2\theta = -\frac{11}{13}$ and $180° < \theta < 270°$.



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**Example** Verify the following identities:

a. $(\sin x + \cos x)^2 = \sin 2x + 1$

b. $\frac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$



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**Examples** Simplify the expressions.

a. $\sin^2 15° - \cos^2 15°$

b. $\sin^2 \frac{3\pi}{5} - \cos^2 \frac{3\pi}{5}$



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**Example** Write $\cos 3x$ in terms of $\cos x$



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**Lesson: Product-to-Sum Identities**

Show what happens when we add the identities $\cos(A+B)$ to $\cos(A-B)$

Now show what happens when we subtract them:



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**Examples** Write the following as the sum or difference of 2 trig functions.

a. $2\sin 48° \cos 104°$

b. $8\sin 9x \sin 5x$



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**Examples** Write the expression as a product of trigonometric functions.

a. $\cos 6x + \cos 9x$

b. $\sin 106° - \sin 96°$



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**Lesson: Half-Angle Identities**

Show how we derive the identity for $\sin\frac{A}{2}$:

Show how we derive the identity for $\cos\frac{A}{2}$:

Show the first way we can derive an identity for $\tan\frac{A}{2}$:



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Show the second way we can derive an identity for $\tan\frac{A}{2}$:

Show the third way we can derive an identity for $\tan\frac{A}{2}$:

Summarize the half-angle identities below:



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**Example** Show how to use the half-angle identity to evaluate: $\sin 12.5°$

**Example** Find the exact value of $\cos 165°$

**Example** Find the exact value of $\tan 75°$ using the identity $\tan A = \frac{\sin A}{1+\cos A}$



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**<u>Example</u>** Given $\cos s = \frac{2}{7}$ with $0 < s < \frac{\pi}{2}$, find $\sin\frac{s}{2}$, $\cos\frac{s}{2}$, and $\tan\frac{s}{2}$



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**<u>Example</u>** Simplify each expression.

a. $\sqrt{\frac{1-\cos 40°}{2}}$

b. $\frac{1-\cos 64.72°}{\sin(64.72°)}$

c. $\pm\sqrt{\frac{1+\cos 10x}{2}}$



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**Example** Verify the identities.

a. $\cot^2 \frac{x}{2} = \frac{(1+\cos x)^2}{\sin^2 x}$

b. $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$



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**Lesson: Summarizing All of the Identities**

Using all the trig identities that we've discussed, verify the following:

a. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

b. $1 - \tan^2 \frac{x}{2} = \frac{2\cos x}{1+\cos x}$



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c. $\tan\left(\frac{a}{2}+\frac{\pi}{4}\right)=\sec a+\tan a$

d. $\sin 7x+\sin 5x+\sin 3x+\sin x=4\sin 4x\cos 2x\cos x$



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e. $\tan 4x = \frac{2\tan 2x}{2-\sec^2 2x}$



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**Lesson: Reference Angles and Non-Acute Angles**

Review: what is a quadrantal angle?

What is associated with every non-quadrantal angle?

Show how we can find reference angles by drawing the reference graphs in the video:

**<u>Example</u>** Find the reference angle for each angle.

a. $274°$

b. $822°$



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Draw how to find the reference angle based on the quadrant (follow along with the video and draw the diagram on your own):

Now let's talk about using $30°$, $45°$, and $60°$ as reference angles.

**<u>Example</u>** Find the values of the 6 trig functions for $240°$. Draw a reference!!! (This class is all about pictures!!)

Don't forget the helpful device: All Students Take Calculus 😊



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Let's outline the steps for finding trig function values for any non-quadrantal angle:

1.

2.

3.

4.

How can you avoid making an error with finding a trig function value?

**<u>Examples</u>** Find the exact values:

a. $\sin(-150°)$

b. $\cos(-510°)$



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**Example** Evaluate: $\sin^2 120° + \cos^2 120°$

**Example** Find all values of $\theta$, if $\theta$ is in the interval $[0°, 360°)$ and has the given function value: $\cos\theta = \frac{\sqrt{3}}{2}$

**Example** Is the following true or false? $\cos 60° = 2\cos 30°$



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**Examples: Reference Angles and Finding Trig Value Functions**

Find the exact values of all 6 trig functions of each angle. Rationalize when applicable.

a. $315°$

b. $-855°$

Evaluate

a. $\cot^2 120° + \tan^2 60° - \cos^2 330°$

b. $\cot^2 135° - \tan^4 45° - \sin^2 150°$



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**Examples: Finding the Value of Theta**

Find all values of $\theta$ if $\theta$ is in the interval $[0°, 360°)$ and has the given value.

a. $\sec\theta = -\sqrt{2}$

b. $\cot\theta = -\frac{\sqrt{3}}{3}$

c. $\cos\theta = \frac{1}{2}$



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**Lesson: How to Approximate Trigonometric Function Values**

We will discuss using a calculator to do approximations of trig functions.

If you are going to use a calculator, what is key?

**<u>Examples</u>** Use a calculator to evaluate the following expressions.

a. $\cos 32°15'$

b. $\sec 55.321°$

c. $\sin(-105°)$

d. $\frac{1}{\csc(233.45°)}$

To find the value of $\theta$ (certain angle measures), we can use inverse trig functions. What are the three inverse trig functions that a calculator will typically have?

What do these functions means?



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**<u>Example</u>** Show how to use completely proper notation to find the angle $\theta$ in the interval $[0°, 90°)$.

a. $\cos\theta = 0.931145$

b. $\cot\theta = 1.7538932$



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**Examples: Solving for Theta with a Calculator**

Find all the values of $\theta$ in $[0, 360°)$ where $\tan\theta = 0.9845331$



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**Lesson: Significant Digits and Solving Triangles**

What is an exact number?

We've seen that many values for trig functions are not exact. Consider this: a room is 12ft x 18ft. We are doing a project where we need to know the length of the diagonal. What is it?

What's the issue with reporting this number?

What do we call the number of digits obtained by actual measurement?

Show some examples of significant digits:



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**Significant Digits for Angles**

| Angle Measure to Nearest | Number of Significant Digits Used | Write an Example |
|---|---|---|
| | | |
| | | |
| | | |
| | | |

How do you perform calculations in this case?

What should you keep in mind about the answer in this case?



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We can practice significant digits by solving triangles.

**Example** Solve the right triangle presented in the video.

**Example** Solve the right triangle presented in the video.



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**Examples: Significant Digits and Rounding**

How many significant digits are there in each?

| | |
|---|---|
| a. 421 | e. 400.01 |
| b. 420 | f. 400.00 |
| c. 400 | g. $72°31'$ |
| d. 400. | h. $72°30'$ |

Your answer needs to be 2 significant digits. You calculate the answer 421. What should your final answer be for significant digits?

Your answer needs to be 4 significant digits. You calculate the answer 421. What should your final answer be for significant digits?

Your answer needs to be 3 significant digits. You calculate the answer $42°18'$. What should your final answer be for significant digits?



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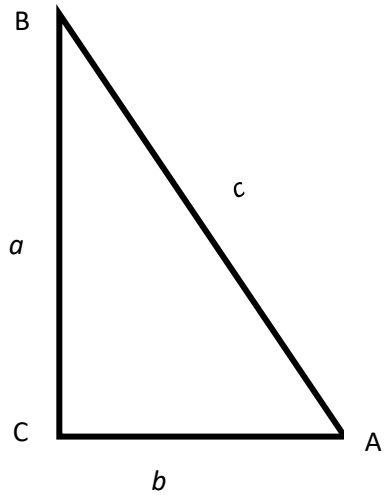
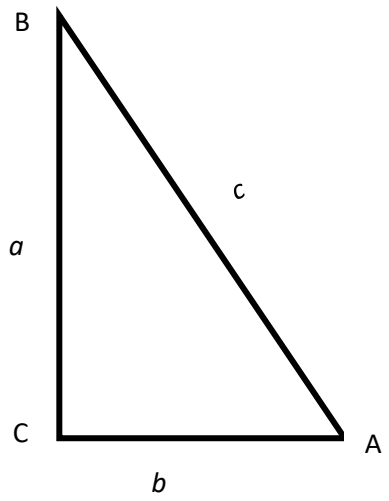
**Examples: Solving Triangles**

Find all missing lengths and angle measures for the given triangles. Use significant figures.

a. $A = 28.3°, c = 18.5$




b. $b = 85m, c = 112m$



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**Lesson: Angles of Elevation or Depression**

What is an angle of elevation and what is its context?

What is an angle of depression?

What should you be careful when interpreting either?



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**Example** Outline the general steps to solving the application problem: the angle of elevation from a point on the ground 122ft from the base of a flagpole is $25°20'$. Find the height of the flagpole.

Step 1 (write out the general approach and then show how to do it for each step)

Step 2:

Step 3:



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**Example** A 14.00m fire truck ladder is leaning against a building. What is the distance that the ladder goes up the wall if the ladder makes an angle of $42°25'$ with the horizontal?

**Example** A company safety committee installs a floodlight to illuminate the exit of a building, with the dimensions shown in the video. Draw the situation, and then find the angle of depression to the nearest minute.



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**Examples: Angles of Elevation or Depression**

A pyramid has a square base with side 600 ft long and height 300 ft. Find the angle of elevation made from the sides of a corner to the center of the pyramid. (Hint: The base of the triangle in the figure is half the diagonal of the square base of the pyramid.



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**Lesson: Applications of Right Triangles with Bearing**

What is bearing?

What are some examples of bearing?

Explain method 1 to express bearing:

What is key with these types of problems?



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**Example** Two radar stations are on an east-west line, 9.2 km apart from one another. The first station detects a plane on a bearing of $63°$. The second station detects the same plane on a bearing of $333°$. Find the distance from the second station to the plane.



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Explain method 2 to express bearing:

Show some examples:

**Example** A ship leaves port and sails on a bearing of N 35° E for 3 hours. Then it turns and sails on a bearing of S 55° E for 2.5 hours. If the ships rate is 22 knots (nautical miles per hour), find the distance that the ship is from the port.



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**Examples: Applications of Right Triangles with Bearing**

Two lighthouses are located on a north-south line. From lighthouse A, the bearing of a ship 3520 meters away is 139°43^′. From lighthouse B, the bearing of a ship is 49°43^′. Find the distance between the lighthouses.

The bearing from A to C is S 52°$E$. The bearing from A to B is N 84° $E$. The bearing from B to C is S 38°$W$. A plane flying at 375 mph takes 1.5 hours to go from A to B. Find the distance from B to C.



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**Lesson: Applications of Right Triangles with Distance and Height**

What is the subtense bar method?

**<u>Example</u>** Draw a situation between points P and Q, then solve the following:

a. Find d when $\theta = 2°3'15''$ and $b = 3.5000$cm.

b. How much change would there be in the value of d if $\theta$ measured $1''$ larger?



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We will now examine a few other problems involving distances and heights.

**Example** Copy the illustration shown in the video of the angle of elevation from a point on the ground to the top of a pyramid. The angle of elevation is $32°15'$. The angle of elevation from a point 120ft farther back is $20°25'$. Find the height of the pyramid.

**Example** In a certain area, the lowest angle of elevation of the sun in winter is $22°15'$. Find the minimum distance x that a plant needing full sun can be placed from a fence $4.75$ ft high.



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**Example: Modeling Distance with Right Triangles**

A highway curve connects 2 straight sections road and may be circular. Use the diagram in the video to illustrate the situation.

a. Find the distance between A and C if r=925ft and $\theta = 34°$.

b. Find an expression in terms of r and $\theta$ for the distance from P to Q.



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**Lesson: Oblique Triangles and the Law of Sines**

First, we should review some facts about congruent triangles. When are 2 triangles congruent?

**<u>Congruence Axioms</u>**

Side-Angle-Side (SAS)

Angle-Side-Angle (ASA)

Side-Side-Side (SSS)



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When does Side-Angle-Angle (SAA) hold?

When a triangle is <u>not</u> a right triangle, what is it it called?

When can a triangle be solved?

What data is required for solving oblique triangles?

Case 1:

Case 2:

Case 3:

Case 4:



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Is there an Angle-Angle-Angle congruency?

When Case 1 or Case 2 is involved, what is required?

When Case 3 or Case 4 is involved, what is required?

This Law of Cosines will be discussed in another video.

Show how we derive the Law of Sines:



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State the Law of Sines:

What is an alternative form of the law of sines?



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**Example** Solve Triangle ABC if $A = 28.5°, C = 102.3°, c = 25.1$ in.

Recall the formula for the area of a triangle:

Using SAS, how else can you find the area of a triangle?



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**Example** Reference the sketch in the video. Find the area of the triangle.

**Example** Find the area of a triangle DEF if $D = 58°10', e = 31.2in, F = 73°30'$.



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**Lesson: Applications of the Law of Sines**

**<u>Example</u>** An engineer wishes to measure the distance across a river. Reference the sketch in the video.
If $C = 117.4°, A = 30.0°, b = 75.8$ ft



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Review the two methods of bearing:

**<u>Example</u>** The bearing of a lighthouse from a ship was found to be $N\ 52°$ W. The ship sailed 12.3 miles south, and the new bearing was $N\ 23°W$. Find the distance, to the nearest mile, between the ship and the lighthouse at each location.



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**Example** A balloonist is directly above a straight road, 2 miles long that joins the 2 villages. The closer town to him is an angle of depression of $45°$, and the farther town is at angle of depression of $41°$. How high above the ground is the balloon?



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**Lesson: The Ambiguous Case of the Law of Sines**

Describe the ambiguous case of the law of sines as describe in the video. There's going to be quite a bit of information presented – watch the entire explanation of why we need the law of sines and do your best to describe the various cases to sum up. Summarize the explanation so that it makes sense to YOU.



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What were some tips for applying the law of sines?

**<u>Example</u>** Solve triangle ABC if $a = 16.2cm, c = 14.1\ cm$, and $C = 72°30'$.



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**<u>Example</u>** Solve triangle DEF if $D = 62°8', a = 34.5in, b = 38.1in.$



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Explain the number of triangles satisfying the ambiguous case (SSA):

**<u>Examples</u>** Outline the steps to solve these triangles. You don't have to solve completely, but do enough work to determine how many possible triangles there are. Include sketches.

a. $B = 80°, C = 35°, b = 27$

b. $A = 60°, B = 70°, c = 5$



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**Examples** Outline the steps to solve these triangles. You don't have to solve completely, but do enough work to determine how many possible triangles there are. Include sketches.

c. $a = 15, b = 24, B = 33°$

d. $A = 25°, b = 8, a = 11$



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**Examples** Outline the steps to solve these triangles. You don't have to solve completely, but do enough work to determine how many possible triangles there are. Include sketches.

e. $A = 61.4°, b = 35.5, b = 39.1$

f. $a = 17.1, c = 13.1, C = 76°$



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**<u>Example</u>** Without using the law of sines, explain why no triangle ABC exists satisfying $B = 95°, b = 40cm, c = 46cm$.



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**Lesson: The Law of Cosines**

What is the property of triangle side length restriction?

Show how to derive the law of cosines:



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State the Law of Cosines:

Where is one place where we've seen this?

**<u>Example</u>** Two boats leave a harbor at the same time, traveling on courses that make an angle of $81°30'$ between them. When the slower boat has traveled $65.2$ miles, the faster boat has traveled $84.3$ miles. At that time, what is the distance between the boats?



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**Example** Solve triangle ABC if $B = 72.4°, a = 31.4ft, c = 44.3ft$.

**Example** Solve triangle DEF if $d = 24.2in, b = 42.6in, c = 59.1in$.



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**Example** The simplest type of roof truss is a triangle. Refer to the sketch in the video. Find angle C to the nearest degree for the truss shown.

Explain the 4 Cases of Solving Oblique Triangles



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**Example: The Law of Cosines with SSS Triangles**

Find the angles of the triangles given the following sides:

a. $a = 3ft, b = 5ft, c = 6ft$

b. $a = 324m, b = 421m, c = 298m$



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**Lesson: Heron's Formula for the Area of a Triangle**

Explain how to use Heron's formula (as explained in the video):

**<u>Example</u>** Find the area of triangle ABC where $a = 12m, b = 16m, c = 25m$



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**Example** Find the area of the Bermuda Triangle if the sides of the triangle have approximate lengths 800mi, 875mi, and 1250mi.



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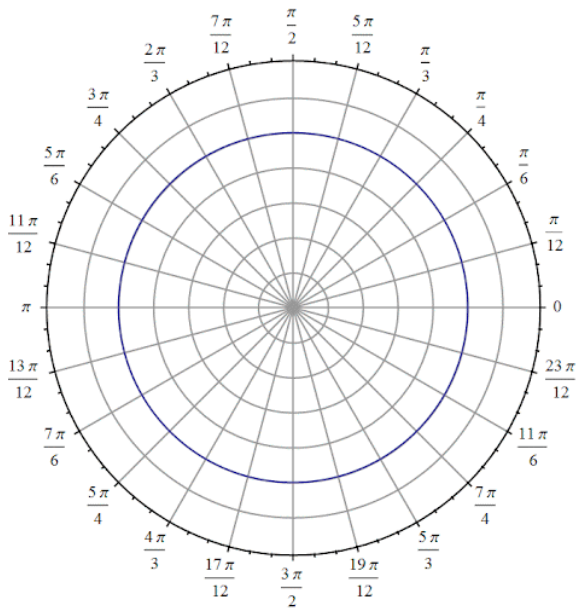
**Lesson: Polar Equations and Graphs**

There are other coordinate systems besides the rectangular coordinate system.

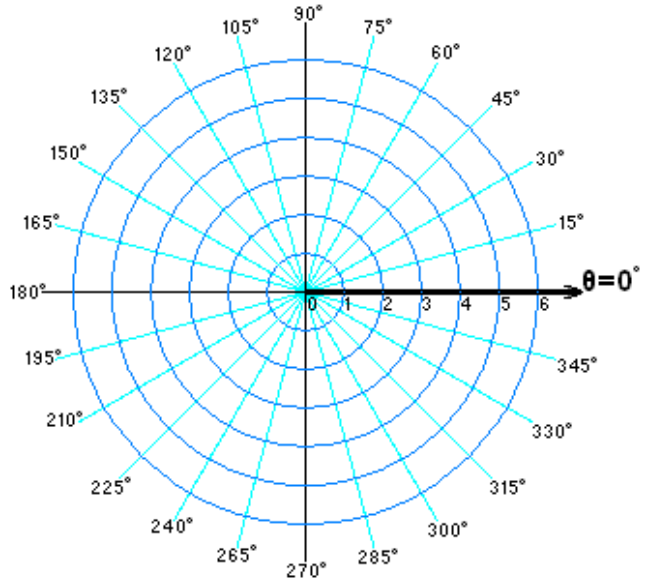
We will discuss the polar coordinate system!

What is a pole? What is the polar axis? Draw a sketch.

Here is what the polar coordinate system looks like:




It can also be in degrees:



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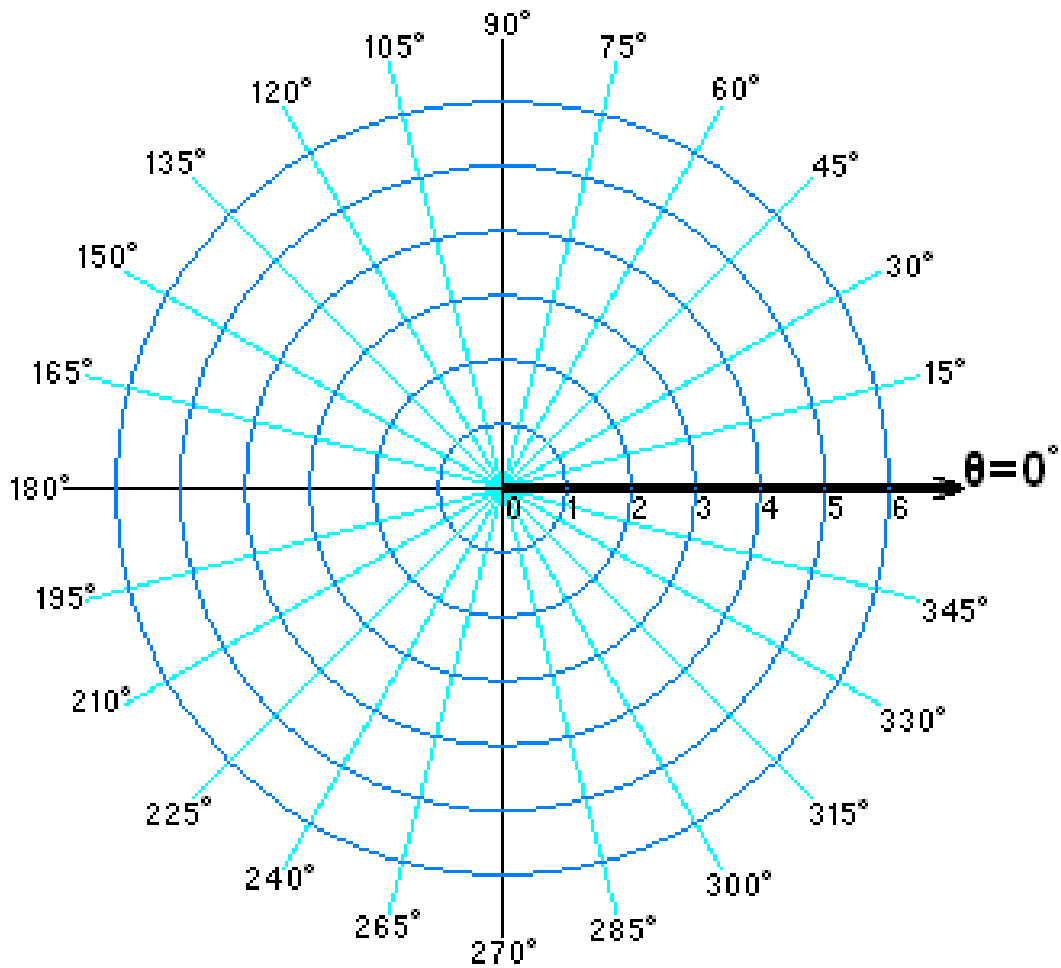
How do we convert between rectangular and polar coordinates?

**<u>Examples</u>** Show how to plot each point, and then determine the rectangular coordinates of each point.

a. $P(3, 45°)$



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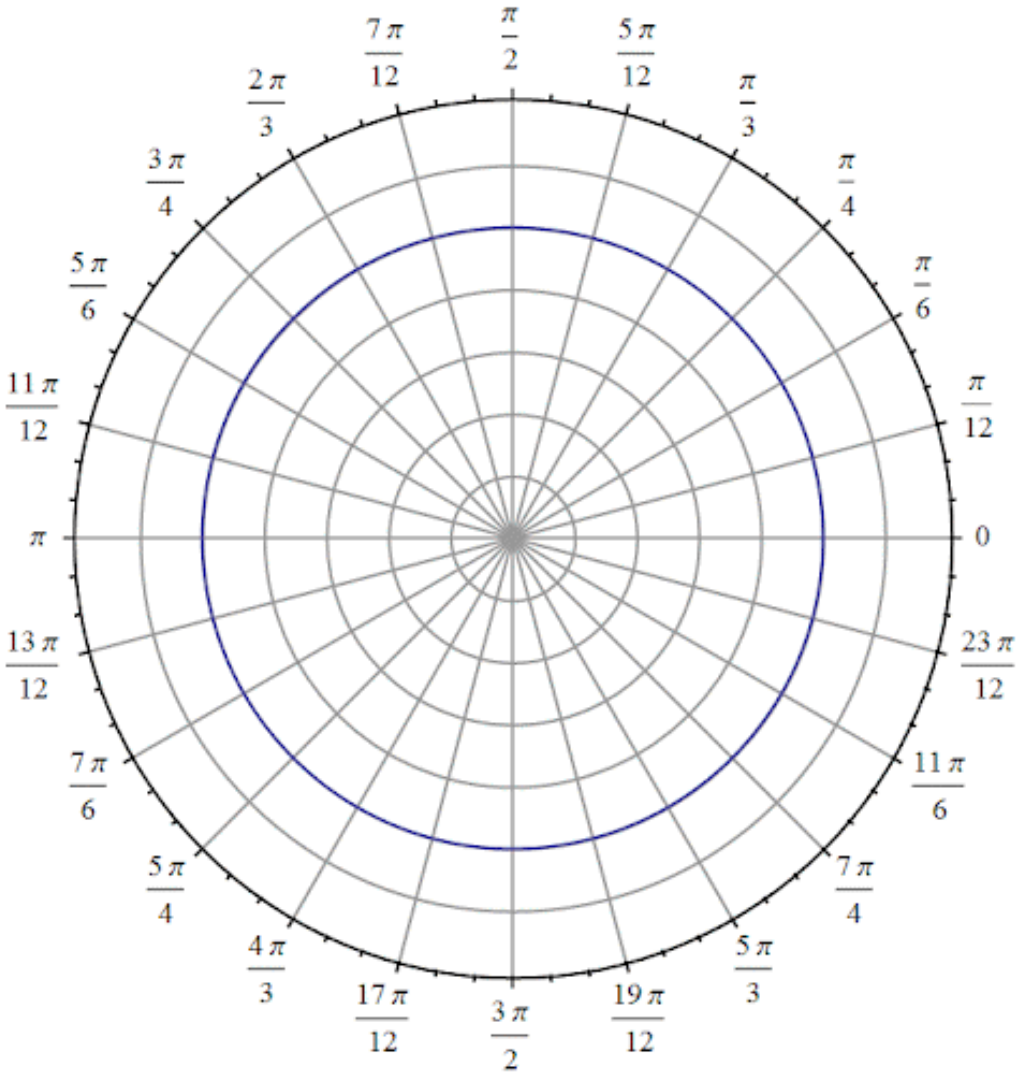
**Examples** Show how to plot each point, and then determine the rectangular coordinates of each point.

b. $P\left(-4, \frac{\pi}{3}\right)$



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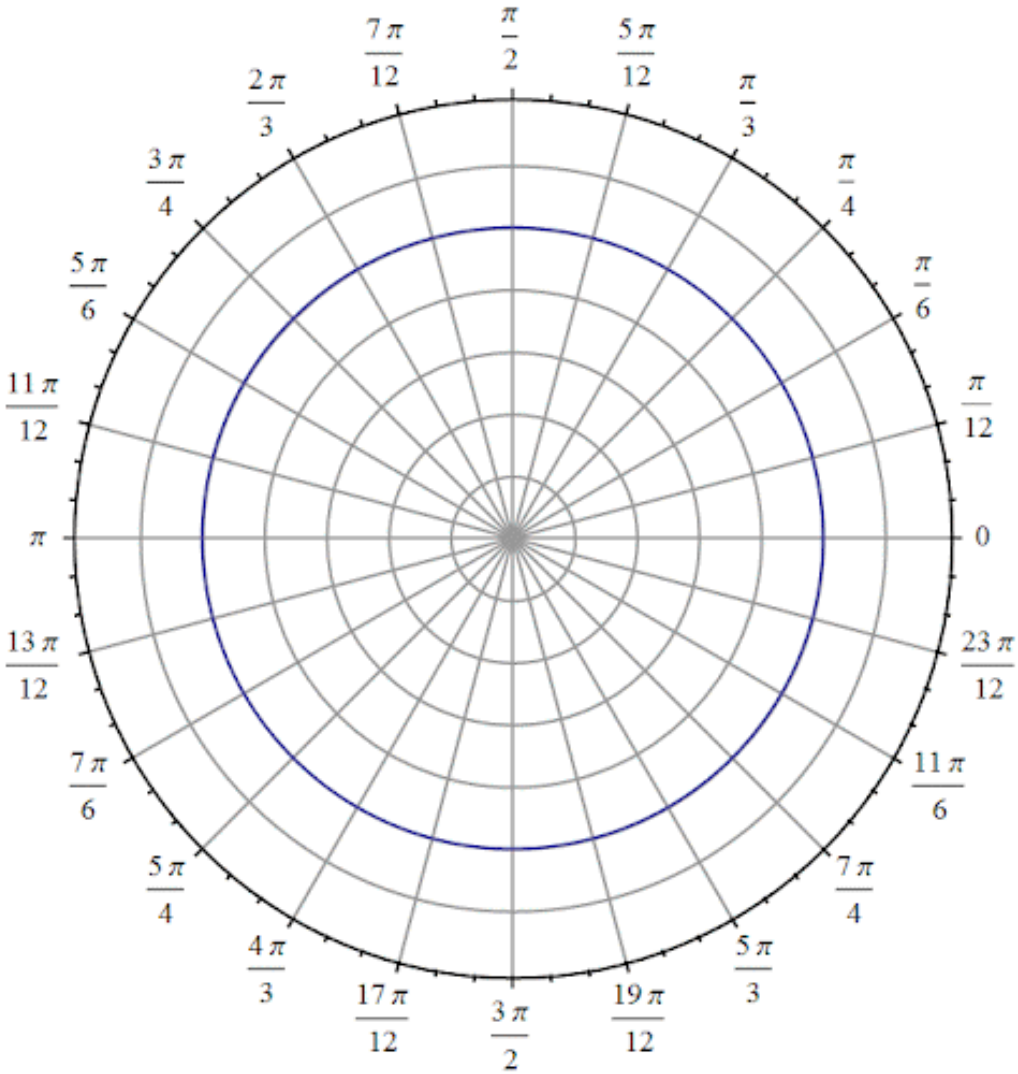
**Examples** Show how to plot each point, and then determine the rectangular coordinates of each point.

c. $R\left(3, -\frac{\pi}{2}\right)$



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While a given point in the plane can only have 1 pair of rectangular coordinates, how many pairs of polar coordinates can it have?

**<u>Examples</u>** Determine the following:

a. Three other pairs of coordinates for the point $P(3, -110°)$.

b. Two pairs of polar coordinates for the point with rectangular coordinates $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$



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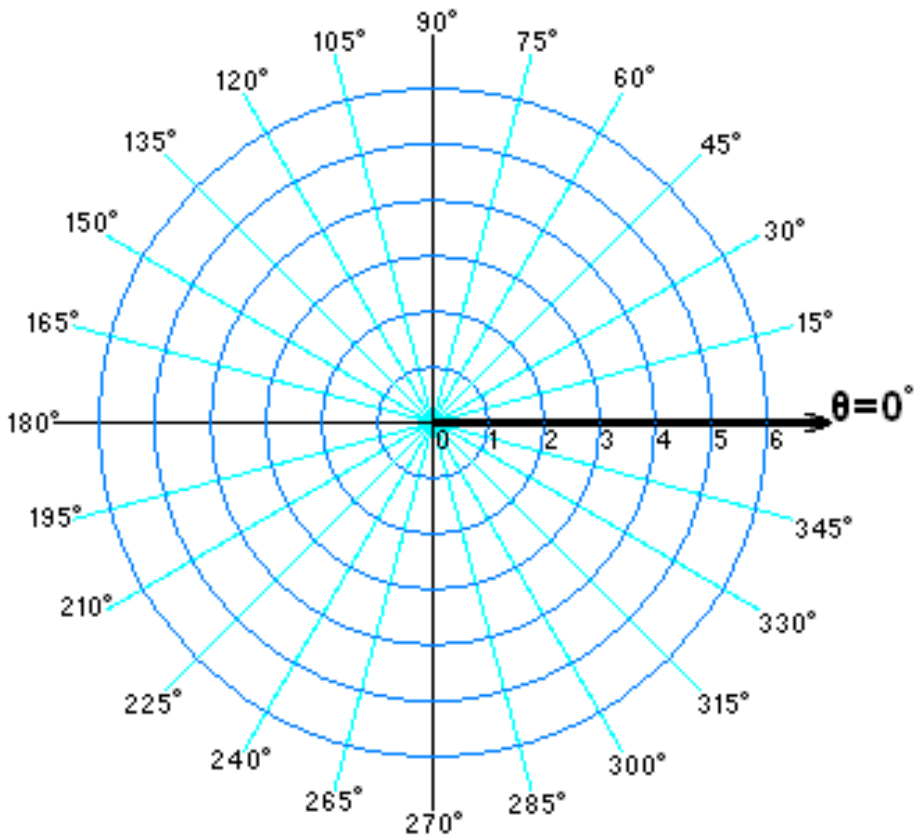
What makes an equation a “polar equation” ?

**<u>Example</u>** Convert the equation to a polar equation, and then graph it: $y = x - 1$



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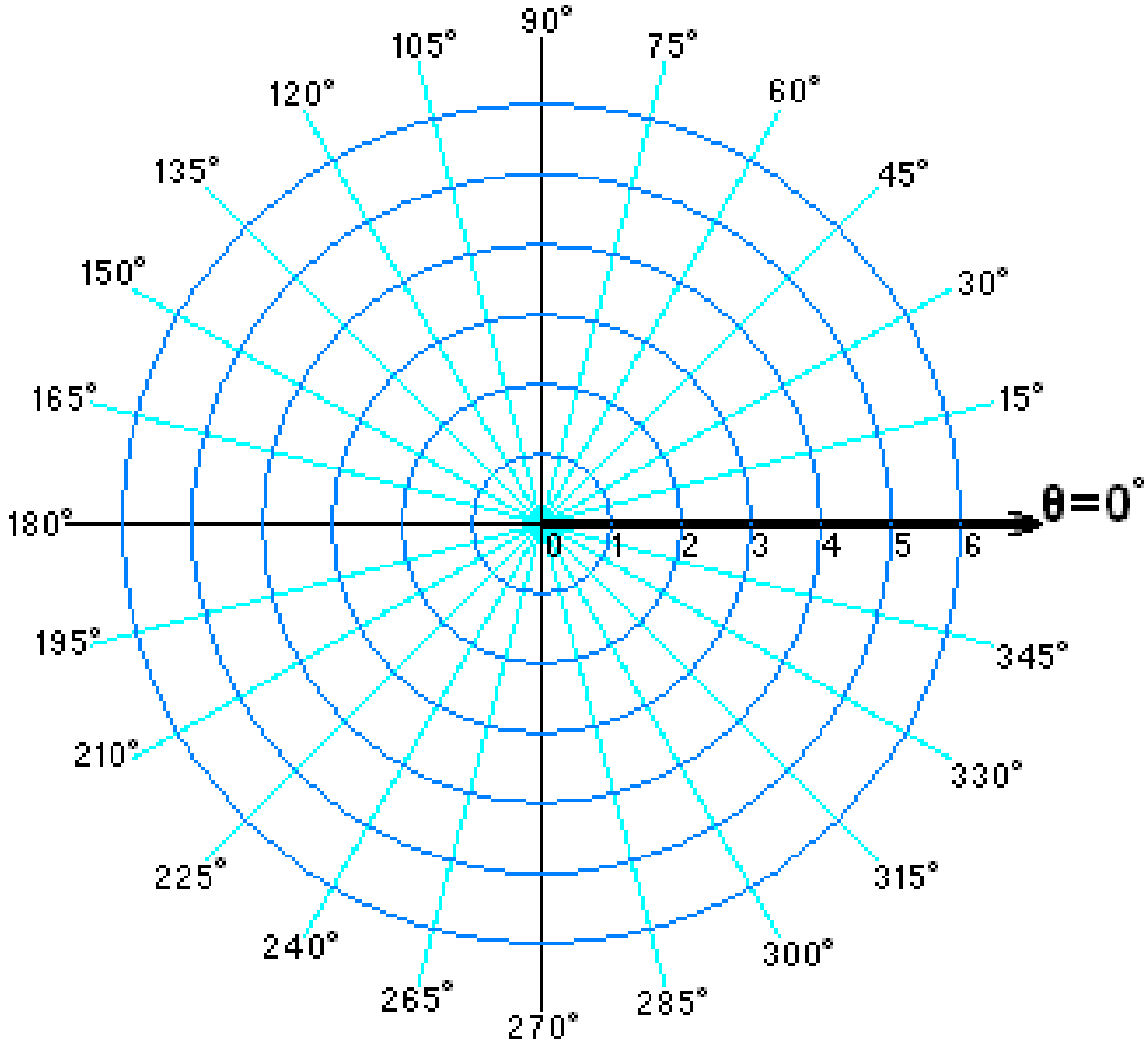
**Example** Convert the equation to a polar equation, and then graph it: $x^2 + y^2 = 25$



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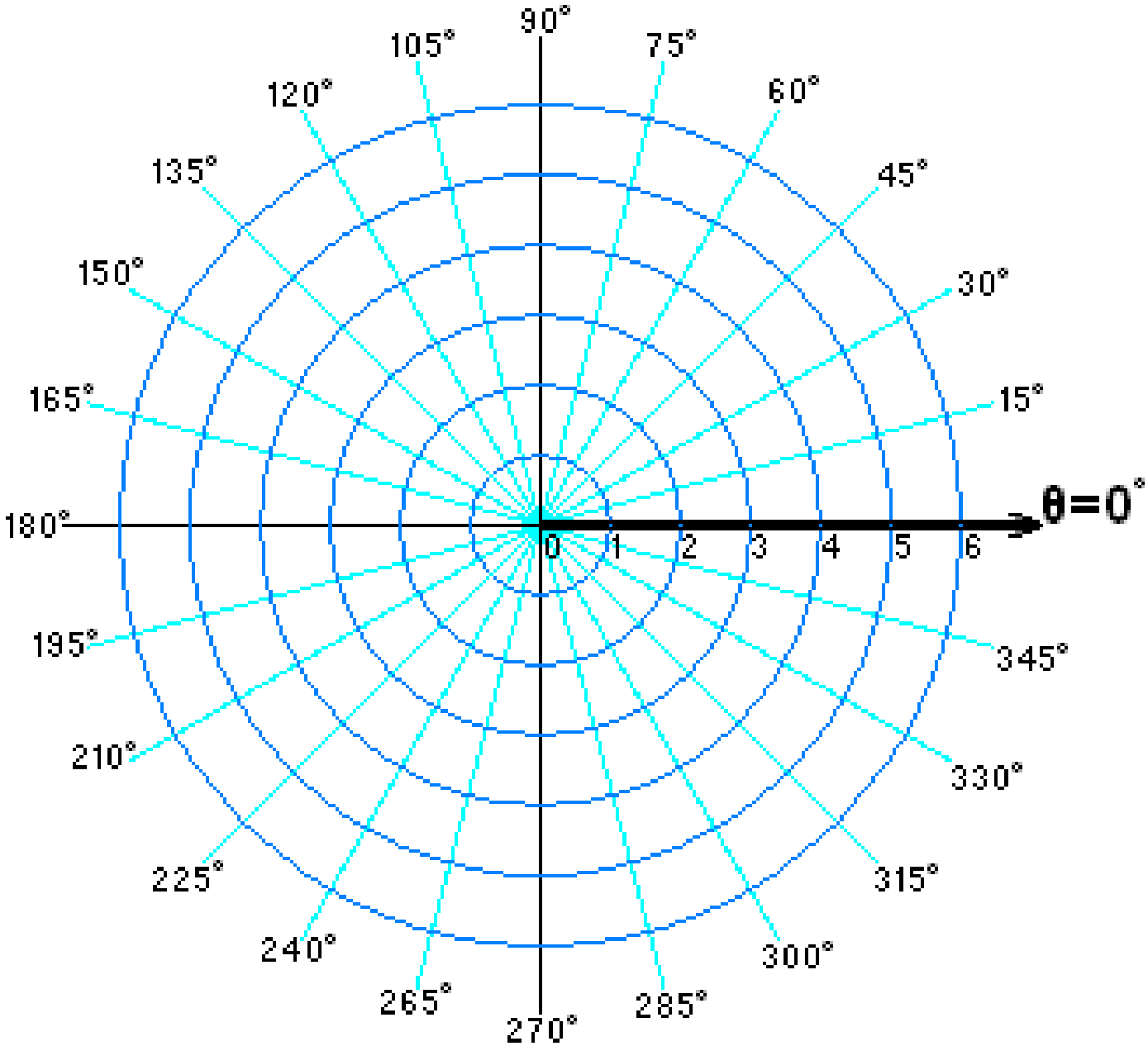
**Example** Graph $r = 1 - \sin\theta$



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**Lesson: Trigonometric (Polar) From of Complex Numbers**

Complex numbers cannot be ordered like real numbers can. What is one way that we can organize complex numbers?

Show how we can graph complex numbers:

Complex numbers correspond to vectors. For a complex number $x + yi$, show how this corresponds to a vector (sketch), then label the sketch.

**<u>Examples</u>** Add $3 + 2i$ and $2 - 5i$, then graph both numbers and their resultant.



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Define the relationships between $x, y, r, \theta$ below (with a sketch)

What is the trigonometric form (or polar form) of a complex number?

How do we refer to the number $r$?

How do we refer to $\theta$?



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**Example** Write $3(\cos 300° + i \sin 300°)$ in rectangular form.

How do you convert from rectangular from to trigonometric form?

**Examples** Write in trigonometric form.

a. $-\sqrt{2} + i$

b. $-12$



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**<u>Examples</u>** Write the complex number in its alternative form.

a. $8(\cos 200° + i \sin 200°)$

b. $-8 + 5i$



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**Lesson: The Product and Quotient Theorems**

First, let's review how to multiply $\left(2+i\sqrt{3}\right)(-2\sqrt{3}+2i)$

Let's rewrite both of those complex numbers in their trigonometric form:

Now multiply their trigonometric forms together:



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What do you notice about the result?

This problem demonstrates the product theorem.

State the Product Theorem:

**<u>Example</u>** Find the product of $3(\cos 45° + i \sin 45°)$ and $5(\cos 120° + i \sin 120°)$. Then write the answer in rectangular form.



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(PART A) Show how to find the quotient of $(2\sqrt{6} - 2i\sqrt{2})$ and $\left(\sqrt{2} - i\sqrt{6}\right)$

Now rewrite these complex numbers in their trigonometric form.

How can the "answer" to PART A be rewritten in trigonometric form?



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What do you notice about this result?

This demonstrates the quotient theorem.

State the Quotient Theorem.

**<u>Example</u>** Find the quotient $\frac{35\ cis\ 45°}{7\ cis\ (-90°)}$. Write the result in rectangular form.



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**Example** Use a calculator to find the following. Put your answer in rectangular form.

a. $(7.2\ cis\ 38.5°)(3.5\ cis\ 49.8°)$

b. $\dfrac{15.42\ (\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})}{5.21\ (\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}$



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**Lesson: De Moivre's Theorem and Powers and Roots of Complex Numbers**

Find the product: $[r\ (\cos\theta + i\sin\theta)]^2$

This is one example of De Moivre's theorem.

State De Moivre's Theorem:

**<u>Example</u>** Find $\left(\sqrt{3} + i\sqrt{5}\right)^8$ and write the answer in rectangular form.



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We can extend this theorem to find all nth roots of complex numbers.

Explain the concept of finding the nth root of a complex number.

State the nth Root Theorem



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**Example** Find the three cube roots of -8

**Example** Find all complex number solutions to $x^4 + 1 = 0$.



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**Lesson: Vectors**

What is a scalar?

What are vector quantities?

What is a vector?

Draw some vectors showing the differences with magnitude and directions.

Explain some of the notation we use regarding these concepts:



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When are two vectors equal?

Is the sum of two vectors still a vector? (circle) Yes No

How do you find the sum of two vectors?

Vector addition is commutative. What does this mean?

For every vector, there is an opposite vector. Explain what this means:

The sum of **v** and **-v** results in what?



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What happens when you multiply a vector and a scalar?

Explain the geometric properties of parallelograms:



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**Examples** For each pair of vectors and the angle $\theta$ between them, sketch the result.

a. $|\boldsymbol{u}| = 12, |\boldsymbol{v}| = 22, \theta = 25°$

b. $|\boldsymbol{u}| = 20, |\boldsymbol{v}| = 40, \theta = 30°$

**Examples** Use the parallelogram in the video to find the magnitude of the resultant force for the 2 forces shown in the figure in the video.



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**Lesson: Vectors and Applications of Magnitude**

**Example** A boat is being towed by two other boats, with the first boat having a force of 840lbs and the second have 925lb. The angle between these forces is 26°. Find the direction and magnitude of the equilibrant.

**Example** Two forces of 150 lbs and 275 lbs act at a point. The resultant force between them is 350 lbs. Find the angle between the forces.



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**Lesson: Vectors and Applications of Navigation**

**<u>Example</u>** A ship leaves port on a bearing of $27°$ and travels $62.4$miles. The ship then turns due east and travels $85.2$ miles. How far is the ship from the port? What is its bearing from port?



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**Lesson: Vectors and Applications of Incline**

**<u>Example</u>** Find the force required to keep 2700-lb car parked on a hill that makes a $13°$ angle with the horizontal.

**<u>Example</u>** A force of $18.00$ lb is required to hold a 74.0 lb crate on a ramp. What angle does the ramp make with the horizontal?



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**Lesson: Vectors (Algebraic Interpretation)**

Vectors are defined algebraically in the rectangular coordinate system with what notation?

Draw how we interpret this vector below.

In a previous video, we discussed how to define vectors geometrically. How is this definition different from our algebraic interpretation?

Show how to define a direction angle.



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How do we calculator the magnitude of a vector?

How do we calculation the direction angle of the same vector $\boldsymbol{u} = \langle a, b \rangle$?

**<u>Example</u>** Calculate the magnitude and direction for $\boldsymbol{u} = \langle 5, 2 \rangle$.



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Draw how we can view the horizontal and vertical components of vectors.

**Example** A vector has magnitude $36.0$ and direction angle $35°$. Find the horizontal and vertical components.

**Example** Copy the sketch in the video. Write the vectors in the form $\langle a, b \rangle$.



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What are the vector operations?

**<u>Example</u>** Let $\boldsymbol{u} = \langle -3, 4 \rangle, \boldsymbol{v} = \langle 5, 7 \rangle$. Find and illustrate each of the following.

a. $\boldsymbol{u} + \boldsymbol{v}$

b. $-3\boldsymbol{v} + 2\boldsymbol{u}$



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What is the i,j form of vectors?

**<u>Example</u>** Write $\langle 3, 5 \rangle$ in i,j form

**<u>Example</u>** Write $2\boldsymbol{i} - 4\boldsymbol{j}$ in bracket vector form.



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**Lesson: The Dot Product**

What is the dot product?

What is another name for the dot product?

Which is true: the dot product results in a vector the dot product results in a real number

**<u>Example</u>** Find the dot products.

a. $\langle 3, 5\rangle \cdot \langle 2, -1\rangle$

b. $\langle 2, -3\rangle \cdot \langle -4, -5\rangle$

What are the properties of the dot product?



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**<u>Examples</u>** Let $\boldsymbol{u} = \langle 1, 3 \rangle, \boldsymbol{v} = \langle 2, 4 \rangle, \boldsymbol{w} = \langle -1, 2 \rangle$. Evaluate the following.

a. $\boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{w}$

b. $(3\boldsymbol{u}) \cdot \boldsymbol{w}$

What are the possible outcomes of the dot product?

What is the geometric interpretation of the dot product? Draw a sketch.



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**Example** Find the angle between the vectors.

a. $\boldsymbol{u} = \langle 1, 3 \rangle, \boldsymbol{v} = \langle 2, 4 \rangle$

b. $\boldsymbol{u} = \langle 1, -2 \rangle, \boldsymbol{v} = \langle -1, -3 \rangle$

Explain how to interpret the dot product in the following scenarios …

| The Dot Product is … | The Angle between the Vectors is … |
|---|---|
| Positive | |
| Negative | |
| Zero | |

When the dot product is 0, what does this mean?

What's the better word for perpendicular? (fancier, makes you sound smart, you should definitely use it to impress people at parties):

**Example** Determine if the pair of vectors is orthogonal: $\boldsymbol{u} = \langle 1, 2 \rangle, \boldsymbol{v} = \langle -6, 3 \rangle$



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**Lesson: Summarizing the Equations and Graphs of Conic Sections**

Re-summarize how the equations and graphs of parabolas work.



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Re-summarize how the equations and graphs of circle and ellipses work.



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Re-summarize how the equations and graphs of hyperbolas work.



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**Lesson: The General Form of a Conic Section**

What is the general form of a conic section?

How do you interpret this equation to decide what type of conic section you have?



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**Example** Identify the type of conic section and graph it (you will need to put each equation in standard form).

a. $x^2 - 6x + y = 0$

b. $2x^2 - 8x + 2y^2 + 20y = 12$



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**Example** Identify the type of conic section and graph it (you will need to put each equation in standard form).

c. $y^2 + 6y - 4x^2 + 8x = -6$

d. $9x^2 + 9y^2 = 81$



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**Lesson: Eccentricity and Conic Sections**

How can we define conic sections in terms of eccentricity? Draw an illustration to explain this.

Summarize the value of the e's based on the conic section

| Conic Section | Eccentricity e |
|---|---|
| Parabola | |
| Circle | |
| Ellipse | |
| Hyperbola | |



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**Examples** Find the eccentricity of each conic section.

a. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b. $x^2 + 2x + y = 0$

c. $\frac{(x+3)^2}{25} + \frac{(y-1)^2}{9} = 1$



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**Lesson: An Intro to Conic Sections**

What are the 4 conic sections we will discuss? Draw an illustration with it.



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**Lesson: Parabolas as Conic Sections**

What is the previous equation we used for parabolas opening upwards or downwards? Draw a illustration to go with this.

How can we represent parabolas in similar notation for parabolas that open sideways?



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**<u>Examples</u>** Give the domain and range of the following parabolas, then graph the parabola.

a. $x + 1 = (y + 1)^2$

b. $-x = 3y^2 + 6y + 2$



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When dealing with conic sections, we shift how we think about things.

What is the geometric definition of a parabola? Draw an illustration to accompany this definition.

What equations do we use that align with the geometric definition?



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**Examples** Give the focus, directrix, vertex, and axis of symmetry for each parabola. Then use this information to graph the parabola.

a. $x^2 = 4y$

b. $(y-2)^2 = 24(x-3)$



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**Lesson: Writing Equations of Parabolas**

**<u>Example</u>** Write the equation of a parabola with the following characteristics.

a. focus $\left(0, -\frac{1}{2}\right)$ with the vertex at the origin

b. vertex (4, 3) and focus is (4, 5)

c. vertex (1, 2), directrix x=4



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**Lesson: Ellipses as Conic Sections**

Give the geometric definition of an ellipse. Draw an illustration to accompany this.



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What are the standard forms of an ellipse? Describe the parts and draw VERY CLEAR illustrations.

What is important to note with this equation?



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**Example** Graph each ellipse. Give the domain, range, center, vertices, endpoints of the minor axis, and focus.

a. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

b. $9x^2 + y^2 = 81$



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**<u>Example</u>** Graph each ellipse. Give the domain, range, center, vertices, endpoints of the minor axis, and focus.

c. $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} = 1$



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**Lesson: Writing Equations of Ellipses**

**Example** Write the equation of an ellipse with the following characteristics.

a. major axis 6, focus at (0,2) and (0, -2)

b. center at (-2, 7), major axis vertical with length 10, and c=2



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**Lesson: Eccentricity with Ellipses**

To discuss the concept of eccentricity, we need to discuss what the formal mathematical definition of conic sections is.

Define a conic:

The intuitive idea behind eccentricity can change slightly with each conic section, but it is generally a ratio of different measures within the conic section.

What does the eccentricity of an ellipse tell us?

How do we represent eccentricity?

How do you calculate the eccentricity of an ellipse?



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How does the eccentricity of a parabola compare to the eccentricity of an ellipse? Summarize the explanation from the video.

**<u>Examples</u>** Find the eccentricity of each ellipse.

a. $\frac{x^2}{9} + \frac{y^2}{4} = 1$

b. $x^2 + 25y^2 = 25$



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**<u>Example</u>** Find the equation of an ellipse with $e = \frac{3}{4}$ with foci at $(-2,0), (2,0)$



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**Examples: Using Complete the Square to Write an Ellipse Equation in Standard Form**

Rewrite the following equations in standard form.

a. $x^2 + 12y^2 - 14x - 48y = -85$

b. $9x^2 + 36x + 16y^2 - 160y + 292 = 0$



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**Lesson: Hyperbolas as Conic Sections**

What is the geometric definition of a hyperbola? Draw an illustration with your definition.



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State the equations for the standard form of the hyperbola. Draw illustrations with it.



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**<u>Examples</u>** Graph each hyperbola. Give the domain, range, center, vertices, foci, and equations of the asymptotes.

a. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

b. $9x^2 - 25y^2 = 225$



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**<u>Examples</u>** Graph each hyperbola. Give the domain, range, center, vertices, foci, and equations of the asymptotes.

c. $\frac{(y-1)^2}{9} - \frac{(x+3)^2}{4} = 1$



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**Lesson: Writing Equations of Hyperbolas**

**<u>Examples</u>** Write the equation of the hyperbola with the following characteristics.

a. vertices at (0, 6) and (0, -6) with asymptotes $y = \pm\frac{1}{2}x$

b. foci at $\left(-3\sqrt{5}, 0\right), (3\sqrt{5}, 0)$ with asymptotes $y = \pm 2x$



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**Examples** Write the equation of the hyperbola with the following characteristics.

c. center at (1, -2) and focus at (-2, -2) and vertex at (-1, -2)



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**Lesson: Eccentricity of Hyperbolas**

How do you calculate the eccentricity of a hyperbola?

What can we say about the eccentricity of the hyperbola? What does it mean?

**<u>Example</u>** Find the eccentricity of the hyperbolas.

a. $\frac{x^2}{36} - \frac{y^2}{9} = 1$

b. $8y^2 - 16x^2 = 16$



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**<u>Example</u>** Find the equation of a hyperbola with $e = \frac{25}{9}$ and foci at (9, -1) and (-11, -1)



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**Examples: Writing the Standard Form of a Hyperbola**

Write each hyperbola in standard form.

a. $x^2 - 6x - y^2 + 4y = 1$

b. $y^2 - 14y - x^2 + 14x = 16$



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**Lesson: A Quick Review of the Basics of Solving Systems of Linear Equations**

Systems of equations are 2 or more equations in 2 or more variables. They can look like:

$\begin{cases} 2x + 3y = 6 \\ x - 3y = 3 \end{cases}$ or $\begin{cases} 2x - y + z = 8 \\ 5x + 2y - z = 9 \\ x - 4y + z = 10 \end{cases}$ or $\begin{cases} 3x + 2y - z = 7 \\ x - 4y + 2z = 8 \end{cases}$

A solution of a system is an ordered pair that is true in each equation of the system. Some systems have no solutions, and some systems have infinite solutions.

In this lesson, we will briefly review the main techniques to solve systems. We will focus on systems that have a solution.

**<u>Example</u>** Use the substitution method to solve: $\begin{cases} 3x - y = -4 \\ x + 3y = 12 \end{cases}$

**<u>Example</u>** Use the elimination method to solve: $\begin{cases} 3x - y = -4 \\ x + 3y = 12 \end{cases}$



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**<u>Example</u>** Solve: $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 4 \\ \frac{3x}{2} + \frac{3y}{2} = 15 \end{cases}$



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**<u>Example</u>** Solve: $\begin{cases} 2x + y + z = 9 \\ -x - y + z = 1 \\ 3x - y + z = 9 \end{cases}$



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**Examples: Systems of 3 Linear Equations with 3 Variables**

Solve: $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$



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Solve: $\begin{cases} x + 2y + 3z = -11 \\ 3x - y + z = 0 \\ -2x + 3y - z = 4 \end{cases}$



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**Lesson: Systems with No Solutions or Infinite Solutions**

**<u>Example</u>** Solve: $\begin{cases} x + 4y - 3z = 2 \\ 2x - 5y + 2z = -8 \\ -3x - 12y + 9z = 7 \end{cases}$

**<u>Example</u>** Solve: $\begin{cases} 7x + 2y = 6 \\ 14x + 4y = 12 \end{cases}$



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**<u>Example</u>** Solve: $\begin{cases} 5x - 4y + z = 9 \\ y + z = 15 \end{cases}$

**<u>Example</u>** $\begin{cases} 3x - 2y + z = 15 \\ x + 4y - z = 11 \end{cases}$



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**<u>Example</u>** Solve: $\begin{cases} 2x + y - 3z = 0 \\ 4x + 2y - 6z = 0 \\ x - y + z = 0 \end{cases}$



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**Lesson: Solving Systems of Equations by Graphing**

We will now turn our attention to systems of equations.

## System of Equations

A system of equations refers to 2 or more equations that share 2 or more variables.

## Solution of a System of Equations

A solution of a system of equations is an ordered pair that is true in all equations.

Determine if the following pairs are solutions to the systems.

a. $(1,3)$
$x - 2y = -5$
$2x - 3y = -4$

b. $(-4, 3)$
$x + 3y = 5$
$2x + 3y = 1$

There are several ways to solve systems. The first method we will focus on is how to solve by graphing.

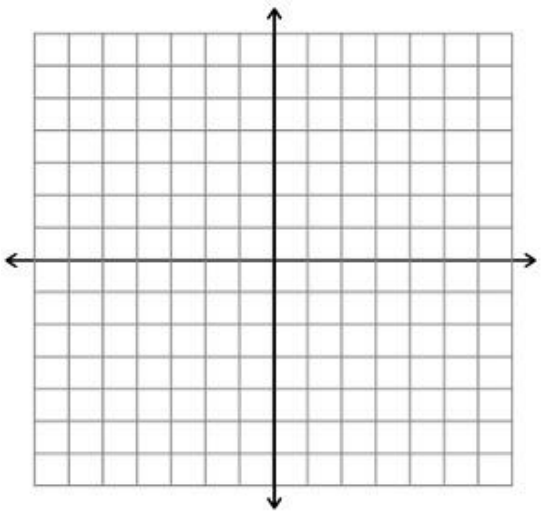
## Solving by Graphing

In this method, we graph both lines and determine the point of intersection if possible. We will examine all the possibilities.

## Example

Graph each line and determine the point of intersection to find the solution.

$y = 2x + 3$
$y = -\frac{1}{2}x - 2$





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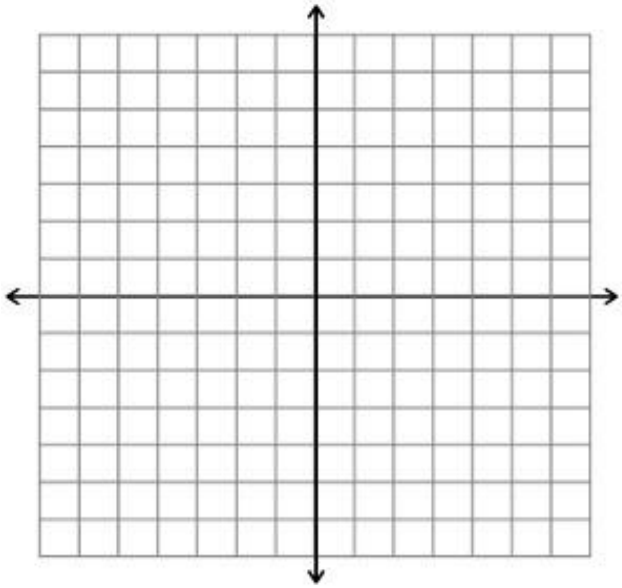
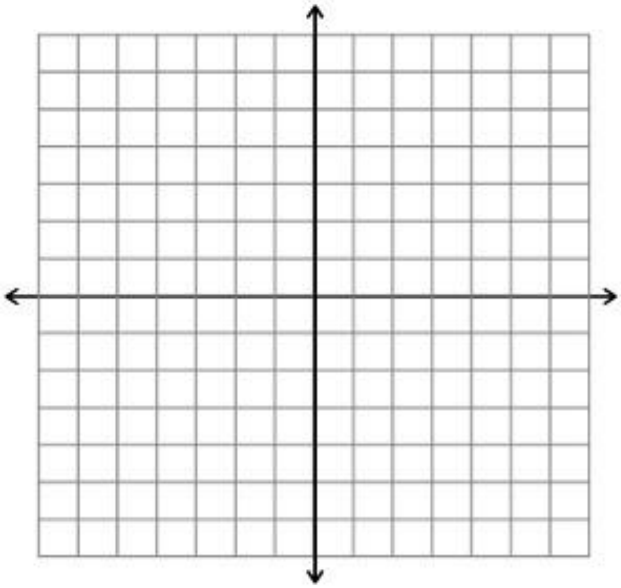
**Example**

a. $y = -3x + 1$
$y = 2x - 4$



b. $y - \frac{1}{2}x = -4$
$y + 2x = 1$





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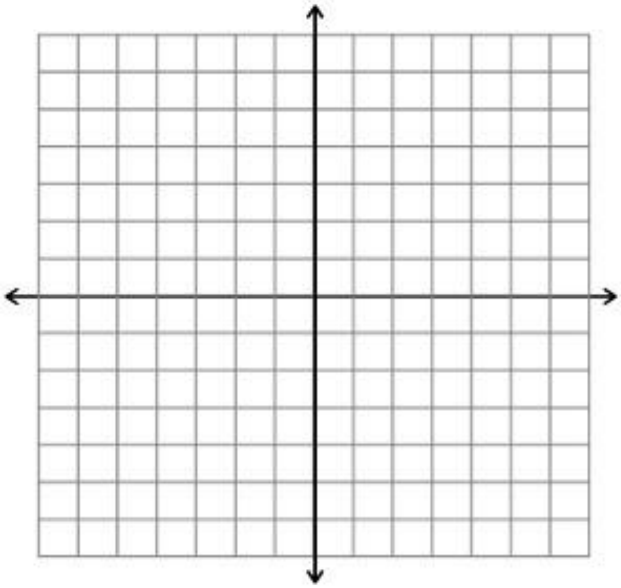
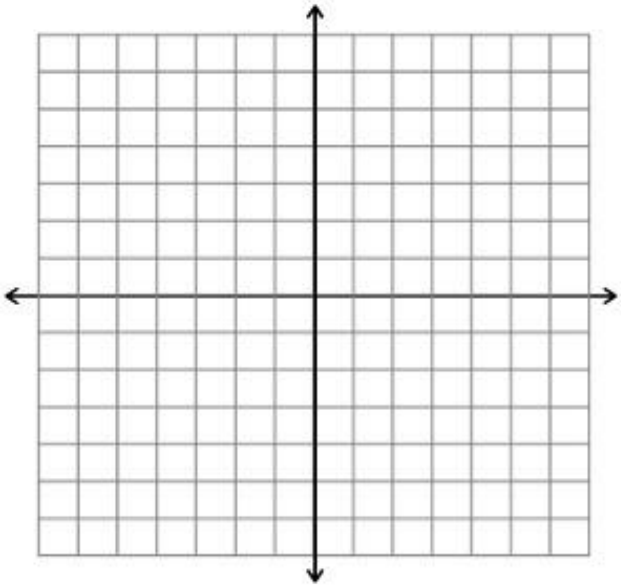
There are two special situations that can occur with graphing.

**Example**

a. $y = -3x + 2$
$9x + 3y = 6$



b. $y = -\frac{1}{2}x + 4$
$2x + 4y = 12$





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**Lesson: Solving Systems of Equations by Substitution**

In this lesson, we will focus on the second method for solving systems: substitution.

Show how to use substitution with this example:

$2x + y = 6$

$3x + 5y = 9$

**<u>Example</u>** Solve the system using substitution

$x - 2y = 10$

$5x - 7y = 20$



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Below are 2 more complicated problems using substitution. Outline the steps shown in the video:

$-2x + 4y = 4$

$-2x + 6y = 6$

$2x - 5y = -4$

$8x - 9y = 6$



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Now let's see what happens when we get a problem with no solution or infinite solutions.

a. $9y - 18x = 5$

$2x - y = 3$

b. $x + 2y = -3$

$4x + 12 = -8y$



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**Lesson: Solving Systems of Equations by Elimination**

In this lesson, we will focus on the third method for solving systems: elimination.

Show how to use elimination with this example:
$3x - y = 8$

$2x + y = -3$

**<u>Example</u>** Use elimination to solve the system.

$4x + 2y = -6$
$3x - 2y = 6$



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Now let's examine a more complicated case of elimination. Write down how to approach these problems:

a. $3x - 6y = 12$

$5x - 2y = -4$

b. $5x - 10y = 5$
$2x - 3y = 6$



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Now let's investigate what happens in the situation where there is no solution or infinite solutions.

a. $5x - 2y = 1$
$-10x + 4y = 3$

b. $3x + y = 5$
$-9x - 3y = -15$



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**Lesson: Solving Systems of Equations that Require Clearing of Fractions**

**<u>Examples</u>**

a. $\frac{1}{6}x - \frac{1}{3}y = \frac{1}{2}$
$\frac{2}{5}x - \frac{3}{5}y = -\frac{6}{5}$

b. $-\frac{1}{2}x - \frac{3}{4}y = 1$
$\frac{1}{3}x + \frac{3}{2}y = -\frac{2}{3}$



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c. $\frac{1}{2}x + \frac{3}{5}y = 6$

$\frac{1}{4}x - \frac{1}{2}y = -9$



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**Lesson: Solving Systems of Equations that Require Clearing of Decimals**

**<u>Examples</u>**

a. $0.02x + 0.1y = 0.05$
$4x + 20y = 10$

b. $0.1x - 0.05y = 0.2$
$0.3x - 0.25y = 0.2$

c. $-0.4x + 1.3y = 1.$
$1.6x - 0.9y = 6.9$



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**Lesson: Applications of Systems**

The top of a rectangular desk is twice as long as it is wide. The perimeter of the desk is 180 inches. What are the dimensions of the desk?

Sarah bought 3 shirts and a scarf for $53. Marie bought 2 shirts and 3 scarves for $54. Find the cost of each item.

How many milliliters of a 5% acid solution and how many milliliters of a 17% acid solution must be mixed to obtain a 60 mL solution that is 15% acid?



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**Lesson: Systems of Equations with 3 Equations**

Show how to solve the system:

$x + 2y + 3z = -11$

$3x - y + z = 0$

$-2x + 3y - z = 4$



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Sometimes you will have missing variables. Show how to solve the following system that is missing some variables:

$x - 2y = -3$

$x + 6z = 3$

$2y - 4z = 6$



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Now let's look at an example where there is no solution versus infinite solutions:

$8x + 20y - 4z = 16$

$-2x - 5y + z = -4$

$10x + 25y - 5z = 20$

$x + 4y - 3z = 2$

$2x - 3y + 2z = -5$

$3x + 12y - 9z = 2$



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**Examples: More Examples of Systems of 3 Equations**

**<u>Example</u>** Solve the system:

$x + 3y - z = -2$

$-x + 3y - 2z = 5$

$2x + 4y + 2z = 4$



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**<u>Example</u>** Solve the following system:

$x - 2z = 2$

$-4y + 2z = -2$

$-x + 3y = -3$



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**<u>Example</u>** Solve the following system:

$x - 4y + 2z = -7$

$3x - 8y + z = 7$

$6x - 12y + 3z = 12$



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**Lesson: Using Matrices to Solve Systems of Linear Equations**

What is a matrix?

Show how to setup the associated **augmented matrix** for the system: $\begin{cases} 3x + 4y = 4 \\ x - y = 13 \end{cases}$

What are the dimensions of this matrix?

In general, how do we state dimensions of a matrix (circle): rows x columns columns x rows

**<u>Example</u>** What is the dimension of

a.

$$\begin{bmatrix} 3 & 2 & 4 & 1 \\ 7 & 8 & 1 & 2 \\ 9 & 5 & -2 & -3 \end{bmatrix}$$

b.

$$\begin{bmatrix} 2 & 4 \\ 5 & 3 \\ 8 & 3 \\ -1 & -4 \end{bmatrix}$$



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With matrices, there are 3 **row transformations** that we can make. We are not allowed to do anything else to a matrix. Write down these 3 transformations.

Now, recall, our goal in this lesson is to use a matrix to solve systems of linear equations. What do we call this specific solving process?

What do we call the “ideal form” of a matrix in its solved form?



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The Gauss-Jordan method has a goal of transforming a matrix into reduced-row echelon form or diagonal form. Draw a few examples of how the FINAL matrix should look after we transform it:

When we use the Gauss-Jordan method, which direction do we want to work on the matrix?

What is the general strategy to remember with this method?



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**Example** Outline the steps for the Gauss-Jordan method for the system: $\begin{cases} -2x + 4y = -4 \\ x - y = 4 \end{cases}$

**Example** Use Gauss-Jordan method for the system: $\begin{cases} 3x + 4y = 4 \\ x - y = 13 \end{cases}$



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**Lesson: Using The Gauss-Jordan Method for a System with 3 Equations**

**<u>Example</u>** Use the Gauss-Jordan method for the system: $\begin{cases} x + y + z = 2 \\ 2x + y - z = 5 \\ x - y + z = -2 \end{cases}$



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Another page for you to show your work …



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**Lesson: Using Matrices to Solve Systems of Linear Equations when There is No Solution or Infinite Solutions**

**<u>Example</u>** Use the Gauss-Jordan method to solve: $\begin{cases} 3x + 5y - z = -2 \\ 4x - y + 2z = 1 \\ -6x - 10y + 2z = 0 \end{cases}$

**<u>Example</u>** Use the Gauss-Jordan method to solve: $\begin{cases} 3x + 5y = -2 \\ 9x + 15y = -6 \end{cases}$



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**<u>Example</u>** Use the Gauss-Jordan method to solve: $\begin{cases} x - 2y + 3z = 6 \\ 2x - y + 2z = 5 \end{cases}$



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**<u>Example</u>** Use the Gauss-Jordan method to solve: $\begin{cases} 5x - 4y + z = 0 \\ x + y = 0 \\ -10x + 8y - 2z = 0 \end{cases}$



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**Examples: Using Matrices to Solve Systems of Linear Equations (any type of solutions)**

Use the Gauss-Jordan method to solve the following.

a. $\begin{cases} x + 4y - z = 6 \\ 2x - y + z = 3 \\ 3x + 2y + 3z = 9 \end{cases}$



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b. $\begin{cases} 2x + y - 3z = 0 \\ 4x + 2y - 6z = 0 \\ x - y + z = 0 \end{cases}$



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c. $\begin{cases} 3x - 2y + z = 15 \\ x + 4y - z = 11 \end{cases}$



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**Lesson: How to Calculate the Determinate of a Matrix**

Every n x n matrix has a real number associated with it called the **determinant.** What is the notation to represent this?

**<u>Example</u>** Show, in detail, how to calculate the determinants of the following matrices.

a. $A = \begin{bmatrix} 2 & 4 \\ -5 & 7 \end{bmatrix}$

b. $B = \begin{bmatrix} -5 & 1 \\ -3 & 4 \end{bmatrix}$



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**<u>Example</u>** Show, with illustrations, how to calculate the determinant of $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -3 \\ 3 & 4 & -2 \end{bmatrix}$

The 2x2 "sub" matrices that we calculate are called minors. The symbol $M_{ij}$ represents the minor that results when row i and column j are eliminated. Using the matrix above, describe the following minors.

$M_{11} =$

$M_{32} =$



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The number that we multiply the minor against is called the **cofactor.** Draw an illustration to show whether a cofactor will be positive or negative:

**<u>Example</u>** Using the matrix below, describe the minor associated with each cofactor AND determine whether the cofactor is positive or negative. You do not need to calculate the determinant.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ -3 & -5 & -1 \end{bmatrix}$$

a. 2

b. 3

c. -3



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**<u>Example</u>** Show how to efficiently calculate the determinants of the following matrices. EXPLAIN YOUR REASONING FOR WHY YOU DID IT THE WAY THAT YOU DID.

a. $A = \begin{bmatrix} 2 & 7 & 8 \\ -1 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}$

b. $B = \begin{bmatrix} 9 & -3 & 1 \\ 2 & 3 & 0 \\ 5 & 2 & 0 \end{bmatrix}$



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**Lesson: Determinant Theorems**

**Theorem 1:** If every element (or column) of a matrix is 0, then the determinant will be 0.

**<u>Example</u>**: find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 8 & 9 \\ 3 & 2 & 8 & 4 \\ 9 & 3 & 1 & 7 \end{bmatrix}$

**Theorem 2:** If the rows of a matrix A are the corresponding columns of a matrix B, then the determinants are equal.

**<u>Example:</u>** Calculate both determinants: $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$



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**Theorem 3:** If any 2 rows (or columns) of a matrix are interchanged, then the determinants are negatives of one another.

**<u>Example</u>** Find the determinants of: $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 2 & 4 \end{bmatrix}$

**Theorem 4:** Suppose matrix B is formed by multiplying every element in a row or columns in a matrix A by a real number k. Then the determinant is $|B| = k|A|$.

**<u>Example:</u>** Find the determinants of: $A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 4 \\ 3 & 6 \end{bmatrix}$



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**Theorem 5:** If two rows (or columns) of a matrix are identical, then the determinant will be 0.

**<u>Example</u>** Find the determinant of: $A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

**Theorem 6:** Changing a row (or column) or a matrix by adding it to a constant times another row (or column) does not change the determinant of the matrix.

**<u>Example</u>** Find the determinant of $A = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$. Then create a matrix B by taking adding $3 *$ (row 3) to row 1. Find the new determinant.



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**Theorem 7:** If a matrix is in triangular form (all 0s either above or below the diagonal), then the determinant is found by multiplying the elements down the diagonal.

**<u>Example</u>**: Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 8 & 7 & 3 & 1 \end{bmatrix}$

**<u>Examples</u>** Use the determinant theorems to find the following determinants.

a. $\begin{vmatrix} -4 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{vmatrix}$

b. $\begin{vmatrix} 1 & 0 & 8 \\ 4 & 0 & 2 \\ 0 & 0 & 8 \end{vmatrix}$



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**Lesson: Properties of Matrices: Equality, Addition, Subtraction, and Scalar Multiplication**

When are 2 matrices equal?

**<u>Example</u>** What values of x, y, z, and w will make the matrices equal?

$$\begin{bmatrix} 2+x & 3-w \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2-y & 2+z \end{bmatrix}$$

When can you add 2 matrices?



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What is the definition of matrix subtraction?

What is scalar multiplication with a matrix?

List the properties of scalar multiplication for matrices:



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**<u>Example</u>** Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 8 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 7 \\ 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, E = \begin{bmatrix} 7 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Complete the following operations, if possible.

a. $A + C$

b. $B - D$

c. $B - E$

d. $2B + E$

e. $2A - 3C$



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**Lesson: Matrix Multiplication**

First, let's review dimension. What is the dimension of the following matrices?

$A = \begin{bmatrix} 3 & 2 \\ 1 & 8 \end{bmatrix}$

$B = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}$

To understand how to multiply matrices, let's use the following 2 examples:

a. $\begin{bmatrix} 2 & 7 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 7 & 2 \end{bmatrix}$



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**<u>Example</u>** Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 8 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 7 \\ 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, E = \begin{bmatrix} 7 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Complete the following operations, if possible.

a. $AC$

b. $CA$

Using the above example, what can we say about matrix multiplication and commutativity?



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List the properties of matrix multiplication:

**<u>Example</u>** Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 8 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 7 \\ 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, E = \begin{bmatrix} 7 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Complete the following operations, if possible.

a. $BC$

b. $ED$

c. $CB$



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**Lesson: Inverse Matrices**

What is the 2x2 identity matrix?

What is the 3x3 identity matrix?

How do we describe the general nxn identity matrix?



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**<u>Example</u>** Multiply $A = \begin{bmatrix} 3 & 2 & 1 \\ 8 & 1 & 7 \\ 3 & 1 & 0 \end{bmatrix}$ by the identity matrix as follows:

a. $AI$

b. $IA$

What can we generalize about the identity matrix?



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How do we describe a multiplicative inverse of a real number?

Some matrices have inverses. For a matrix A, what notation denotes its inverse?

What is the property of inverse matrices?

What are the steps for finding an inverse matrix?



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**<u>Example</u>** Find the inverses of the following matrices. Prove your answer by checking it: $\begin{bmatrix} 6 & -2 \\ 2 & 4 \end{bmatrix}$



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**<u>Example</u>** Find the inverses of the following matrices. Prove your answer by checking it: $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$



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**<u>Example</u>** Find the inverses of the following matrices. Prove your answer by checking it: $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$



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**Lesson: Using Inverse Matrices to Solve Systems**

We've previously talked about using matrices to solve systems. We will look at another way to do that in this lesson. Show another way to set up the system using matrices (as well as the relevant notation):

$$\begin{cases} x + y + z = 6 \\ 2x + 3y - z = 7 \\ 3x - y - z = 6 \end{cases}$$

Now, using our setup above, explain the reasoning for how inverse can help solve a system:



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**<u>Example</u>** Use inverse matrices to solve: $\begin{cases} -x + y = 1 \\ 2x - y = 1 \end{cases}$



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**<u>Example</u>** Solve the system: $\begin{cases} x + y + z = 6 \\ 2x + 3y - z = 7 \\ 3x - y - z = 6 \end{cases}$ using inverse matrices.



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**Lesson: Sequences**

Define a sequence.

**<u>Examples</u>** Write the first five terms of each sequence.

a. $a_n = \frac{1}{n}$

b. $a_n = \frac{n\mp 1}{n+3}$

c. $a_n = (-1)^{n+1} * (2n + 1)$



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What does it mean for a sequence to converge?

What does it mean for a sequence to diverge?

What is a recursively defined sequence?



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**Examples** Find the first four terms in the sequence.

a. $a_1 = 3; a_n = 2a_{n-1} - 4$

b. $a_1 = 2; a_n = a_{n-1}^2 + n - 2$



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**Lesson: Series**

What is the summation sign and what does it mean? What do we called the “i” in this formula?

Define a finite series.

Define an infinite series.



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**<u>Examples</u>** Evaluate the following sums.

a. $\Sigma_{i=1}^{4}\left(3^i + 1\right)$

b. $\Sigma_{j=3}^{6} a_j$

c. $\Sigma_{k=1}^{4}(4x_k - 1)$ if $x_1 = 2, x_2 = 3, x_3 = 5, x_4 = 8$

d. $\Sigma_{i=1}^{3} f(x_i)\Delta x$, if $f(x) = 3x, x_1 = 1, x_2 = 4, x_3 = 7, \Delta x = 3$



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What are the summation properties?

What are the summation rules for summing up terms?



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**Examples** Use the summation properties and rules to evaluate each series.

a. $\Sigma_{i=1}^{50} 3$

b. $\Sigma_{k=1}^{25} 3k$

c. $\Sigma_{j=1}^{33} (4j^2 + 1)$



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**Lesson: Arithmetic Sequences**

What is an arithmetic sequence?

Give some examples of an arithmetic sequence.

**<u>Example</u>** Determine the common difference, d, for the arithmetic sequence:

$$-7, -5, -3, -1, \ldots$$



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**Example** Find the first five terms of each arithmetic sequence.

a. The first term is 3, and the common difference is -2.

b. $a_1 = -5, d = 7$

How do we define the general nth term of an arithmetic sequence?



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**Example** Determine $a_{11}$ and $a_n$ for the arithmetic sequence $-2, 2, 6, 10, \ldots$

**Example** Determine $a_n$ and $a_{20}$ for the arithmetic sequence having $a_3 = 9$, and $a_4 = 16$.

**Example** An arithmetic sequence has $a_7 = -15$ and $a_{14} = -35$. Determine $a_1$.



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**Lesson: Arithmetic Series**

When we sum up the terms of an arithmetic sequence, how can we define that sum? (sum of an arithmetic sequence)

**<u>Example</u>** Consider the arithmetic sequence $-7, -3, 1, 5, \ldots$

a. Evaluate $S_{10}$

b. Evaluate the sum of the first 50 positive integers.



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**Example** The sum of the first 15 terms of an arithmetic sequence is 345. If $a_{15} = 65$, find $a_1$ and d.

**Examples** Evaluate each sum.

a. $\Sigma_{i=1}^{19}(-3 - 5i)$

b. $\Sigma_{k=4}^{15}(2 + 4k)$



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**Lesson: Geometric Sequences and Series**

Define a geometric sequence:

**<u>Example</u>** Determine $a_6$ and $a_n$ for the geometric sequence 4, 12, 36, 108, ….

**<u>Example</u>** Determine r and $a_1$ for the geometric sequence with $a_2 = -18$ and $a_5 = 486$



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How do we define the sum of the first n terms of a geometric sequence?

**<u>Example</u>** Evaluate: $\Sigma_{i=1}^{8} 3 * 2^i$

If we start with 2 and use a common ratio of $\frac{1}{2}$, what eventually happens as the series continues infinitely? Summarize the explanation below.



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**Examples** Evaluate the following infinite sums.

a. The first term is 1 and the common ratio is $\frac{1}{3}$

b. $\Sigma_{i=1}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{3}{4}\right)^{i-1}$

c. $\Sigma_{i=1}^{\infty}\left(\frac{3}{7}\right)^{i}$



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**Lesson: The Binomial Theorem**

Below we can observe the pattern for binomial expansion:

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Focusing on the VARIABLES x and y only, how can we generalize the exponents for the variables?

Pascal's Triangle can explain how to find the coefficient. Draw out Pascal's triangle below.



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Having to draw out this pattern would be very difficult for larger expansions, so we will explore a few ways to simplify how to expand this pattern.

First, we need to understand factorials.

**n-factorial:** for any positive integer n,

$$n! = n(n-1)(n-2)\dots(3)(2)(1)$$

By definition: $0! = 1$

**<u>Examples</u>** Evaluate:

a. $5!$

b. $8!$

Now that we understand factorials, we can discuss the formula to find binomial coefficients.

For nonnegative integers $n$ and $r$, $r \leq n$, the binomial coefficient is defined as follows:

**<u>Examples</u>** Evaluate each binomial coefficient:

a. $\binom{5}{2}$

b. $\binom{6}{0}$



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We can summarize how to expand $(x+y)^n$ as follows:

1. There are n+1 terms in the expansion

2. The first and last terms are $x^n$ and $y^n$ respectively

3. In each succeeding term, the exponent on x decreases by 1 and the exponent on y increase by 1

4. The sum of the exponents on x and y in any term is n.

5. The coefficient of the term with $x^r y^{n-r}$ or $x^{n-r} y^r$ is $\binom{n}{r}$

These observations are captured in the binomial theorem.

State **The Binomial Theorem:**

**<u>Example</u>** Find the first, last, and seventh term of the binomial expansion: $(3x+2y)^{10}$



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**<u>Example</u>** Use the binomial theorem to expand: $(a-b)^5$



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**Examples: Finding Average Rates of Change**

Calculate the average rate of change of the functions over the given intervals.

a. $f(x) = x^2 + 3x; [0, 3]$

b. $f(x) = 2 + \sin x; \left[\frac{\pi}{2},\ \pi\right]$

c. $f(x) = \sqrt{4x + 1}; [0, 2]$



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**Examples: Determining the Slope at a Point and the Tangent Line**

Find the slope of the curve at the given point and find the equation of the tangent line at that point. Do this only using the formula for the slope of the secant line.

a. $f(x) = x^2 - 1, (2, 3)$

b. $f(x) = x^2 - 4x; (2, -4)$

c. $y = \frac{1}{x}; (2, \frac{1}{2})$



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**Lesson: An Intro to Limits**

Graph $f(x) = \frac{(x^2-4)}{x-2}$ below. What does it look like this graph should equal when x=2?

Explain this previous question now in mathematical terms

**<u>Example</u>** Find $\lim_{x \to 1} f(x)$ when $f(x) = \begin{cases} x^2 + 1 & x \neq 1 \\ -2 & x = 1 \end{cases}$. Draw a picture of this situation.

What is the value of $f(1)$?



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**Example** What is $\lim_{x \to 2} f(x)$ when $f(x) = x$? Draw a graph to accompany your answer.

What do we call the function $f(x) = x$?

What can we say about the limit of these types of functions?

**Example** What is $\lim_{x \to -3} f(x)$ when $f(x) = 4$? Draw a graph to accompany your answer.

What do we call the function $f(x) = c$, where c is a constant?

What can we say about the limit of these types of functions?



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Write out all the limit laws on this space:

Write out the other 2 handy theorems:

Theorem 1: Limits of Polynomials

Theorem 2: Limits of Rational Functions



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**Example** Using the limit laws, calculate the following limits:

a. $\lim_{x \to 3}(2x + 1)$

b. $\lim_{x \to 1} \frac{5x+2}{x-4}$

c. $\lim_{x \to 2} \sqrt{5x + 4}$

d. $\lim_{x \to -3} 5\sqrt{2 - 3x}$



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**Examples: Intro to Calculating Limits**

Calculate the following limits:

a. $\lim_{x \to 3}(x^2 + 2x + 1)$

b. $\lim_{x \to 0}(8 - 2x)^{\frac{2}{3}}$

c. $\lim_{x \to \frac{2}{5}}(5x + 2)(3x - 1)$

d. $\lim_{x \to \frac{1}{3}} \sqrt[3]{3x + 7}$



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**Lesson: Troubleshooting Limits With Algebraic Tricks (When You Get 0/0)**

Trick #1 (name this trick here):

Note what happens if we try to evaluate $\lim_{x \to 2} \frac{x^2-4}{x-2}$

Draw what this situation looks like graphically.

How can we solve this problem?

**<u>Example</u>** Evaluate $\lim_{x \to 3} \frac{(x^3-27)}{x-3}$



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Trick #2 (name it):

**<u>Examples</u>** Evaluate:

a. $\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x}$

b. $\lim_{x \to 1} \frac{2-\sqrt{x^2+3}}{x-1}$



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Trick #3 (name it):

**<u>Examples</u>** Evaluate

a. $\lim_{x\to 3} \frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{x-3}$

b. $\lim_{x\to 1} \frac{\left(1-x^{-1}\right)}{1-x}$



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**Examples: Algebraic Tricks for Limits**

Evaluate the following:

a. $\lim_{x \to -4} \frac{\left(\frac{1}{\sqrt{12+x}} - \frac{1}{3}\right)}{x+4}$

b. $\lim_{x \to 0} \frac{\left(\frac{1}{3+x} - \frac{1}{3}\right)}{x}$

c. $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$



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**Lesson: Limits that Do Not Exist**

List the types of limits that do not exist, as well as an example and graph



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**Lesson: The Sandwich Theorem and Basic Trig Limits**

Explain the idea of the Sandwich Theorem:

State the Sandwich Theorem

**<u>Example</u>** Suppose there is a function $f(x)$ where $3 - x^2 \leq f(x) \leq 3\cos x$. Find $\lim_{x \to 0} f(x)$.



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Show how to use the Sandwich Theorem to evaluate the following limits:

a. $\lim_{x \to 0} \sin x$

b. $\lim_{x \to 0} \cos x$



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**Examples: Basic Trig Limits**

Evaluate the following:

a. $\lim_{x \to 0} \frac{3+x+\sin x}{4\cos x}$

b. $\lim_{x \to \frac{\pi}{3}} \sec x$

c. $\lim_{x \to \pi} \sin^2 x$



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**Lesson: Finite Limits as $x \to \infty$ and Limits That Go To Infinity, the Conceptual Idea**

Recreate the graph below, then find the following limits:

a. $\lim_{x \to \infty} f(x)$

b. $\lim_{x \to -\infty} f(x)$

c. $\lim_{x \to 3^+} f(x)$

d. $\lim_{x \to 2^-} f(x)$

e. $f(2)$

f. $\lim_{x \to 2^+} f(x)$

g. $\lim_{x \to 3} f(x)$

h. $\lim_{x \to -4} f(x)$



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**Lesson: Finite Limits as $x \to \infty$ and Rational Functions**

Do the limit laws apply to limits as $x \to \infty$? Yes No

How do you evaluate limits at infinity for rational functions?

In the following examples, we will see all the different ways this trick might be carried out:

**<u>Examples</u>** Evaluate the following:

a. $\lim_{x\to\infty} \frac{4x+3}{x^2+4x+3}$

b. $\lim_{x\to-\infty} \frac{5x^2+3x+1}{7x^2+5x-4}$

c. $\lim_{x\to\infty} \frac{6x^2+3x+1}{x+2}$



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This trick can be extended.

**Examples**

a. $\lim_{x\to\infty} \sqrt{\frac{x^2+5}{3x^2-6}}$

b. $\lim_{x\to-\infty} \frac{x^{\frac{2}{3}}-x^{\frac{1}{3}}+2}{x^2+3}$

c. $\lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+3}}$



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**Examples: Limits as $x \to \infty$**

Find the horizontal asymptotes / evaluate the following:

a. $\lim_{x\to\infty} \frac{\sin x}{x^2}$

b. $\lim_{x\to\infty} \frac{7x^4-2x^3+8}{9x^4+2x^2+x-1}$

c. $\lim_{x\to-\infty} \frac{\sqrt{25x^2+x}}{x}$



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d. $\lim_{x\to\infty}\left(x-\sqrt{x^4+2x^2}\right)$

e. $\lim_{x\to-\infty}(5x^7+2x^3+8)$

f. $\lim_{x\to-\infty}\frac{x^3+2x+5}{x^2+3x+1}$



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**Lesson: One-Sided Limits**

Use the following function to describe one-sided limits: $f(x) = \begin{cases} 3, & x > 2 \\ 1, & -1 \leq x \leq 2 \\ -2, & x < -1 \end{cases}$

Evaluate the following:

a. $\lim_{x \to 2^+} f(x)$

b. $\lim_{x \to 2^-} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to -1^+} f(x)$

e. $\lim_{x \to -1^-} f(x)$

f. $\lim_{x \to -1} f(x)$



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Draw the graph presented in the video then evaluate the following:

a. $\lim_{x \to 3^+} f(x)$

b. $\lim_{x \to 3^-} f(x)$

c. $f(3)$

d. $\lim_{x \to -2^+} f(x)$

e. $\lim_{x \to 4^-} f(x)$

f. $\lim_{x \to 4^+} f(x)$

g. $\lim_{x \to 4} f(x)$

h. $f(4)$



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Write down the theorem that explains when $\lim_{x \to c} f(x)$ exists:

**<u>Examples</u>** Evaluate the following:

a. $\lim_{x \to 2^+} \sqrt{\frac{x-2}{x+1}}$

b. $\lim_{x \to 2^-} \left(\frac{1}{x+1}\right)\left(\frac{x-3}{x}\right)\left(\frac{4}{4-x}\right)$

c. $\lim_{x \to -3^-} \frac{(x+1)|x+3|}{x+3}$



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**Examples: Basic One-Sided Limits**

a. $\lim_{x \to 2^+} \sqrt{x^2 - 4}$

b. $\lim_{x \to 0^+} \frac{\sqrt{2x^2+3x+1}-1}{x}$

c. $\lim_{x \to 3^-} \begin{cases} 4, & x > 3 \\ -2, & x \leq 3 \end{cases}$

d. $\lim_{x \to 5^-} \frac{x-5}{|x-5|}$



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**Examples: One-Sided Limits with Absolute Value**

a. $\lim_{x \to 2^+} \frac{2-x}{|x-2|}$

b. $\lim_{x \to 5^-} \frac{(x-4)|x-5|}{x-5}$

c. $\lim_{x \to 2^+} \frac{\sqrt{3x}(x-2)}{|x-2|}$

d. $\lim_{x \to 0^-} \frac{\sin x}{|\sin x|}$



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**Lesson: The $\boldsymbol{lim_{x\to 0}\frac{sin\ x}{x} = 1}$**

This lesson is about the tricky but very important limit:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Or equivalently:

$$\lim_{x\to 0}\frac{x}{\sin x} = 1$$

The proof is interesting and worth noting, you can write it down if you wish but you should minimally attempt to understand the reasoning behind it. It's difficult, but difficult things will expand your mind!



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**Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\ x}{x} = 1}$**

a. $\lim\limits_{x\to 0}\frac{\sin 3x}{5x}$

b. $\lim\limits_{x\to 0} 12x^2(\cot 2x)\ (\csc 3x)$

c. $\lim\limits_{x\to 0}\frac{x^3-x+\sin 5x}{3x}$

d. $\lim\limits_{x\to 0}\frac{\sin x}{\sin 3x}$



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**Step-by-Step Examples: Limits using $\boldsymbol{lim_{x\to 0}\frac{sinx}{x}=1}$**

a. $\lim\limits_{x\to 0}\frac{\sin 7x}{x}$

b. $\lim\limits_{x\to 0} x\cot 3x$



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**More Examples: The $\boldsymbol{lim_{x\to 0}\frac{sin\,x}{x} = 1}$**

a. $\lim\limits_{x\to 0}\frac{1}{x^2}\sin^2\left(\frac{x}{2}\right)$

b. $\lim\limits_{x\to 0}\frac{\sin x}{\sqrt{x}}$

c. $\lim\limits_{x\to 0}\frac{1-\cos 2x}{x}$

d. $\lim\limits_{x\to 0}\frac{\sin 3x}{\mathrm{x}\cos 5x}$



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Examples: Troubleshooting with $\lim_{x \to 0} \frac{\sin x}{x}$ problems – examples of what not to do

a. $\lim_{x \to 0} \frac{\tan 5x}{x}$

b. $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$

**Lesson: Continuity**

The idea of continuity is probably intuitive at the point. Draw the example in the video to demonstrate continuity.

For a point c, define when a function is:

1. Continuous

2. Right – continuous

3. Left – continuous



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Draw the graph in the video to demonstrate these ideas.

**<u>Example</u>** Draw the graph in the video, then describe the continuity at the following points:

$x = 1$ $x = 3$ $x = -4$ $x = -2$



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A function is continuous at a point c is it meets these 3 conditions:

1. f(c) exists

2. the limit $\lim_{x \to c} f(x)$ exists

3. $\lim_{x \to c} f(x) = f(c)$

Describe the different kinds of discontinuities below:

What does it mean for a function to be continuous?

If a function is not continuous, then it is _____

If $f(x)$ and $g(x)$ are continuous functions at the point x=c, then the following are continuous at c:

1. sum/differences:

2. constant multiples (for any number k):

3. products:

4. quotients:

5. powers:

6. roots:

7. compositions:



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**Example** Describe where the following functions are continuous:

a. $f(x) = \frac{x+3}{x-2}$

b. $f(x) = \sqrt{2x}$

What can we say about the limits of a composition of continuous functions $f(x)$ and $g(x)$?

**Example** Find the limit $\lim_{x \to 0} (x+2)e^{\sin x}$



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**Examples: Continuity**

Determine where the following functions are continuous:

a. $f(x) = \frac{1}{(x+3)^2} - 2$

b. $g(x) = \begin{cases} 1 + x, & x \geq 3 \\ 4, & -2 \leq 0 < 3 \\ 2 - x, & x < -2 \end{cases}$

c. $h(x) = \begin{cases} |x + 1|, & x \geq -2 \\ -5, & x < -2 \end{cases}$



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**Examples: Using Continuity with Limits**

Find the limits.

a. $\lim_{x \to 0} \cos\left(x + \cos\left(\frac{\pi}{2} + x\right)\right)$

b. $\lim_{x \to \frac{\pi}{2}} \sin(\cos(\tan x))$

c. $\lim_{x \to 0} \sin^{-1}\left(e^{\cos\left(\frac{\pi}{2} + x\right)}\right)$



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**Lesson: The Intermediate Value Property and Continuous Functions**

What is the Intermediate Value Property?

There are 2 consequences of this:

1. Connectedness:

2. Root Finding:



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**Example** Determine the value of a that would make the function continuous at every x:

$$f(x) = \begin{cases} x^2 + 1, & x > 2 \\ ax, & x \leq 2 \end{cases}$$

**Example** Show that $x^3 - 14x + 2 = 0$ has 3 solutions in the interval $[-4, 4]$



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**Examples: Continuous Extensions**

Determine the value of a that would make the following functions continuous:

a. $f(x) = \begin{cases} 2x, & x > 1 \\ ax^2, & x \leq 1 \end{cases}$

b. $g(x) = \begin{cases} \frac{2x+a}{a+2}, & x > 0 \\ x^2 + a, & x \leq 0 \end{cases}$



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**Step-by-Step Examples: Continuous Extensions**

Use the space below to find a (or a and b) to make the following functions continuous:

a. $f(x) = \begin{cases} x + a, & x \geq 1 \\ 3, & x < 1 \end{cases}$

b. $f(x) = \begin{cases} 3, & x \geq 1 \\ ax + b, & -2 < x < 1 \\ -3, & x \leq -2 \end{cases}$



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**Lesson: The Formal Definition of the Derivative**

One of the things we learn in pre-calculus is about secant lines. You can write a review of this below:

When the distance between two points for a secant line goes to zero, what do we get?

What is the formal definition of the derivative?

What is the alternate formula for the derivative?

Which formula should you use?



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How do you find the derivative at a specific point?

What 4 things does the derivative at a point represent?

1.

2.

3.

4.

**<u>Example</u>** Calculate the derivative of $f(x) = x^2 + 2$

**<u>Example</u>** What is the value of the derivative of $f(x) = x^2 + 2$ at $(1, 3)$?



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**Example** Find the equation of the tangent line for $f(x) = x^2 + 2$ at $(1, 3)$

List all the different ways to represent a derivative below:

Write out the theorem that discusses differentiability and continuity

What types of functions will not have a derivative at a point?



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**Examples: Calculating Derivatives Using the Formal Definition Only and Finding Tangent Lines**

Use the formal definition of a derivative to calculate the follow derivatives of each function. Then determine the tangent line at the given point.

a. $f(x) = 4 - x^2; (-1, 3)$

b. $y = \frac{1-x}{2x}, (1, 0)$

c. $g(x) = x^{\frac{3}{2}}, (4, 8)$



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**Examples: Using the Alternative Formula for Derivatives**

Use the alternative formula for derivatives to calculate the derivative of the following function. Compare this calculation to the original formula.

a. $y = x^3 - x^2$

b. $y = \frac{1}{x+3}$

c. $f(x) = 2 + \sqrt{x}$



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**Step-by-Step Examples: Comparing the Definition of the Derivative to the Alternative Formula**

a. $f(x) = x + \frac{3}{x}$

b. $g(x) = x^3 - 5x^2 + 1$



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**Example: Finding the Derivative of a Square Root with the Definition of the Derivative**

Find the tangent line at (6,5) for $y = \frac{5}{\sqrt{7-x}}$ using only the definition of the derivative.



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**Lesson: Other Notes about the Derivative**

Define right-hand derivatives

Define left-hand derivatives

Explain how $y = |x|$ is differentiable on each side



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Interpreting Graphs of Derivatives

The derivative of $f(x) = x^2$ is $f'(x) = 2x$. Compare their graphs below:



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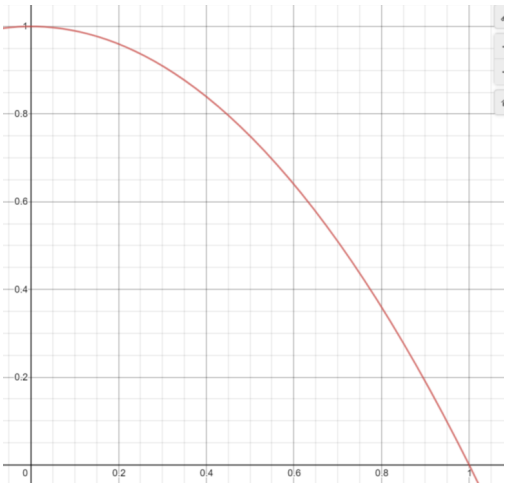
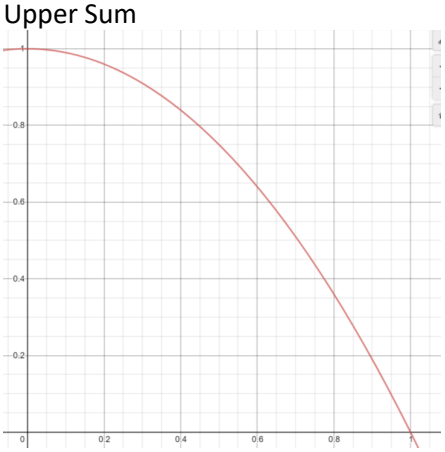
**Area and Estimating with Finite Sums**

First we will try to understand the idea behind this. Draw out one way to find the area of the curve $y = 1 - x^2$ below.

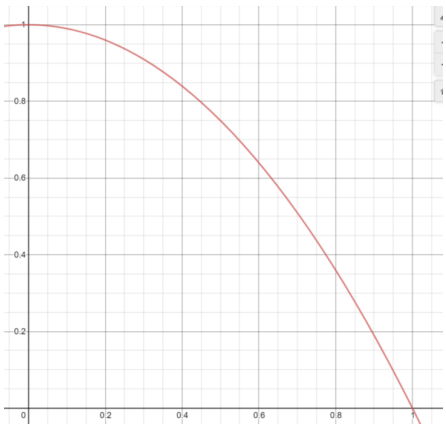



Now we need more formality around this idea. Show the idea of how to create sums using a more algorithmic approach. Detail what each sum means:

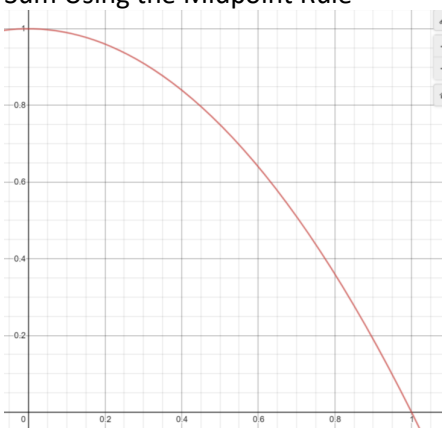
Upper Sum



Lower Sum



Sum Using the Midpoint Rule





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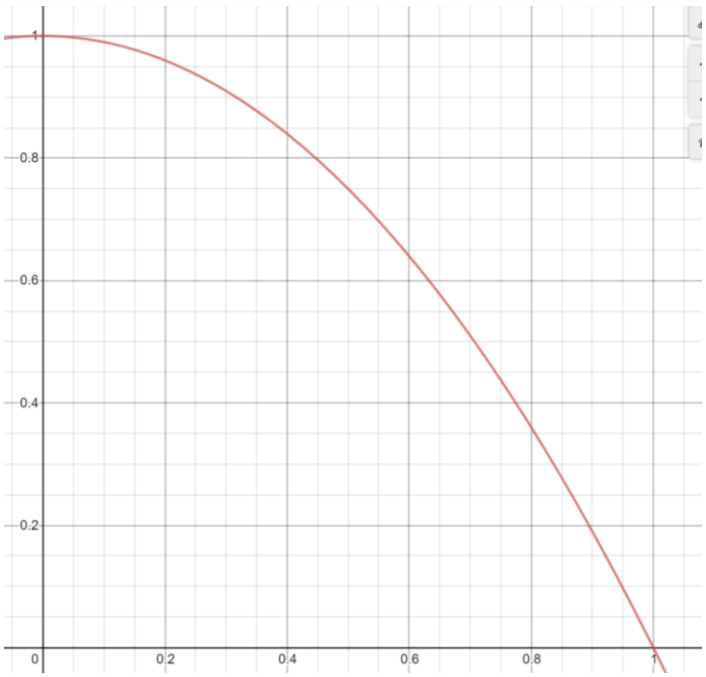
Now we need some vocabulary. Separate the graph in 4 parts and then label it with the correct terminology:




Define a Riemann Sum:

What is a left Riemann sum?

What is a right Riemann sum?

What is a midpoint Riemann sum?



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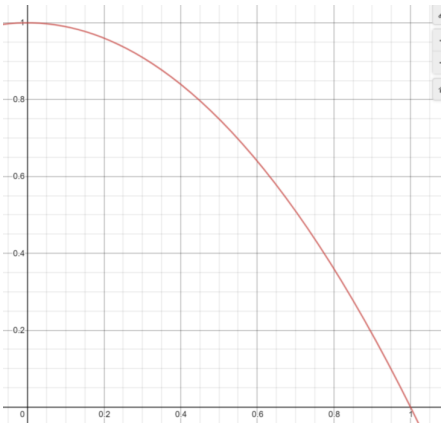
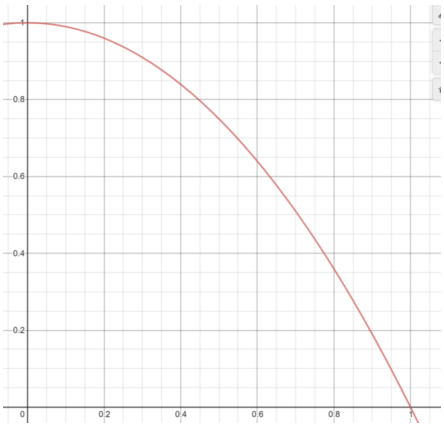
Show how you can make a more precise sum:

8 Subintervals

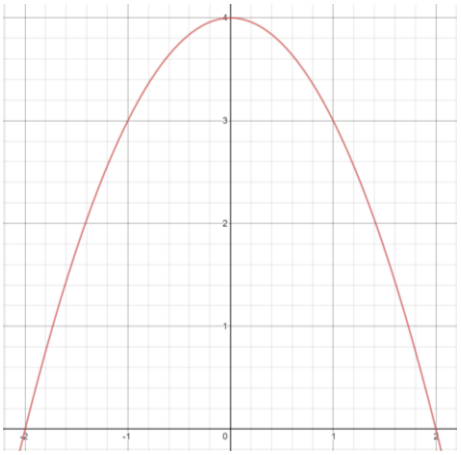


16 Subintervals

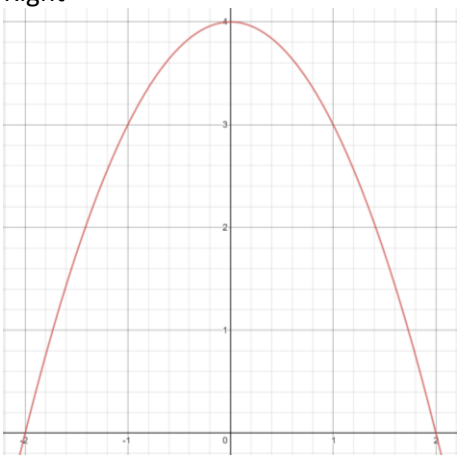


**<u>Example</u>** Compute the left, right, and midpoint Riemann sums for $f(x) = 4 - x^2$ between $[-2, 2]$ using 4 subintervals

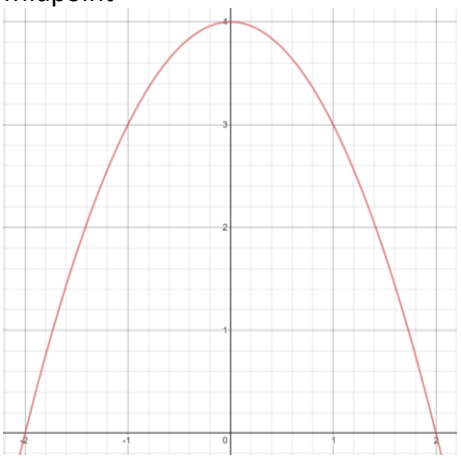
Left



Right



Midpoint



*Graphs were created using Desmos.com. They have the greatest graphing software for free, check them out!*



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**Examples of Riemann Sums**

Compute the left, right, and midpoint Riemann sum for $f(x) = 3 + x$ from $[1,3]$ using 4 subintervals.



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**Examples of Riemann Sums – When It Goes Negative?**

Compute the left, right, and midpoint Riemann sum for $f(x) = x^3$ from $[-2,2]$ using 4 subintervals.



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**Examples: Different Ways to Compute Area Estimations**

Using $f(x) = -x^2 + 2x + 3$, compute an area estimate using 4 subintervals from x=-1 to x=3.

Compute this estimate using an upper, lower, left-hand, and right-hand sum.



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## Not included (no notes file):

004 (Examples of the Greatest Common Factor)

005 (Examples of Factoring Basic Trinomials via Trial and Error)

006 (Examples of Grouping)

007 (Examples of the Sum and Difference of Cubes)

008 (Examples of Factoring Harder Trinomials)

013 (Examples of Simplifying Rational Expressions)

014 (Examples of How to Use the Multiplicative Property of -1 to Simplify Rational Expressions)

015 (Examples of Finding Domains of Rational Expressions)

018 (Examples of Multiplying and Dividing Rational Expressions)

022 (Examples of Finding Common Denominators for Rational Expressions)

112 (Lesson Introducing Where Calculus Starts)